STABILITY OF THE THEORY OF EXISTENTIALLY CLOSED S-SETS OVER A RIGHT COHERENT MONOID S

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Abstract. Let L_S denote the language of (right) S-sets over a monoid S and let $\sum_S$ be a set of sentences in L_S which axiomatises S-sets. A general result of model theory says that $\sum_S$ has a model companion, denoted by T_S, precisely when the class $\mathcal{E}$ of existentially closed S-sets is axiomatisable and in this case, T_S axiomatises $\mathcal{E}$. It is known that T_S exists and is stable if and only if S is right coherent.

In the study of stable first order theories, superstable and totally transcendental theories are of particular interest. We describe types over T_S algebraically and use our result to confirm that T_S is stable. We prove that T_S is superstable if and only if S satisfies the maximal condition for right ideals. The situation for total transcendence is more complicated but a usable result is obtained for the case where U-rank coincides with Morley rank.

1. Introduction

In this paper we are concerned with the investigation of stability properties of certain complete theories of S-sets. Stability properties (see Sections 2 and 5 for the relevant definitions) arose from the question of how many models a theory (a set of sentences of a first order language) has of any given cardinality. The seminal work of Shelah shows that an unstable theory, indeed a non-superstable theory, has 2^λ models of cardinality λ for any $\lambda > |T|$ [28]. The philosophy then is that, in these cases, there are too many models to attempt to classify by means of a sensible structure theorem. It is reasonable therefore for the algebraist to consider for a given class of algebras 'how stable' is the theory associated with it, before embarking on the search for structure or classification theorems.

For a monoid S, an *S-set* is simply a set A upon which S acts on the right with the identity of S acting as the identity map on A—properly, our S-sets should be referred to as *right S-sets*. Thus A is a unary algebra where the fundamental operation symbols form the monoid S. When studying unary algebras, little is lost by concentrating on S-sets and much is gained. First, for a given S, the category of S-sets and their morphisms has very pleasant properties, in particular, it is a topos. Second, working with S-sets means that the extensive algebraic

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theory of semigroups is available as a tool. Finally, we can think of an S-set as being analogous to a module over a ring; this observation inspires the approach of much of the paper.

The model theory of modules has been and continues to be extensively investigated (see [26]), yielding both structure results for modules and giving concrete realisations of model theoretic concepts. In contrast, only a few studies have been made of the model theory of S-sets. Some results in the latter theory are close parallels of corresponding results for modules. There are, however, several major differences between the two theories. Essentially, these differences arise since right congruences on monoids cannot be determined by right ideals (as is the case for rings). For the model theorist, this means that atomic formulae without parameters cannot be replaced by formulae involving parameters. A notable difference between the model theory of modules and that of S-sets is that, as demonstrated by Mustafin [20], for some monoids S, there are S-sets which have unstable theories whereas all complete theories of modules are stable. Mustafin goes on to describe all monoids S for which every S-set has a stable theory or superstable theory. The thrust of his later papers in this area is to move toward a description of those monoids S over which all S-sets are ω-stable [3, 21]. On the other hand Stepanova [29] has characterised monoids such that all *regular* S-sets have stable, superstable or ω-stable theories.

The present paper is somewhat in the spirit of Stepanova's approach. Rather than imposing conditions on the theories $Th(M)$ for *all* S-sets M over a given S, we are concerned with the case where M is an existentially closed S-set and S is a right coherent monoid. Following Wheeler [30] the notion of right coherence for monoids was introduced in [10] where it is shown that the theory of all S-sets (for fixed S) has a model companion T_S if and only if S is right coherent. It follows that the models of T_S are precisely the existentially closed S-sets, and further, that T_S is a complete theory so that $T_S = Th(M)$ for any existentially closed S-set M. Ivanov [14] argues that T_S is a normal theory (see [23]) and hence stable.

Given that T_S is stable it is natural to investigate conditions under which it satisfies the stronger stability properties of being superstable, ω-stable or totally transcendental. In [5] the corresponding questions in module theory are posed and answered. This work both inspired and heavily influenced the present paper. For a right coherent ring R, the model companion of the theory of all R-modules is denoted by T_R. In [5] complete types are characterised by pairs consisting of a right ideal of R and an R-homomorphism. This is the key to a thorough analysis of complete types and so to finding for which rings R the theory T_R is superstable or totally transcendental.

In a ring R, a right congruence is determined by a right ideal, but as remarked above, this is not true for monoids in general. For this reason, in the case of right S-sets, complete types are characterised by triples consisting of a right ideal of S, a right congruence on S and an S-morphism. It is this result which allows us to translate model theoretic properties of T_S into algebraic properties of S and hence to apply the theory of semigroups. An immediate consequence is that we can easily find upper bounds for the number of types. This enables us to deduce Ivanov's result [14] (Proposition 1.4) that the theory T_S is stable. Further, if every right ideal of S is finitely generated, then T_S is superstable, and if in addition S is countable and has at most $\aleph_0$ right congruences, then T_S is ω-stable.

To obtain the converse of these results we use the U-rank of types and the fact that a complete theory is superstable if and only if the U-rank of each type is defined (see [25]). Our approach is similar to but slightly more complicated than that of Bouscaren. The end results are that T_S is superstable if and only if every right ideal of S is finitely generated and that for a countable S, T_S is ω-stable if and only if S has at most $\aleph_0$ right congruences and every right ideal of S is finitely generated. The superstability result is also a straightforward consequence of [14] (Theorem 2.4). In these results there is of course the underlying assumption that S is right coherent, for this is needed for the theory T_S to exist. Right coherence does not follow from the property that every right ideal is finitely generated as shown by Example 3.1 in [11]. The equivalent results for modules are that superstability and total transcendence of T_R are both equivalent to R being right noetherian.

Another important rank of types is the Morley rank. This is used to define the concept of total transcendence, a complete theory T being totally transcendental if and only if every complete type over a subset of a model of T has Morley rank. Morley rank is always greater than U-rank, so that a totally transcendental theory is certainly superstable. In fact a countable theory T is totally transcendental if and only if it is ω-stable [25].

For a complete theory T of modules, the Morley rank of a type (when it exists) coincides with the U-rank of the type [26]. This is not the case for S-sets and we find necessary and sufficient conditions on S for the theory T_S to be totally transcendental with the Morley rank of any type being equal to its U-rank. The final section of the paper is devoted to a study of monoids which satisfy these conditions. If S is such a monoid and is weakly periodic, then S is finite. On the other hand, the infinite cyclic monoid satisfies the conditions.

2. Preliminaries

This paper is intended to be accessible to any readers with some familiarity with the basic ideas of first-order logic and, with the exception of the final section, only a very little semigroup theory. We recommend [7] and [9] for the former and [13] for the latter. With a view to encouraging algebraists, we give a slightly more leisurely account of rank notions than is usual in the stability literature. Full accounts of the stability theory we use can be found in the books [1, 16, 24, 25, 26, 5]; we extract the key ideas and main results which we need. Any unreferenced results may be found in these texts.

Let L be a first order language. We use the standard notation that if $\phi(x_1, ..., x_n)$ is a formula of L, then the free variables of $\phi(x_1, ..., x_n)$ lie in $\{x_1, ..., x_n\}$. Models will be denoted by letters M, N, P; we use the same notation for their universes. The letters $A, B, \ldots$ are used for subsets of models. For a set A, the language $L(A)$ is obtained from L by adding a new constant symbol to L for each element a of A. Again, we follow the usual practice and do not distinguish elements of A from the constants of $L(A)$ which they label.

The notion of a *type* is crucial to our investigations. To define this, it is useful to employ the so-called monster model of a theory. Let T be a complete theory in L. We fix a model $\mathbf{M}$ of T, saturated of cardinality κ for some cardinal κ much bigger than all other cardinals under consideration; $\mathbf{M}$ is the *monster model* of T. We make the convention that all models of T will be elementary substructures of $\mathbf{M}$ with universes of cardinality less than κ and all sets of parameters will be subsets of $\mathbf{M}$ of cardinality less than κ. Justification of the use of the monster model can be found in [6].

Let A be a subset of $\mathbf{M}$ and let $c \in \mathbf{M}$. Then

$$\mathrm{tp}(c/A) = \{\phi(x) \in L(A) : \mathbf{M} \models \phi(c)\}$$

is a *(complete) 1-type over* A. Clearly $\mathrm{tp}(c/A)$ is a set $p(x)$ of sentences of $L(A, x)$, that is *consistent* with $\mathrm{Th}(\mathbf{M}, a)_{a \in A}$ and is *complete* in the sense that for any formula $\phi(x)$ of $L(A)$, either $\phi(x)$ or $\neg\phi(x)$ is in $p(x)$; we say that $p(x)$ is *realised* by c. Conversely, if $p(x)$ is a set of formulae satisfying these conditions, then the saturation of M gives that $p(x) = \mathrm{tp}(b/A)$ for some $b \in \mathbf{M}$.

For us the term 'type' will mean complete 1-type. The *Stone space* $S(A)$ of A is the collection of all types over A; $S(A)$ is equipped with a natural topology, which comes into play in the definition of Morley rank (see Section 5).

