Recent Advances in Mathematics

Quantum Communication

Example Sheet 3: Solutions

1. (i) $\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\left(\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle\right)$

$$= \frac{3}{5\sqrt{2}}|0\rangle|0\rangle + \frac{3}{5\sqrt{2}}|1\rangle|0\rangle + \frac{4}{5\sqrt{2}}|0\rangle|1\rangle + \frac{4}{5\sqrt{2}}|1\rangle|1\rangle$$

Since $|0\rangle|0\rangle$ is the same as $|00\rangle$, etc., the probability of the result 01 is $\left(\frac{4}{5\sqrt{2}}\right)^2 = \frac{8}{25}$.
(since the state is normalised).

(ii) $|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

so $\langle\Psi_+|\psi\rangle = \frac{1}{\sqrt{2}}\cdot\frac{3}{5\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\frac{4}{5\sqrt{2}} = \frac{7}{10}$

where $|\psi\rangle$ is the state in (i). The probability is therefore $\left(\frac{7}{10}\right)^2 = 0.49$.

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2. We have to express the state of Alice's qubit in terms of $|X\rangle$ & $|Y\rangle$. Solving the given eq^n's,

$$|0\rangle = \frac{3}{5}|X\rangle - \frac{4}{5}|Y\rangle$$

$$|1\rangle = \frac{4}{5}|X\rangle + \frac{3}{5}|Y\rangle$$

$\therefore$ The given state is

$$|\psi\rangle = \frac{1}{2}\left\{\left(\frac{3}{5}|X\rangle - \frac{4}{5}|Y\rangle\right)\left(|0\rangle + |1\rangle\right) + \left(\frac{4}{5}|X\rangle + \frac{3}{5}|Y\rangle\right)\left(|0\rangle - |1\rangle\right)\right\}$$

$$= |X\rangle\left(\frac{7}{10}|0\rangle - \frac{1}{10}|1\rangle\right) + |Y\rangle\left(-\frac{1}{10}|0\rangle - \frac{7}{10}|1\rangle\right)$$

$|\psi\rangle$ is normalised, since $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$, so the probability that Alice will get the result X is $\left(\frac{7}{10}\right)^2 + \left(-\frac{1}{10}\right)^2 = \frac{1}{2}$. If she does get this result, the state of the other qubit will be $\frac{7}{10}|0\rangle - \frac{1}{10}|1\rangle$.

If she gets the result Y, the state of the other qubit will be $-\frac{1}{10}|0\rangle - \frac{7}{10}|1\rangle$.

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3. Not all of the coefficients a, b, c, d can be zero, since we have to divide by $|a|^2 + |b|^2 + |c|^2 + |d|^2$. Suppose $a \neq 0$; then $d = \frac{bc}{a}$, so

$$|\Psi\rangle = a|0\rangle|0\rangle + b|0\rangle|1\rangle + c|1\rangle|0\rangle + \frac{bc}{a}|1\rangle|1\rangle$$

$$= \frac{1}{a}\left(a^2|0\rangle|0\rangle + ab|0\rangle|1\rangle + ac|1\rangle|0\rangle + bc|1\rangle|1\rangle\right)$$

$$= \frac{1}{a}\left(a|0\rangle + c|1\rangle\right)\left(a|0\rangle + b|1\rangle\right)$$

$$= |\varphi\rangle|\psi\rangle \quad \text{with } |\varphi\rangle = |0\rangle + \frac{c}{a}|1\rangle,$$

$$|\psi\rangle = a|0\rangle + b|1\rangle.$$

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4. If the measurement of the first qubit gets the result 0 (probability $\frac{1}{2}$), the second qubit goes into the state

$$|0\rangle = \cos\theta|X\rangle - \sin\theta|Y\rangle$$

Then the probabilities of the results X & Y for the second qubit are $\cos^2\theta$ & $\sin^2\theta$. If the first measurement gets result 1, the second qubit goes into the state $|1\rangle = \sin\theta|X\rangle + \cos\theta|Y\rangle$... Thus the four probabilities are

$0X : \frac{1}{2}\cos^2\theta$ $\qquad$ $0Y : \frac{1}{2}\sin^2\theta$

$1X : \frac{1}{2}\sin^2\theta$ $\qquad$ $1Y : \frac{1}{2}\cos^2\theta$.

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5. As in the lectures (handout 1.5),

$$P(a=0, b=0) - P(a=0, c=0) - P(b=0, c=1)$$

$$= \cos^2\beta - \cos^2\gamma - \sin^2(\beta-\gamma).$$

Call this $B(\beta,\gamma)$. For a maximum,

$$\frac{\partial B}{\partial \beta} = 0 \Rightarrow -2\cos\beta\sin\beta - 2\sin(\beta-\gamma)\cos(\beta-\gamma) = 0$$

$$\frac{\partial B}{\partial \gamma} = 0 \Rightarrow 2\cos\gamma\sin\gamma + 2\sin(\beta-\gamma)\cos(\beta-\gamma) = 0$$

$$\therefore \quad \sin 2\beta = -\sin 2(\beta-\gamma) = \sin 2\gamma$$

$$\therefore \quad 2\beta = 2\gamma \quad \text{or} \quad 2\beta = \pi - 2\gamma$$

$$\& \quad 2\beta = -2(\beta-\gamma) \quad \text{or} \quad 2\beta = 2(\beta-\gamma) + \pi$$

$$(4\beta = 2\gamma) \qquad\qquad (2\gamma = \pi)$$

All these equations must be taken mod 2π.

The possibilities are:

$2\beta = 2\gamma$ & $4\beta = 2\gamma$ $\Rightarrow$ $2\beta = 2k\pi$, $2\gamma = 2l\pi$

$B(\beta,\gamma) = 0$

$2\beta = 2\gamma$ & $2\gamma = \pi$ $\Rightarrow$ $\beta = (k+\frac{1}{2})\pi$, $\gamma = (l+\frac{1}{2})\pi$

$B(\beta,\gamma) = 0$

$2\beta = \pi - 2\gamma$ & $4\beta = 2\gamma$ $\Rightarrow$ $\beta = \frac{1}{6}(2k+1)\pi$, $\gamma = \frac{1}{3}(2l+1)\pi$

$$\therefore \quad \beta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \text{ or } \frac{11\pi}{6}$$

$$\gamma = \frac{\pi}{3}, \pi \text{ or } \frac{5\pi}{3}$$

$\cos^2\beta = 0$ or $\frac{3}{4}$, $\cos^2\gamma = \frac{1}{4}$ or 1, $\sin^2(\beta-\gamma) = \frac{1}{4}$ or $\frac{3}{4}$ or 1

Positive $B(\beta,\gamma)$ can only be obtained if $\cos^2\beta = \frac{3}{4}$, $\cos^2\gamma = \frac{1}{4}$ & $\sin^2(\beta-\gamma) = \frac{1}{4}$ (e.g. $\beta = \frac{\pi}{6}$, $\gamma = \frac{\pi}{3}$).

The final possibility is $2\beta = 2\gamma = \pi$, when $B(\beta, \gamma) = 0$. Hence the maximum value of $B(\beta, \gamma)$ is $\frac{1}{4}$, given by $\beta = \frac{\pi}{6}$, $\gamma = \frac{\pi}{3}$ as in the lectures.