For a cardinal κ, T is *κ-stable* if for every subset A of a model of T with $|A| \leq \kappa$ we have $|S(A)| \leq \kappa$. If T is κ-stable for some infinite κ, then T is *stable* and T is *superstable* if T is κ-stable for all $\kappa \geq 2^{|T|}$. If T is not stable, then it is said to be *unstable*.

For a countable theory T we have the following theorem of Morley which shows that ω-stable theories satisfy a stronger condition than that needed for superstability.

Theorem 2.1 ([19]). *Let T be a complete theory in a countable language. Then T is ω-stable if and only if T is κ-stable for every infinite κ.*

We now give some brief details concerning S-sets. Further details may be found in the comprehensive [15].

Let S be a monoid. A *(right) S-set* is a set A on which S acts on the right, that is, there is a map $\cdot$ from $A \times S$ to A satisfying :

$$(a \cdot s) \cdot t = a \cdot (st) \text{ and } a \cdot 1 = a$$

for all $s, t \in S, a \in A$. We usually write as for $a \cdot s$. Clearly we can think of the elements of S as unary operation symbols and A as a unary algebra in the sense of universal algebra. We thus have all the standard concepts and results of universal algebra at our disposal (see, for example [18]). In particular, we have *S-subsets*, *S-morphisms*, *congruences on S-sets* and *quotient S-sets* A/ρ where A is an S-set and ρ is a congruence on A. For an S-subset B of an S-set A, the relation ρ_B is defined by $a_1 \rho_B a_2$ if and only if $a_1 = a_2$ or a_1, a_2 are both in B. It is easy to see that ρ_B is a congruence on A; the quotient S-set A/ρ_B is usually denoted by A/B and is called the *Rees quotient* of A by B. We differ from standard semigroup terminology in that we make the convention that the empty set $\emptyset$ is an S-subset of every S-set.

For any congruence ρ on an S-set A we denote the ρ-class of an element a of A by $a\rho$. For an S-morphism $f : A \to B$ we denote by $\mathrm{Ker} f$ the congruence on A determined by

$$(a, b) \in \ \mathrm{Ker} f \text{ if and only if } f(a) = f(b).$$

The multiplication in a monoid S makes S itself into a right S-set. The S-subsets of S are called *right ideals* of S and S-set congruences on S are called *right congruences* on S, to distinguish them from semigroup congruences on S.

The category of S-sets and S-morphisms has arbitrary products and coproducts. Another property enjoyed by this category which is useful for our purposes is the *strong amalgamation property.* This asserts that if A, B are S-sets with common S-subset U, then there is an S-set C and injective S-morphisms $f : A \to C, g : B \to C$ such that $f|U = g|U$ and $f(A) \cap g(B) = f(U)$.

Let I be a right ideal of a monoid S and ρ be a right congruence on S. The *ρ-closure* of I, denoted by $I\rho$, is defined by

$$I\rho = \{s \in S : s \,\rho\, t \text{ for some } t \in I\}.$$

It is easy to see that $I\rho$ is a right ideal of S containing I and that $(I\rho)\rho = I\rho$. We say that a right ideal J of S is *ρ-saturated* if $J\rho = J$; thus $I\rho$ is ρ-saturated for any right ideal I. If ν, ρ are right congruences on S and $\nu \subseteq \rho$, then any ρ-saturated right ideal is also ν-saturated.

When I is a ρ-saturated right ideal of S we say that the pair (I, ρ) is a *congruence pair.* We denote by $\mathcal{C}(S)$ or $\mathcal{C}$ the set of all congruence pairs of S.

The language L_S of the theory of S-sets consists of a unary function symbol f_s for each element s of S. We follow the usual convention and write as for $f_s(a)$. The axioms for the theory of S-sets are simply those sentences of L_S which assert that an L_S-structure is an S-set.

Let A be a subset of an S-set M. Recall that the *diagram* of A is the set $\mathrm{Diag}\, A$ of all atomic and negated atomic sentences of $L_S(A)$ which are true in M, or equivalently, are true statements about the S-subset of M generated by A.

A monoid S is *right coherent* if for any finitely generated right congruence ρ on S, every finitely generated S-subset of S/ρ is finitely presented. As mentioned already, this concept and related ones arose in the investigation of model companions of theories of S-sets in [10]. A careful study of right coherence for S-sets is made in [11].

An *equation* over an S-set A is an atomic formula of $L_S(A)$ and has one of the forms:

$$xs = xt, \ xs = yt, \ xs = a$$

where $s, t \in S$ and $a \in A$. An *inequation* over A is simply the negation of an equation over A.

A set $\sum$ of equations and inequations over A is *consistent* if $\sum$ has a solution in some S-set containing A. An S-set A is *existentially closed* if every consistent finite set of equations and inequations over A has a solution in A. Since the class of S-sets is inductive, that is, is closed under unions of chains, every S-set is contained in an existentially closed S-set.

To say that the theory of all S-sets has a model companion is equivalent to saying that the class of all existentially closed S-sets is the class of models of a theory T_S; then T_S is the required model companion. In [10] it is proved that T_S exists if and only if S is right coherent. Given two existentially closed S-sets A, B it is certainly the case that A, B can be embedded in an S-set C (the coproduct of A and B for example) and C can be embedded in an existentially closed S-set. It follows from this and the model completeness of T_S that T_S (when it exists) is complete (Proposition 3.1.9 of [7]). That is, for any sentence ϕ of L_S either $\phi \in T_S$ or $\neg\phi \in T_S$; equivalently, $T_S = \text{Th}(M)$ for any of its models M. Since the theory of all S-sets is universal and as T_S is actually the model completion of this theory [10], we have by Theorem 13.2 in [27] that T_S admits elimination of quantifiers.

These properties ensure that T_S is precisely the kind of theory most amenable to the application of stability theory.

Let A be an S-set. As remarked above, A is an S-subset of an existentially closed S-set M and M is an elementary substructure of the monster existentially closed S-set $\mathbf{M}$. A standard argument using quantifier elimination gives that the deductive closure of $T_S \cup \text{Diag}\, A$ is $\text{Th}(M, a)_{a\in A}$ $=\text{Th}(\mathbf{M}, a)_{a\in A}$.

3. Types

From now on we shall concentrate on the theory T_S for a fixed right coherent monoid S. The purpose of this section is to characterise types over S-sets. Among other things, this gives an alternative approach to results on the stability of T_S.

If A is an S-set, then an A-*triple* is a triple (I, ρ, f) such that $(I, \rho) \in \mathcal{C}$ and $f : I \to A$ is an S-morphism with $\text{Ker} f = \rho \cap (I \times I)$. We denote the set of all A-triples by $\mathcal{T}(A)$. We show that for an S-set A there is a bijection between the set of types over A and the set of all A-triples. A consequence of this is that T_S is a stable theory.

Let $\mathcal{T} = (I, \rho, f)$ be an A-triple and let $\sum_{\mathcal{T}}$ be the union of the following sets of formulae of $L_S(A)$:

$$\{xs = a : a = f(s), s \in I\}, \{xs \neq a : s \notin I, a \in A\},$$
$$\{xs = xt : (s,t) \in \rho\}, \{xs \neq xt : (s,t) \notin \rho\}.$$

Lemma 3.1. *There is an embedding $h : A \to D(\mathcal{T})$ of A in an S-set $D(\mathcal{T})$ which contains an element c which satisfies $\sum_{\mathcal{T}}$.*

Proof. Let $\rho' = \rho \cap (I \times I)$. Then $I/\rho' \cong f(I)$ and for $s \in I$ we may identify $s\rho'$ with $f(s) \in A$. Amalgamate S/ρ with A over the common S-subset $f(I)$ and take c to be 1ρ. $\square$

As remarked in Section 2, we can embed $D(\mathcal{T})$ of the above result into an existentially closed S-set E and our convention is to regard E as elementarily embedded in the monster model $\mathbf{M}$. Let $p = \mathrm{tp}(c/A)$. Then $T_S \cup \mathrm{Diag}\, A \subseteq p$ and since c satisfies $\sum_{\mathcal{T}}$, we also have $\sum_{\mathcal{T}} \subseteq p$. Quantifier elimination allows us to deduce that p is uniquely determined by $\mathcal{T}$.

Lemma 3.2. *Let $\sum^{\mathcal{T}} = \sum_{\mathcal{T}} \cup T_S \cup \mathrm{Diag}\, A$. Then $\sum^{\mathcal{T}}$ is consistent and its deductive closure is the unique type $p_{\mathcal{T}}$ over A containing $\sum_{\mathcal{T}}$.*

Conversely, given $p \in S(A)$ we obtain an A-triple $\mathcal{T}_p$.

Proposition 3.3. *Let p be a type over an S-set A. Let*

$$I_p = \{s \in S : xs = a \in p \text{ for some } a \in A\},$$
$$\rho_p = \{(s,t) \in S \times S : xs = xt \in p\},$$

and

$$f_p : I_p \to A \text{ be defined by } f_p(s) = a \text{ where } xs = a \in p.$$

Then $\mathcal{T}_p = (I_p, \rho_p, f_p)$ is an A-triple and the maps

$$p \mapsto \mathcal{T}_p, \;\; \mathcal{T} \mapsto p_{\mathcal{T}}$$

are mutually inverse bijections between $S(A)$ and $\mathcal{T}(A)$.

The first two corollaries are immediate consequences of the proposition.

Corollary 3.4. *Let A be an S-set and let $p, q \in S(A)$. Then $p = q$ if and only if $I_p = I_q, \rho_p = \rho_q$ and $f_p = f_q$.*

Corollary 3.5. *There is a bijection between the set of right congruences on S and $S(\emptyset)$.*

Corollary 3.6. *For any congruence pair (I, ρ) on S there is an S-set A and a type p over A with $I_p = I$ and $\rho_p = \rho$.*

Proof. If $I = \emptyset$, put $A = \emptyset$ and $f = \emptyset$; otherwise let $A = I/\rho$ and $f : I \to I/\rho$ be the natural map. Then (I, ρ, f) is an A-triple and so the required type exists. $\square$

Corollary 3.7. *Let p be a type over an S-subset A of B. Then there is a type q over B such that $I_p = I_q, \rho_p = \rho_q$ and $f_q = jf_p$ where $j : A \to B$ is the inclusion map.*

Proof. Clearly (I_p, ρ_p, jf_p) is in $\mathcal{T}(B)$. $\square$

Let A be an S-set and I be a right ideal of S. The number of S-morphisms from I to A is at most $|A|^{|S|}$, the number of right ideals of S is at most $2^{|S|}$ and the number of right congruences on S is at most $2^{|S|^2}$. Hence the number of A-triples is at most $2^{|S|}2^{|S|^2}|A|^{|S|}$. Thus, if we take $\kappa =$max$\{\aleph_0, 2^{|S|}\}$ and $|A| \leq \kappa$, then $|\mathcal{T}(A)| \leq \kappa$ and, in view of Proposition 3.3, $|S(A)| \leq \kappa$.

Now consider an arbitrary subset B of the S-set $\mathbf{M}$. It is easy to see that $|S(B)| = |S(A)|$, where A is the S-subset of $\mathbf{M}$ generated by B (indeed, the Stone spaces are homeomorphic, see [1, 16]). We can therefore deduce that the theory T_S is stable.

We can do better than this when every right ideal of S is finitely generated, that is, when S is *weakly right noetherian.* Then, for any right ideal I, the number of S-morphisms from I to A is at most max$\{\aleph_0, |A|\}$ so that there are no more than $2^{|S|}$max$\{\aleph_0, |A|\}$ A-triples. Hence for any infinite cardinal κ with $2^{|S|} \leq \kappa$ we have that if $|A| \leq \kappa$, then $|S(A)| \leq \kappa$. Now $|T_S| =$max$\{\aleph_0, |S|\}$ so that T_S is superstable.

If we assume that S has at most max$\{\aleph_0, |S|\}$ right congruences in addition to being weakly right noetherian, then we see that the number of A-triples is at most max$\{\aleph_0, |S|\}^2$max$\{\aleph_0, |A|\}$. Thus for any infinite cardinal κ with $|S| \leq \kappa$ we have that if $|A| \leq \kappa$, then $|S(A)| \leq \kappa$. Hence, for a countable S which is weakly right noetherian and has only countably many right congruences we have that T_S is ω-stable. In particular, T_S is ω-stable for any finite monoid S.

A monoid S is *right noetherian* if every right congruence on S is finitely generated; since every right ideal of S is determined by a right congruence, it follows that such a monoid is weakly right noetherian. Moreover, every right noetherian monoid is right coherent [11]. Thus if S is a countable, right noetherian monoid, then T_S is ω-stable.

If S is countably infinite and T_S is ω-stable, then $|S(\emptyset)| \leq \aleph_0$ so that by Corollary 3.5, S has only countably many right congruences.

The following result summarises the above discussion; (1) and (2) are also consequences of results in [14].

Proposition 3.8. *Let S be a right coherent monoid. Then*

(1) the theory T_S is stable;

(2) if S is weakly right noetherian, then T_S is superstable;

(3) if S is weakly right noetherian and has at most max$\{\aleph_0, |S|\}$ right congruences, then T_S is κ-stable for all κ with max$\{\aleph_0, |S|\} \leq \kappa$;

(4) if S is countable, then if S is weakly right noetherian and has at most $\aleph_0$ right congruences, T_S is ω-stable;

(5) if S is finite, then T_S is ω-stable;

(6) if S is countable and right noetherian, then T_S is ω-stable;

(7) if S is countable and T_S is ω-stable, then S has at most $\aleph_0$ right congruences.

The converses of (2) and (4) of the above proposition will be obtained in Section 4.

By an *extension* of a type p in $S(A)$ we mean a type q in $S(B)$ where A is an S-subset of B and $p \subseteq q$. The proof of the following result follows easily from Lemma 3.2.

Proposition 3.9. *Let A be an S-subset of B, $p \in S(A)$ and $q \in S(B)$. Then q is an extension of p if and only if*

(i) $I_p \subseteq I_q$, (ii) $f_q|I_p = f_p$, (iii) $f_q^{-1}(A) = I_p$ and (iv) $\rho_p = \rho_q$.

A consequence of Proposition 3.9 is that if p and q are as in Corollary 3.7, then q is an extension of p.

Proposition 3.10. *Let A be an S-set and $p \in S(A)$. Let J be a ρ_p-saturated right ideal containing I_p. Then there is an S-set B containing A and an extension q of p in $S(B)$ such that $I_q = J$. Moreover, B can be chosen to be existentially closed.*

Proof. Since I_p and J are ρ_p-saturated, I_p/ρ_p is an S-subset of J/ρ_p. We also note that since $\mathrm{Ker} f_p = \rho_p \cap (I_p \times I_p)$ we can define a one-one S-morphism $\overline{f} : I_p/\rho_p \to A$ by putting $\overline{f}(x\rho_p) = f_p(x)$ and so we can regard I_p/ρ_p as an S-subset of A. Since the class of S-sets has the strong amalgamation property there is an S-set B and one-one S-morphisms $h : A \to B, k : J/\rho_p \to B$ such that $h|(I_p/\rho_p) = k|(J/\rho_p)$ and $h(A) \cap k(J/\rho_p) = h(I_p/\rho_p)$. Clearly we may regard $A, J/\rho_p$ as S-subsets of B which intersect in I_P/ρ_p. Now the natural morphism ν from J onto J/ρ_p maps to B, and Ker $\nu = \rho_p \cap (J \times J)$ so that by Proposition 3.3. there is a type q in $S(B)$ such that $I_q = J, \rho_q = \rho_p$ and $f_q = \nu$. Certainly $I_p \subseteq I_q$ and for $s \in I_p$, $f_q(s) = s\rho_p = f_p(s)$ so that $f_q|I_p = f_p$. Now for $s \in J$, $f_q(s) \in A$ if and only if $s\rho_p \in A \cap (J/\rho_p)$, that is, if and only if $s\rho_p \in I_p/\rho_p$. As I_p is ρ_p-saturated, we see that $f_q(s) \in A$ if and only if $s \in I_p$, that is, $f_q^{-1}(A) = I_p$.

Thus by Proposition 3.9, q is an extension of p.

To see that we can choose B to be existentially closed, call upon Corollary 3.7 and Proposition 3.9. □

4. Superstability of T_S

This section concentrates on proving the converse of (2) of Proposition 3.8. We utilise the notion of U-rank of a type, introduced by Lascar in [17], and relate the U-rank of a type p to what we call the ρ_p-rank of the right ideal I_p.

First we recall the *foundation* rank on a set $\mathcal{S}$ partially ordered by $\leq$. We define subclasses $\mathcal{S}_\alpha$ of $\mathcal{S}$ for each ordinal α by transfinite induction :

(I) $\mathcal{S}_0 = \mathcal{S}$;
(II) $\mathcal{S}_\alpha = \bigcap\{\mathcal{S}_\beta : \beta < \alpha\}$, if α is a limit ordinal;
(III) $x \in \mathcal{S}_{\alpha+1}$ if and only if $x < y$ for some $y \in \mathcal{S}_\alpha$.

We thus obtain a nested sequence of subclasses of $\mathcal{S}$ indexed by the ordinals. The *foundation rank* of $x \in \mathcal{S}$, denoted by $\mathrm{R}(x)$, can now be defined as follows :

If $x \in \mathcal{S}_\alpha$ for all ordinals α, then we write $\mathrm{R}(x) = \infty$. Otherwise, $\mathrm{R}(x) = \alpha$ where α is the (unique) ordinal such that $x \in \mathcal{S}_\alpha \setminus \mathcal{S}_{\alpha+1}$; in this case we say that *x has R-rank*.

The convention that $\alpha < \infty$ for all ordinals α simplifies the statements of the following standard proposition (see for example [25], p. 35).

Proposition 4.1. *(i) For any $x \in \mathcal{S}$ and any ordinal α*

$$R(x) \geq \alpha \text{ if and only if } x \in \mathcal{S}_\alpha.$$

(ii) Let $x, y \in \mathcal{S}$ where $x < y$. If $R(y)$ is an ordinal then $R(x) > R(y)$. Moreover, if $R(x)$ is an ordinal then so is $R(y)$.

(iii) For any $x \in \mathcal{S}$, $R(x)$ is an ordinal if and only if there are no infinite chains of the form

$$x = x_0 < x_1 < \dots .$$

For the first application of foundation rank, consider a right congruence ρ on S and put

$$\mathcal{S} = \{J : (J, \rho) \in \mathcal{C}\}.$$

The relation $\leq$ is taken as the usual inclusion order of right ideals. If $J \in \mathcal{S}$ then $\mathrm{R}(J)$ is said to be the *ρ-rank* of J and is written as ρ-$\mathrm{R}(J)$.

Corollary 4.2. *Let $(I, \rho) \in \mathcal{C}$. Then ρ-$R(I)$ is an ordinal if and only if S has the ascending chain condition on ρ-saturated right ideals containing I.*

Our second application of foundation rank is to obtain the *U-rank* $\mathrm{U}(p)$ of a type $p \in S(A)$, where $A \subseteq \mathbf{M} \models T$ and T is a complete, stable theory in a first order language L. First we review some definitions associated with types of T; for more details the reader can consult one of the standard texts.

If $p \in S(A)$, where $A \subseteq \mathbf{M}$, then the *class* of p, written $\mathrm{cl}(p)$, is the set

$$\mathrm{cl}(p) = \{\phi(x, y_1, \dots, y_n) \in L : \text{ for some } a_1, \dots, a_n \in A, \phi(x, a_1, \dots, a_n) \in p\}$$

and C_p is the set

$$C_p = \{\mathrm{cl}(q) : p \subseteq q, q \in S(M), A \subseteq M \models T\}.$$

It is a fact that C_p has a unique minimum element (under inclusion) denoted by $\beta(p)$. Clearly, if $p \in S(M)$ where $M \models T$, then $\mathrm{cl}(p) = \beta(p)$. For $A \subseteq B$ and an extension $q \in S(B)$ of p, it is obvious that $\beta(p) \subseteq \beta(q)$. Then q is a *non-forking extension* of p if $\beta(p) = \beta(q)$; otherwise, q is a *forking extension* of p.

Put

$$\mathcal{S} = \{\beta(p) : p \in S(A) \text{ for some } A \subseteq \mathbf{M}\}.$$

Clearly $\mathcal{S}$ is partially ordered by set inclusion. The *U-rank* of $p \in S(A)$, denoted $\mathrm{U}(p)$, is the foundation rank of $\beta(p)$. If $\mathrm{U}(p)$ is an ordinal, then we say that p *has U-rank.* Clearly, in our discussion of U-rank, we can assume that all types are over L-substructures of models of T.

Corollary 4.3. *Let $p \in S(A)$ where $A \subseteq \mathbf{M}$. Then p has U-rank if and only if there are no infinite ascending chains of the form*

$$\beta(p) = \beta(p_0) \subset \beta(p_1) \subset \beta(p_2) \subset \dots .$$

Our objective in this section is to characterise those monoids S for which T_S is superstable or ω-stable. In other words, our goal is to prove the converses of (2) and (4) of Proposition 3.8. In fact, the converse of (4) follows easily from that of (2) so our effort is directed towards showing that if T_S is superstable, then S is weakly right noetherian. To do this, we use the characterisation of superstable theories in terms of U-rank of types.

Theorem 4.4. [17] *Let T be a complete, stable theory in a first order language. Then T is superstable if and only if all types have U-rank.*

We now turn to the theory T_S. In Proposition 3.9 we gave a criterion in terms of the associated triples for one type to be an extension of another. Our present aim is to refine this result in order to distinguish between forking and non-forking extensions. We start by looking at inclusion between classes of types over models of T_S.

We will need the following property of forking.

Proposition 4.5. [25] *Let T be a complete stable theory in a first order language. If $A \subseteq B$ and $p \in S(A)$, then p has a non-forking extension q in $S(B)$.*

Proposition 4.6. *Let M, N be models of T_S and $p \in S(M), q \in S(N)$. Then $cl(p) \subseteq cl(q)$ if and only if $\rho_p = \rho_q$ and $I_p \subseteq I_q$.*

Proof. Suppose that $\mathrm{cl}(p) \subseteq \mathrm{cl}(q)$. Then certainly $p|\emptyset = q|\emptyset$ and hence $\rho_p = \rho_q$. If $s \in I_p$, then the formula $xs = f_p(s)$ is in p. By assumption, $xs = y$ is represented in q so that $xs = n$ is in q for some $n \in N$. Hence $s \in I_q$ and $I_p \subseteq I_q$.

Conversely, suppose that $\rho_p = \rho_q$ and $I_p \subseteq I_q$. Since $\rho_p = \rho_q$, there is a well defined function $f : f_p(I_p) \to f_q(I_q)$ given by $f(f_p(s)) = f_q(s)$. It is easy to see that f is an embedding. Construct an S-set $\overline{M}$ where $\overline{M} \cap N = f(f_p(I_p))$ and $g : M \to \overline{M}$ is an isomorphism extending f; now embed $\overline{M} \cup N$ in an existentially closed S-set P.

Let $\overline{p} \in S(\overline{M})$ be given by

$$\phi(x, \overline{m}) \in p \text{ if and only if } \phi(x, g(\overline{m})) \in \overline{p}.$$

Clearly $\rho_p = \rho_{\overline{p}}, I_p = I_{\overline{p}}$ and $f f_p = f_{\overline{p}}$. Since $q \in S(N)$ we may use Proposition 4.5 to choose a non-forking extension $\overline{q}$ of q in $S(P)$; then $\mathrm{cl}(q) = \beta(q) = \beta(\overline{q}) = \mathrm{cl}(\overline{q})$, so by the first part of the proof $I_q = I_{\overline{q}}$ and $\rho_q = \rho_{\overline{q}}$: but $q \subseteq \overline{q}$ so we must also have that $f_q = f_{\overline{q}}$.

We claim that $\overline{p} \subseteq \overline{q}$. Certainly $I_{\overline{p}} \subseteq I_{\overline{q}}$ and $\rho_{\overline{p}} = \rho_{\overline{q}}$. Moreover, $f_{\overline{p}} = ff_p$ so that if $s \in I_{\overline{p}}$,

$$f_{\overline{p}}(s) = ff_p(s) = f_q(s) = f_{\overline{q}}(s),$$

so that $f_{\overline{q}}|I_{\overline{p}} = f_{\overline{p}}$. It remains to show that $f_{\overline{q}}^{-1}(\overline{M}) = I_{\overline{p}}$. Let $s \in f_{\overline{q}}^{-1}(\overline{M})$. Then $f_{\overline{q}}(s) \in \overline{M} \cap N = f_q(I_p)$. Thus $f_q(s) = f_{\overline{q}}(s) = f_q(t)$ for some $t \in I_p$; but $I_p = I_{\overline{p}}, \rho_q = \rho_p$ and I_p is ρ_p-saturated, so that $s \in I_{\overline{p}}$ as required. Hence $f_{\overline{q}}^{-1}(\overline{M}) \subseteq I_{\overline{p}}$ and as the opposite inclusion is trivial, $f_{\overline{q}}^{-1}(\overline{M}) = I_{\overline{p}}$ and by Proposition 3.9, $\overline{p} \subseteq \overline{q}$.

Thus cl(p) =cl$(\overline{p})$ $\subseteq$cl$(\overline{q})$ =cl(q) as required.

$\square$

Lemma 4.7. *Let A be an S-set and $p \in S(A)$. Then for any model M of T_S with $A \subseteq M$, the minimum member $\beta(p)$ of C_p is cl(q), where q is the type in $S(M)$ such that $p \subseteq q$ and $I_p = I_q$.*

Proof. By virtue of Corollary 3.7 and Proposition 3.9, $p \subseteq q$ for some (unique) $q \in S(M)$ with $I_p = I_q$. In addition, Proposition 4.5 says that $p \subseteq \overline{p}$ for a non-forking extension $\overline{p}$ of p, where $\overline{p} \in S(M)$. Thus $\rho_q = \rho_p = \rho_{\overline{p}}$ and $I_q = I_p \subseteq I_{\overline{p}}$, so that using the previous result,

$$\beta(p) \subseteq \beta(q) = \mathrm{cl}(q) \subseteq \mathrm{cl}(\overline{p}) = \beta(\overline{p}) = \beta(p),$$

which yields $\beta(p)$ =cl(q) as required. $\square$

The following corollary is an immediate consequence of Proposition 4.6 and Lemma 4.7.

Corollary 4.8. *Let $A \subseteq B$ be S-sets, and let $q \in S(B)$ be an extension of $p \in S(A)$. Then q is a non-forking extension of p if and only if $I_p = I_q$.*

We are now able to characterise the U-rank of types in terms of descending chains of right ideals of S.

Proposition 4.9. *For any S-set A and $p \in S(A)$,*

$$UR(p) = \rho_p\text{-}R(I_p).$$

Proof. We prove by induction that for any ordinal α,

$$\mathrm{UR}(p) \geq \alpha \text{ if and only if } \rho_p\text{-R}(I_p) \geq \alpha.$$

The proposition will then follow.

Clearly, the result is true for $\alpha = 0$. Moreover, at any limit ordinal the inductive step is straightforward.

Suppose now that U$(p) \geq \alpha + 1$ so that by definition, $R(\beta(p)) \geq \alpha + 1$. This means that $\beta(p) \subset \beta(q)$ for some $q \in S(B)$ and $R(\beta(q)) \geq \alpha$. Now $\beta(p) = cl(\overline{p})$ and $\beta(q) = cl(\overline{q})$ for some types $\overline{p}, \overline{q}$ over models. By Proposition 4.6 and Lemma 4.7 we have that

$$I_p = I_{\overline{p}} \subset I_{\overline{q}} = I_q.$$

Also we have $\mathrm{U}(q) \geq \alpha$, and since $\beta(p) \subset \beta(q)$, it follows that $\rho_p = \rho_q$. Our inductive assumption yields

$$\rho_q\text{-R}(I_q) = \rho_p\text{-R}(I_q) \geq \alpha$$

so that as $I_p \subset I_q$, we get $\rho_p\text{-R}(I_p) \geq \alpha + 1$.

Conversely, if $\rho_p\text{-R}(I_p) \geq \alpha + 1$ then $I_p \subset J$ for some ρ_p-saturated right ideal J with $\rho_p\text{-R}(J) \geq \alpha$. By Proposition 3.10 there is a type r with $p \subseteq r$ and $I_r = J$. As $\rho_p = \rho_r$ we have by assumption that $\mathrm{U}(r) \geq \alpha$ and since r is a forking extension of p, $\mathrm{U}(p) \geq \alpha + 1$. □

Corollary 4.10. *For any S-set A and $p \in S(A)$, p has U-rank if and only if the set of ρ_p-saturated right ideals containing I_p satisfies the ascending chain condition.*

Proof. This is an immediate consequence of Proposition 4.1 and and Propostion 4.9. □

Part (1) of the following theorem is also a consequence of [14] (Theorem 2.4).

Theorem 4.11. *Let S be a right coherent monoid.*
(1) The theory T_S is superstable if and only if S is weakly right noetherian.
(2) If S is countable, then the theory T_S is ω-stable if and only if S is weakly right noetherian and has only countably many right congruences.

Proof. (1) If S is weakly right noetherian, then T_S is superstable by (2) of Proposition 3.8. Alternatively, this follows from Theorem 4.4 and Corollary 4.10.

Conversely, if T_S is superstable, then applying Corollary 4.10 to the type in $S(\emptyset)$ corresponding to the identity congruence gives that S is weakly right noetherian.

(2) Suppose that S is countable. If T_S is ω-stable, then it is superstable by Theorem 2.1 and so by (1), S is weakly right noetherian. Also we must have $|S(\emptyset)| \leq \aleph_0$ and hence by Corollary 3.5, the number of right congruences on S is countable. The converse is (4) of Proposition 3.8. □

This theorem allows us to give examples of monoids to illustrate the various possibilities. Thus if $S = \{1\} \cup I$ where 1 acts as an identity and I is an infinite set with $ab = a$ for all $a, b \in I$, then I is a right ideal of S which is not finitely generated ; moreover, it is easy to see that S is right coherent. Hence T_S exists, but is not superstable.

On the other hand, T_G is superstable for any group G. But, for example, the group of rationals Q has $2^{|Q|}$ subgroups (and hence $2^{|Q|}$ (right) congruences) so that T_Q is not ω-stable.

Both the infinite cyclic group and the quasi-cyclic group $Z(p^\infty)$ (p a prime number) have $\aleph_0$ subgroups so they provide specific examples of infinite groups G such that T_G is ω-stable.

Of course, for any *finite* monoid S we have that T_S is ω-stable.

5. Total transcendence of T_S

Having considered U-rank of types in the previous section we now turn our attention to another rank, the Morley rank $\mathrm{MR}(p)$, of a type p. This rank is used to define totally transcendental theories; to be precise a complete theory T is *totally transcendental* if and only if for all subsets A of models of T, all types over A have Morley rank.

For a countable theory T, it is a fact that T is totally trancendental if and only if T is ω-stable [19]. There are, however, uncountable theories T which are not totally transcendental but are κ-stable for all κ with $|T| \leq \kappa$.

When T is a theory of modules, if p is a type over a subset of a model of T such that $\mathrm{MR}(p)$ is defined, then $\mathrm{MR}(p) = \mathrm{U}(p)$ [26]. For S-sets, however, the picture is different and in this section we investigate those monoids S for which $\mathrm{MR}(p) = \mathrm{U}(p) < \infty$ for all types p over subsets of models of T_S. Our algebraic characterisation of such monoids allows us to give examples of S such that T_S is totally transcendental but is such that $\mathrm{U}(p) < \mathrm{MR}(p)$ for some type p. We remark that for a complete, stable theory T, if $p \in S(A)$ and $q \in S(B)$ with $A \subseteq B$, $p \subseteq q$ and $\mathrm{MR}(p)$ an ordinal, then $\beta(p) = \beta(q)$ if and only if $\mathrm{MR}(p) = \mathrm{MR}(q)$ [25].

The two conditions on monoids used in the characterisation theorem are the right noetherian property (that is, all right congruences are finitely generated) and the condition (MU) which we now explain. Let S be a monoid and let (I, ρ) be a congruence pair, that is, $(I, \rho) \in \mathcal{C}$. We say that (I, ρ) is *critical* if there is a finite subset K of $(S \times S) \setminus \rho$ such that for all right congruences θ which saturate I, contain ρ and agree with ρ on I (i.e. $\theta \cap (I \times I) = \rho \cap (I \times I)$) we have

$$K \subseteq (S \times S) \setminus \theta \text{ implies } \rho = \theta \text{ or } \theta\text{-R}(I) < \rho\text{-R}(I).$$

We then say that S *satisfies* (MU) if every congruence pair of S is critical.

Note that for any right congruence ρ, the congruence pair (S, ρ) is critical. In the very special case where S is a group, to show that S satisfies (MU) we need only show that $(\emptyset, \rho)$ is critical for every right congruence ρ. In this case, for *any* right congruence θ, we have that $\theta\text{-R}(\emptyset) = 1$. Thus to show that $(\emptyset, \rho)$ is critical, we need to find a finite set $K \subseteq (S \times S) \setminus \rho$ such that if $\rho \subseteq \theta$ and $K \subseteq (S \times S) \setminus \theta$, then $\rho = \theta$.

For any right coherent monoid S, if $(I, \rho) \in \mathcal{C}$ and $\{s\rho : s \notin I\}$ is finite it is then easy to see that the pair (I, ρ) is critical.

Lemma 5.1. *For any right ideal I of a monoid S with S/I finite, every congruence pair (I, ρ) is critical. In particular, every finite monoid satisfies (MU).*

We now consider a useful sufficient condition for a monoid to satisfy (MU).

Proposition 5.2. *Let $C_r(S)$ be the lattice of right congruences of a monoid S. If $C_r(S)$ satisfies the minimal condition and each $\rho \in C_r(S)$ has only a finite number of covers, then S satisfies (MU).*

Proof. Let (I, ρ) be a congruence pair. If $S = I$, then we have already noted that the pair is critical. Otherwise, ρ cannot be universal since I is ρ-saturated and so the set of right congruences strictly containing ρ contains minimal members which are covers of ρ. Let $\rho_1, ..., \rho_t$ be these covers. For each $i \in \{1, ..., t\}$ choose a pair (a_i, b_i) in $\rho_i \setminus \rho$. Now put

$$K = \{(a_1, b_1), ..., (a_t, b_t)\}.$$

Suppose that $\theta \in C_r(S)$ and $\rho \subseteq \theta$. If $\rho \neq \theta$, then it follows from the minimal condition that $\rho_i \subseteq \theta$ for some i. Thus $(a_i, b_i) \in \theta$ and consequently K is not contained in $(S \times S) \setminus \theta$. Hence the pair (I, ρ) is critical and consequently S satisfies (MU). □

For groups the converse of Proposition 5.2 is true as we now demonstrate.

Proposition 5.3. *A group G satisfies (MU) if and only if the lattice $L(G)$ of subgroups of G satisfies the minimal condition and every subgroup has only finitely many covers in $L(G)$.*

Proof. Suppose that G satisfies (MU) and let

$$\rho_1 \supseteq \rho_2 ...$$

be a decreasing sequence of right congruences. Put $\rho = \bigcap\{\rho_i : i \in \omega\}$. By assumption, $(\emptyset, \rho)$ is critical and so there is a finite set K such that $K \subseteq (G \times G) \setminus \rho$ and if $K \subseteq (G \times G) \setminus \rho_m$, then $\rho = \rho_m$. If $(a, b) \in K$, then $(a, b) \notin \rho_t$ for some t and since K is finite, it follows that for some m we do have $K \subseteq (G \times G) \setminus \rho_m$. Hence $\rho_m = \rho_{m+1} = ...$ and $C_r(G)$ satisfies the minimal condition.

In view of the minimal condition, every $\rho \in C_r(G)$ except $G \times G$ actually does have covers. If $\{\rho_\lambda : \lambda \in \Lambda\}$ is the set of covers of ρ, then $\rho_\lambda \cap \rho_\mu = \rho$ for each $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$. Hence, if $(a, b) \notin \rho$, then (a, b) can belong to at most one of the covers of ρ. Since $(\emptyset, \rho)$ is critical, there is a finite set K such that $K \subseteq (G \times G) \setminus \rho$ and K is not contained in $(G \times G) \setminus \rho_\lambda$ for any cover ρ_λ of ρ. But any given pair in K is in at most one cover of ρ and so there are only finitely many covers of ρ.

Now use the fact that the lattice of right congruences on a group is isomorphic to the lattice of subgroups. □

We note that the quasi-cyclic group $Z(p^\infty)$ where p is a prime number satisfies the conditions of Proposition 5.3 and thus satisfies (MU). On the other hand the infinite cyclic group does not satisfy the minimal condition for subgroups and hence does not satisfy (MU). It is, in fact, easy to show that the congruence pair $(\emptyset, \iota)$ is not critical in this case.

We have introduced the condition (MU) to help in our discussions of Morley rank. To define the latter we use make use of the natural topology on Stone spaces of types.

Let T be a complete theory and let $A \subseteq \mathbf{M}$. Then $S(A)$ may be made into a topological space by specifying the sets

$$\langle \phi(x) \rangle = \{p \in S(A) : \phi(x) \in p\}$$

as a basis of open sets, where $\phi(x)$ is a formula of $L(A)$. The space $S(A)$ has a basis of clopen sets $\langle \phi(x) \rangle$, and is compact and Hausdorff.

If T is a theory which has elimination of quantifiers (for example, T_S), then a routine argument gives that the sets $\langle \theta(x) \rangle$ where $\theta(x)$ is a conjunction of atomic and negated atomic formulae form a basis for the topology of $S(A)$.

Let T be a complete theory in a first order language L and let A be a subset of a model of T. Subsets $MR^\alpha(A)$ of $S(A)$ are defined by induction on the ordinal α as follows:

(I) $MR^0(A) = S(A)$.

(II) If α is a limit ordinal, then

$$MR^\alpha(A) = \bigcap \{MR^\beta(A) : \beta < \alpha\}.$$

(III) For any α, $MR^{\alpha+1}(A) = MR^\alpha(A) \setminus X^\alpha$, where

$$X^\alpha = \{p \in MR^\alpha(A) : \text{ for all } B \supseteq A \text{ and all extensions } q \text{ of } p \text{ on } B,$$

$$q \notin MR^\alpha(B) \text{ or } q \text{ is isolated in } MR^\alpha(B)\}.$$

We may take B to be an L-substructure of a model of T.

For $p \in S(A)$, the *Morley rank* of p is $\mathrm{MR}(p)$ where, if $p \in MR^\alpha(A)$ for all α, then $\mathrm{MR}(p) = \infty$ and otherwise $\mathrm{MR}(p)$ is α where $p \in MR^\alpha(A) \setminus MR^{\alpha+1}(A)$. If $\mathrm{MR}(p) < \infty$, then we say that *p has Morley rank*.

It is a standard result that for all types p, $\mathrm{U}(p) \leq \mathrm{MR}(p)$ [25]; we need this in the proof of the main result of this section. We first note that for any type p over an S-set A, $\mathrm{MR}(p) = 0$ if and only if $I_p = S$, that is $\mathrm{U}(p) = 0$. For if $I_p = S$ and $p \subseteq q$ where $q \in S(B)$, then since $1 \in I_q$, $x = b \in q$ for some $b \in B$ and $\{x = b\}$ isolates q in $S(B)$. Thus $p \notin MR^1(A)$ so that $\mathrm{MR}(p) = 0$. The converse is clear.

Theorem 5.4. *For every type p over an S-set A, $MR(p) = U(p) < \infty$ if and only if S is right noetherian and satisfies (MU).*

Proof. Suppose first that the condition on ranks of types holds. Let (I, ρ) be a congruence pair. By Corollary 3.6, there is an S-set A and a type p over A with $I_p = I, \rho_p = \rho$. Let the associated A-triple be (I, ρ, f) and let p have Morley rank α. Then there is an open set U in $S(A)$ such that $p \in U$ and $\mathrm{MR}(q) < \alpha$ for all q in $U \setminus \{p\}$. Let $U = \langle \phi(x) \rangle$ where $\phi(x)$ is a conjunction of sets of formulae:-

$$\{xr_i = a_i : i \in \Lambda_1\}, \{xs_j = xt_j : j \in \Lambda_2\},$$

$$\{xu_k \neq xv_k : k \in \Lambda_3\}, \{xw_\ell \neq b_\ell : \ell \in \Lambda_4\}$$

where the index sets $\Lambda_1, ..., \Lambda_4$ are all finite. Since $p \in \langle \phi(x) \rangle$, each r_i is a member of I and each pair (s_j, t_j) is in ρ.

Let θ be any right congruence on S which saturates I, properly contains ρ and agrees with ρ on I. Then (I, θ, f) is an A-triple; let $\overline{p}$ be the associated type over A. Certainly each pair (s_j, t_j) is in θ since $\rho \subseteq \theta$. Thus we see that the sets $\{xr_i = a_i : i \in \Lambda_1\}$ and $\{xs_j = xt_j : j \in \Lambda_2\}$ are contained in $\overline{p}$. If the formula $xw_\ell = b_\ell$ is in $\overline{p}$ for some $\ell \in \Lambda_4$, then $w_\ell \in I$ and $f(w_\ell) = b_\ell$ and consequently, $xw_\ell = b_\ell$ is in p, a contradiction. Thus each inequation $xw_\ell \neq b_\ell$ is in $\overline{p}$ and we see that $\phi(x) \in \overline{p}$ if and only if $xu_k \neq xv_k$ is in $\overline{p}$ for each $k \in \Lambda_3$.

Let $K = \{(u_1, v_1), ..., (u_m, v_m)\}$; since $xu_k \neq xv_k$ is in p we certainly have that $K \subseteq (S \times S) \setminus \rho$. If $K \subseteq (S \times S) \setminus \theta$, then we have $\phi(x) \in \overline{p}$ so that $\overline{p} \in \langle \phi(x) \rangle$ and hence $\mathrm{MR}(\overline{p}) < \mathrm{MR}(p)$. But $\mathrm{U}(\overline{p}) = \mathrm{MR}(\overline{p})$ and $\mathrm{U}(p) = \mathrm{MR}(p)$ so that θ-$\mathrm{R}(I) < \rho$-$\mathrm{R}(I)$. Thus (I, ρ) is critical and hence S satisfies (MU).

To see that S is right noetherian we consider the case $I = \emptyset$. Let σ be the right congruence on S generated by $\{(s_j, t_j) : j \in \Lambda_2\}$. Certainly $\sigma \subseteq \rho$ and if p_1 is the type over $\emptyset$ associated with σ, then clearly $p_1 \in \langle \phi(x) \rangle$. Hence, using our assumption on ranks,

$$\mathrm{MR}(p) = \mathrm{U}(p) = \rho\text{-}\mathrm{R}(\emptyset) \leq \sigma\text{-}\mathrm{R}(\emptyset) = \mathrm{U}(p_1) = \mathrm{MR}(p_1) \leq \mathrm{MR}(p),$$

that is, $\mathrm{MR}(p_1) = \mathrm{MR}(p)$. By the choice of $\langle \phi(x) \rangle$, we that have $p = p_1$ so that $\rho = \sigma$ and ρ is finitely generated.

Conversely, suppose that S is right noetherian and satisfies (MU). By Theorem 4.11, T_S is certainly superstable so that for every S-set A, every type p in $S(A)$ has U-rank. We show by induction that for every p, $\mathrm{MR}(p) = \mathrm{U}(p)$.

If $\mathrm{U}(p) = 0$, then $I_p = S$ and so, as already noted, $\mathrm{MR}(p) = 0$.

Now let $p \in S(A)$ and $\mathrm{U}(p) = \alpha$ and suppose that for all S-sets B and all types $q \in S(B)$ with $\mathrm{U}(q) < \alpha$ we have $\mathrm{MR}(q) = \mathrm{U}(q)$. Let $I = I_p$, $\rho = \rho_p$. Certainly $\mathrm{U}(p) \leq \mathrm{MR}(p)$ so we have $p \in MR^\alpha(A)$ and we wish to show that $p \notin MR^{\alpha+1}(A)$, that is, for every S-set B containing A and every extension q of p over B we want either $q \notin MR^\alpha(B)$ or q is isolated in $MR^\alpha(B)$.

So let $q \in S(B)$ where B is an extension of A and $q|A = p$. Suppose that $q \in MR^\alpha(B)$. We have to find an open set U such that $MR^\alpha(B) \cap U = \{q\}$. By Proposition 3.9, we have $I \subseteq I_q$ and $\rho = \rho_q$. Now $\alpha \leq \mathrm{MR}(q)$ and so by the inductive assumption we cannot have $\mathrm{U}(q) < \alpha$. But $\mathrm{U}(q) \leq \mathrm{U}(p) = \alpha$ so that $\mathrm{U}(q) = \alpha$. Now by the definition of U-rank, we must have that q is a non-forking extension of p and so by Corollary 4.8, $I = I_q$.

As S is right noetherian, $I = \bigcup\{w_i S : i \in \Lambda\}$ for some finite set Λ and ρ is generated by a finite subset H of $S \times S$. For each $i \in \Lambda$, let $a_i = f_q(w_i)$. By assumption, the pair (I, ρ) is critical. Let K be the finite subset of $(S \times S) \setminus \rho$ required in the definition of criticality and let $\xi(x)$ be the formula obtained by taking the conjunction of the following sets of formulae:

$$\{xw_i = a_i : i \in \Lambda\}, \{xs = xt : (s,t) \in H\}, \{xu \neq xv : (u,v) \in K\}.$$

Then $q \in \langle \xi(x) \rangle$. Let $r \in \langle \xi(x) \rangle$ and suppose that $\mathrm{MR}(r) \geq \alpha$. Our aim is to show that $r = q$ and this will complete the proof that $\mathrm{U}(p) = \mathrm{MR}(p)$ and hence prove the result by induction.

Note that $I \subseteq I_r$ and $\rho \subseteq \rho_r$ so that I_r is ρ-saturated. If $I \neq I_r$, then, by Proposition 3.10, q has an extension $\overline{q}$ with $I_{\overline{q}} = I_r$ and by Corollary 4.8, $\overline{q}$ is a forking extension of q. Hence, $\mathrm{U}(\overline{q}) < \mathrm{U}(q) = \alpha$. By Proposition 3.9, $\rho_{\overline{q}} = \rho$ and thus

$$\mathrm{U}(r) = \rho_r\text{-R}(I_r) \leq \rho_{\overline{q}}\text{-R}(I_{\overline{q}}) = \mathrm{U}(\overline{q}) < \alpha.$$

The inductive assumption gives $\mathrm{MR}(r) < \alpha$, a contradiction, so that we may suppose that $I = I_r$.

Since f_q and f_r agree on the set of generators $\{w_i : i \in \Lambda\}$ of I, it follows that $f_q = f_r$ and $\rho_r \cap (I \times I) = \ker f_r = \ker f_q = \rho \cap (I \times I)$.

If $\rho \neq \rho_r$, then as $K \subseteq (S \times S) \setminus \rho_r$ we have $\rho_r\text{-R}(I_r) < \rho\text{-R}(I)$ so that $\mathrm{U}(r) < \mathrm{U}(q) = \alpha$ and the inductive assumption gives $\mathrm{MR}(r) < \alpha$, a contradiction. Thus $\rho_r = \rho$ and, as $f_r = f_q$, Corollary 3.4 now gives $r = q$ as desired. $\square$

We have noted already that the infinite cyclic group does not satisfy (MU) although, of course, it is (right) noetherian. On the other hand the group $Z(p^\infty)$ is not (right) noetherian but does satisfy (MU). Thus the two conditions in the theorem are independent. Furthermore, these observations also show that there are monoids S such that T_S is totally transcendental (ω-stable) but such that for some S-set A there is a type p in $S(A)$ with $\mathrm{U}(p) < \mathrm{MR}(p)$.

We can be more precise with our two examples. For any group G and any type p over a G-set A we have $I_p = G$ or $I_p = \emptyset$. In the former case $\mathrm{U}(p) = \mathrm{MR}(p) = 0$ and in the latter case $\mathrm{U}(p) = 1$. It is not difficult to see that if $p \in S(\emptyset)$ (so that necessarily $I_p = \emptyset$), then for any G-set A there is exactly one extension p_A of p in $S(A)$ with $I_{p_A} = \emptyset$. A simple argument using transfinite induction shows that for all ordinals $\alpha \geq 1$, $\mathrm{MR}(p) \geq \alpha$ if and only if $\mathrm{MR}(p_A) \geq \alpha$ for all G-sets A. It follows that $\mathrm{MR}(p) = \alpha$ if and only if $p \in MR^\alpha(\emptyset)$ and p is isolated in $MR^\alpha(\emptyset)$. Moreover, $\mathrm{MR}(p) = \mathrm{MR}(p_A)$ for all G-sets A.

It is now not difficult to show that for the infinite cyclic group G with generator g, if p_n is the type in $S(\emptyset)$ corresponding to the subgroup generated by g^n, then $\mathrm{MR}(p_n) = 1$ for $n \geq 1$ and $\mathrm{MR}(p_0) = 2$. Thus $\mathrm{U}(p_0) < \mathrm{MR}(p_0)$.

Similarly, if $G = Z(p^\infty)$ is regarded as the group of all p^n-th roots of unity for all $n \geq 1$ and if for each n, p_n is the type in $S(\emptyset)$ corresponding to the subgroup generated by a primitive p^n-th root of one, then $\mathrm{MR}(p_n) = 1$. For the type p_∞ in $S(\emptyset)$ corresponding to G itself we find that $\mathrm{MR}(p_\infty) = 2$ so that $\mathrm{U}(p_\infty) < \mathrm{MR}(p_\infty)$.

6. Right noetherian monoids which satisfy (MU)

The main result of the preceding section makes it natural to consider the monoids of the title. As the condition (MU) is rather complicated it is far from clear precisely which monoids satisfy (MU). Of course, any finite monoid is right noetherian and also, by Lemma 5.1, satisfies (MU). One of the main results of this section shows that the converse is true for an extensive class of monoids, namely the weakly periodic monoids. However, not every right noetherian monoid which satisfies (MU) is finite. We will show that an infinite example is the free commutative monoid on one generator.

Our first objective is to show that (right) noetherian groups which satisfy (MU) are finite. To this end we need the lemma below which can be deduced from König's Lemma, but which is very easy to prove directly in much the same way that König's Lemma is proved.

Lemma 6.1. *Let Y be a lattice satisfying the finite chain condition. If every member of Y has only finitely many covers, then Y is finite.*

Proof. Since Y satisfies the descending chain condition, it has a least element x_0. If Y is infinite, then since x_0 has only finitely many covers, x_0 has a cover x_1 such the filter above x_1 is infinite. But x_1 has only finitely many covers, so there must be one of these, say x_2, such that the filter above x_2 is infinite. Continuing in this way we find an infinite chain

$$x_0 < x_1 < x_2 < \dots$$

of elements of Y, contradicting the ascending chain condition. □

Corollary 6.2. *Let G be a right noetherian group which satisfies (MU). Then G is finite.*

Proof. By Proposition 5.3, the lattice $\mathcal{L}(G)$ of subgroups of G satisfies the minimal condition and every subgroup has only finitely many covers in $\mathcal{L}(G)$. Since $\mathcal{L}(G)$ also satisfies the maximal condition, it has the finite chain condition and by Lemma 6.1, $\mathcal{L}(G)$ is finite. As pointed out on pp.170-171 of [2], it follows easily that G is finite. □

The next stage in our argument is to show that any subgroup of a monoid which is right noetherian and satisfies (MU) inherits these properties. To do this we utilise some classical semigroup theory, in particular, basic results about Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$. The relation $\mathcal{L}$ is defined on a monoid S by the rule that for any $a, b \in S$, $a \,\mathcal{L}\, b$ if and only if $Sa = Sb$. The relation $\mathcal{R}$ is defined dually; $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. Note that $\mathcal{L}$ ($\mathcal{R}$) is a right (left) congruence. Details may be found in any of the standard texts. We recommend [13].

Lemma 6.3. *If the monoid S is right noetherian, then so is every subgroup.*

Proof. Let G be a subgroup of S. For any right congruence ρ on G, let $\overline{\rho}$ denote the right congruence *on S* generated by ρ. If $a, b \in S$ and $a \,\overline{\rho}\, b$, then $a = b$ or there exists a sequence

$$a = c_1 t_1, d_1 t_1 = c_2 t_2, \dots, d_\ell t_\ell = b,$$

where $(c_i, d_i) \in \rho, 1 \le i \le \ell$. Notice in particular that $a \,\mathcal{L}\, b$. Suppose now that $a, b \in G$. We claim that $\overline{\rho} \cap (G \times G) = \rho$. Let e be the identity of G. Then we certainly have

$$a = c_1(et_1), d_1(et_1) = c_2(et_2), \ldots, d_\ell(et_\ell) = b,$$

Taking inverses in G we have

$$et_1 = c_1^{-1}a \in G.$$

This gives that $a \,\rho\, d_1(et_1)$. Now

$$et_2 = c_2^{-1}d_1(et_1) \in G,$$

so that $a \,\rho\, d_2(et_2)$. Continuing in this manner we obtain $a \,\rho\, b$. Thus G is $\overline{\rho}$-saturated and $\overline{\rho} \cap (G \times G) = \rho$ as required. It is now easy to see that if S is right noetherian, so also is G. □

Lemma 6.4. *If the monoid S is right noetherian and satisfies (MU), then so does every maximal subgroup.*

Proof. Let G be a maximal subgroup of S, so that G is a group $\mathcal{H}$-class. We already know from Lemma 6.3 that G is (right) noetherian. Suppose now that S satisfies (MU). To show that G satisfies (MU) we need only prove that the pair $(\emptyset, \rho)$ is critical for any right congruence ρ on G.

Let e be the identity of G, let

$$I = \cup\{SaS : SaS \subset SeS\} \text{ and } J = SeS.$$

Then I and J are ideals of S. From Theorem 1.3 of [12] we know that the principal factor J/I is completely 0-simple or completely simple. Let $\overline{\rho}$ be defined as in Lemma 6.3; since $\rho \subseteq \mathcal{L}$ and $\mathcal{L}$ is a right congruence, we have that $\overline{\rho} \subseteq \mathcal{L}$. Thus any ideal of S is $\overline{\rho}$-saturated. Let ν_I be the Rees congruence associated with I, so that for any $a, b \in S$, $a \,\nu_I\, b$ if and only if $a = b$ or $a, b \in I$. Since I is $\overline{\rho}$-saturated and ν_I-saturated, it is clear that $\tilde{\rho} = \overline{\rho} \cup \nu_I$ is a right congruence saturating I. Moreover, for any $a, b \in S$, if $a \ne b$ and $a \,\tilde{\rho}\, b$, then either $a, b \in I$ or $a, b \in J \setminus I$. In the latter case, we have $a \overline{\rho}\, b$ and so, since J/I is completely 0-simple, it follows that $a \,\mathcal{H}\, b \,\mathcal{R}\, e$. Consequently, any right ideal containing I is $\tilde{\rho}$-saturated. Thus if θ is any right congruence on G, then $\tilde{\rho}$-R$(I) = \tilde{\theta}$-R(I).

The congruence pair $(I, \tilde{\rho})$ is critical; let $K \subseteq (S \times S) \setminus \tilde{\rho}$ be a finite set of pairs guaranteed by the fact that $(I, \tilde{\rho})$ is critical. We need to pick a set of pairs of elements of G that will enable us to show that $(\emptyset, \rho)$ is critical.

For any pair

$$(a, b) \in K \cap \mathcal{H} \cap (R_e \times R_e)$$

choose and fix $c = c_{(a,b)} \in J \setminus I$ with $ac, bc \in G$. It follows from the fact that J/I is completely (0)-simple that $(ac, bc) \notin \rho$. We now put

$$H = \{(ac, bc) : (a, b) \in K \cap \mathcal{H} \cap (R_e \times R_e)\},$$

so that $H \subseteq (G \times G) \setminus \rho$.

Let θ be a right congruence on G containing ρ and such that $H \subseteq (G \times G) \setminus \theta$. Certainly $\tilde{\rho} \subseteq \tilde{\theta}$, I is $\tilde{\theta}$-saturated and $\tilde{\rho} \cap (I \times I) = \tilde{\theta} \cap (I \times I)$. If $K \nsubseteq (S \times S) \setminus \tilde{\theta}$, then there exists $(a, b) \in K \cap \tilde{\theta}$. But $(a, b) \notin \tilde{\rho}$, so we are forced to deduce that $a, b \in R_e$ and $a \, \mathcal{H} \, b$. Consequently,

$$(ac, bc) \in \tilde{\theta} \cap (G \times G) = \overline{\theta} \cap (G \times G) = \theta.$$

But $(ac, bc) \in H$, a contradiction. Thus $K \subseteq (S \times S) \setminus \tilde{\theta}$. Now by the definition of critical pair, $\tilde{\rho} = \tilde{\theta}$ or $\tilde{\rho}$-R$(I) < \tilde{\theta}$-R(I). But the latter is impossible by previous comments on saturation of right ideals. We conclude that $\tilde{\rho} = \tilde{\theta}$ and consequently, $\rho = \theta$ as required. □

From Lemmas 6.2, 6.4 we deduce the following.

Theorem 6.5. *If S is a right noetherian monoid which satisfies (MU), then all subgroups of S are finite.*

A semigroup S is *weakly periodic* if for every element s of S there is a positive integer $n = n(s)$ such that $I^2 = I$ where $I = S^1 s^n S^1$. If S is a semigroup which satisfies the minimal condition for principal ideals or for principal right (or left) ideals or if S is periodic, then S is weakly periodic. Regular and eventually regular (some power of any element is regular) semigroups are weakly periodic as are semisimple semigroups, that is, semigroups with no null principal factors.

Corollary 6.6. *If S is a weakly periodic right noetherian monoid which satisfies (MU), then S is finite.*

Proof. By Theorem 6.5, all subgroups of S are finite. Hence by Theorem 2.3 of [12], S is finite. □

Corollary 6.7. *Let S be a right noetherian monoid which satisfies (MU). If the relation $\mathcal{R}$ is a congruence on S and there are only finitely many trivial $\mathcal{R}$-classes, then S is finite.*

Proof. We show that S is weakly periodic so that the result follows from Corollary 6.6. Let $a \in S$ and consider the sequence $S \supseteq aS \supseteq a^2S \supseteq \ldots$. Let $I = \bigcap\{a^iS : i \in \omega\}$, ρ be the Rees right congruence associated with I and ρ_i that associated with a^iS. If $I = \emptyset$, then we take ρ to be ι. The pair (I, ρ) is critical and so there is a finite subset K of $(S \times S) \setminus \rho$ such that for any right congruence θ with $K \subseteq (S \times S) \setminus \theta$ where θ saturates I, agrees with ρ on I and contains ρ, we have either $\rho = \theta$ or θ-R$(I) < \rho$-R(I). Since K is finite, $K \subseteq (S \times S) \setminus \rho_n$ for some n. By hypothesis, $a^pS = I$ for some p, or there is an element a^m with $n \leq m$ whose $\mathcal{R}$-class is non-trivial.

In the latter case, suppose that $a^mS \neq I$. Let x, y be distinct elements in the $\mathcal{R}$-class of a^m and let ν be the right congruence generated by the set $\rho \cup \{(x, y)\}$. It is easy to see that if $(u, v) \in \nu$ and $u \neq v$, then $u, v \in a^mS$ and either $u\mathcal{R}v$ or $u, v \in I$. Thus $\rho \subset \nu \subseteq \rho_m$ and hence $K \subseteq (S \times S) \setminus \nu$. Furthermore, ν

saturates I and agrees with ρ on I and consequently, ν-R$(I) < \rho$-R(I). But all right ideals which contain I are both ρ-saturated and ν-saturated since as noted above, if $(u, v) \in \nu$ and $u, v \notin I$, then $u\mathcal{R}v$. Hence ν-R$(I) = \rho$-R(I), a contradiction. It follows that if $a \in S$ then the descending chain of principal right ideals $S \supseteq aS \supseteq a^2S \supseteq ...$ is finite. Thus $a^qS = I$ for some q so that $a^qS = a^{q+1}S = ...$ Hence $a^q = a^{2q}s$ for some $s \in S$ and so $a^qS = (a^qS)^2$. It follows that $Sa^qS = (Sa^qS)^2$, and S is weakly periodic. $\square$

On a commutative monoid the relations $\mathcal{H}, \mathcal{R}$ and $\mathcal{L}$ coincide and $\mathcal{R}$ is automatically a congruence. The following result is thus an immediate consequence of Corollary 6.7.

Corollary 6.8. *Let S be a noetherian commutative monoid which satisfies (MU). If S has only finitely many trivial $\mathcal{H}$-classes, then S is finite.*

We now give an example of an infinite noetherian commutative monoid which satisfies (MU). Of course, in view of Corollary 6.8, our example must have infinitely many trivial $\mathcal{H}$-classes.

Proposition 6.9. *The additive monoid $\mathbb{N}$ of non-negative integers is noetherian and satisfies (MU).*

Proof. It is well known and easy to show directly that $\mathbb{N}$ is noetherian. If I is a non-empty ideal of $\mathbb{N}$, then $\mathbb{N}/I$ is finite so that it follows from Lemma 5.1 that any congruence pair (I, ρ) is critical. It remains to consider pairs $(\emptyset, \rho)$. If $\rho = \iota$, then ι-R$(\emptyset) = \omega$. When $\rho \neq \iota$, let r, m be the smallest integers such that $(r, r+m) \in \rho$ and $m \geq 1$. In fact, from page 137, exercise 5 of [8] we know that ρ is generated by $(r, r+m)$. It is then easy to see that ρ-R$(\emptyset)$ is finite so that $(\emptyset, \iota)$ is critical by choosing $K = \emptyset$. Further, putting

$$K = \{(s, s+n) : 0 \leq s \leq r, 0 \leq n \leq m\} \setminus \{(r, r+m)\},$$

it is clear that $K \subseteq (S \times S) \setminus \rho$. But if $\rho \subset \theta$, then $K \cap \theta \neq \emptyset$ and consequently the pair $(\emptyset, \rho)$ is critical. Thus $\mathbb{N}$ satisfies (MU). $\square$

In our final result we show that $\mathbb{N}$ is the only infinite commutative cancellative principal ideal monoid which is both noetherian and satisfies (MU).

Proposition 6.10. *Let S be a commutative, cancellative principal ideal monoid. Then S is noetherian and satisfies (MU) if and only if S is a finite group or is isomorphic to $\mathbb{N}$.*

Proof. Suppose that S is noetherian and satisfies (MU). If S is finite, then since it is cancellative, it must be a group.

If S is infinite, then by Corollary 6.8, S must have infinitely many trivial $\mathcal{H}$-classes. Let a be a unit of S so that $a\mathcal{H}1$. For any element $c \in S$, we have $ac\mathcal{H}c$ since $\mathcal{H}$ is a congruence on S. If $a \neq 1$ then $ac \neq c$ since S is cancellative and so H_c is non-trivial unless $a = 1$. Thus the group of units of S is trivial. It follows from Theorem 12 of [4] that S is isomorphic to $\mathbb{N}$. $\square$

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