AN INTRODUCTION TO COVERS FOR SEMIGROUPS

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This paper is based on a short lecture course for graduate students given at Coimbra in May 2001. I have tried to retain the same level of presentation for the article, with the intention that it will be accessible to anyone with a modest knowledge of semigroup theory and inverse semigroup theory, as provided, for example, by the first two chapters of [21], and the first two sections of Chapter 5 of the same book. For the most part, proofs are given in full detail, and where a proof is omitted, a reference is given. The aim of the paper is to introduce the main concepts associated with covers of semigroups, to give some of the key results, and to illustrate some ways in which some of the theory can be applied.

Before we give details of the content of the paper, we discuss the main concept to be considered. Let T be a subsemigroup of a semigroup S. A *T-cover of S* over a group G is a semigroup $\widehat{S}$ with subsemigroup $\widehat{T}$ such that there are surjective morphisms α, β such that, in the diagram

$$\begin{tikzcd}
\widehat{T} \arrow[r] \arrow[d, "\alpha"'] & \widehat{S} \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & \\
T \arrow[r] & S & G
\end{tikzcd}$$

where the horizontal arrows are inclusion maps, the restriction of α to $\widehat{T}$ is an isomorphism and $1\beta^{-1} = \widehat{T}$. The homomorphism α is called the covering homomorphism.

There are a number of questions which immediately spring to mind. First, what conditions must S and T satisfy in order for appropriate $\widehat{S}$ and G to exist? Secondly, if they do exist, what can we say about the structure of $\widehat{S}$ in terms of G and $\widehat{T}$? Finally, and perhaps most importantly, what is the point of studying covers as described above. We consider these questions in the following pages, but first we give a brief explanation of why there are no covering theorems in group theory.

When S is a group and T is a subgroup, it is natural to want $\widehat{S}$ to be a group. Then $\widehat{T}$ has to be a normal subgroup of $\widehat{S}$ and consequently, T is a normal subgroup of S. Thus we may take $\widehat{S}$ to be S, $\widehat{T}$ to be T and G to be S/T, and we are left with the problem of describing S in terms of G and T, that is, the synthesis problem in the theory of group extensions.

In general, however, $\widehat{S}$ will be different from S. One of the early illustrations of this occurs in the work of McAlister [26], [27] on inverse semigroups in the mid 1970s. Groups and semilattices are the natural pieces into which to split an inverse semigroup and this leads to considering the above situation with S inverse and T the commutative subsemigroup of idempotents of S, denoted by $E(S)$ in the sequel. McAlister obtained a covering theorem in which he showed the existence of an inverse semigroup $\widehat{S}$ where we can take $\widehat{T}$ to be $E(\widehat{S})$ and G to be the maximum group homomorphic image of $\widehat{S}$. The finite case of this result was also obtained by Tilson. The semigroup $\widehat{S}$ is said to be E-unitary because $E(\widehat{S})$ is a unitary subset of $\widehat{S}$, and we say that $\widehat{S}$ is an E-unitary cover of S over G. In the cited papers, McAlister gave a description of E-unitary inverse semigroups in terms of semilattices and groups, in what is now known as the P-theorem.

McAlister's work has been extended in various ways by many authors, including Szendrei, Takizawa, Trotter, Fountain, Almeida, Pin and Weil. Some of these extensions will be discussed in the ensuing pages. In the following section, we consider covers in the contexts of relational morphisms, subdirect products and semidirect products of various kinds. In Section 2, we find necessary and sufficient conditions on a subsemigroup T of a semigroup S for a T-cover of S to exist, and show how such a cover can be constructed. Section 3 is devoted to E-dense semigroups, and starts with some generalities about such semigroups. We introduce the least full weakly self-conjugate subsemigroup $D(S)$ of an E-dense semigroup S, and apply the construction of Section 2 to obtain an E-dense D-unitary cover of S. We concentrate on inverse semigroups in Section 4 starting with a discussion of factorisable monoids which are used to give a short alternative proof of the existence of an E-unitary inverse cover of an inverse semigroup S. We follow this with a brief discussion of McAlister's P-theorem. Next we describe characterisations, due to McAlister and Reilly, of E-unitary inverse covers of an inverse semigroup over a specific group in terms of prehomomorphisms, and dual prehomomorphisms. We conclude the section with an application of covers and the P-theorem to the structure theory of inverse semigroups by giving a new proof of Reilly's theorem on bisimple inverse ω-semigroups. Section 5 also offers an application of covers, this time to orthodox semigroups. Using an appropriate covering theorem, McAlister gave a simple criterion for a finite orthodox semigroup to be a member $\mathbf{A} \vee \mathbf{G}$ where $\mathbf{A}$ is the pseudovariety of all finite aperiodic semigroups, and $\mathbf{G}$ is the pseudovariety of all finite groups. In the final section, we briefly describe some other aspects of covers. First, we mention covers for left ample and weakly left ample semigroups. Secondly, we discuss some work of Auinger and Trotter [5] which extends the results of Section 3 by considering covers of E-dense semigroups over groups in a specific variety of groups. Finally, we say a little about finite covers of finite semigroups, describing just enough to relate the topic to results of Ash [3], Ribes and Zalesskiĭ [36] and Herwig and Lascar [19] which are described in other articles in this volume.

1. Generalities

1.1. **Subdirect products and relational morphisms.** Let A, B be semigroups. A *subdirect product* of A and B is (a semigroup isomorphic to) a subsemigroup S of $A \times B$

such that S projects onto both A and B. We have the following semigroup version of a standard universal algebra result.

Proposition 1.1. *A semigroup S is a subdirect product of semigroups A and B if and only if there are congruences δ_A, δ_B on S such that $S/\delta_A \cong A$, $S/\delta_B \cong B$ and $\delta_A \cap \delta_B = \iota$.*

We now recall the associated notion of relational morphism, introduced by Tilson in [8]. Let A and B be semigroups. A *relational morphism* τ from A to B is a mapping $\tau : A \to 2^B$ such that

(1) $a\tau \neq \emptyset$ for all $a \in A$,
(2) $(a_1\tau)(a_2\tau) \subseteq (a_1a_2)\tau$ for all $a_1, a_2 \in A$.

The notation $\tau : A \mathrel{-\!\!\!\circ\!\!\!\rightarrow} B$, introduced by Manuel Delgado, is used to indicate that τ is a relational morphism from A to B. The *graph* of τ, that is,

$$\mathrm{gr}(\tau) = \{(a, b) \in A \times B : b \in a\tau\}$$

is a subsemigroup of $A \times B$ which projects onto A. For an element b of B, we put $b\tau^{-1} = \{a \in A : b \in a\tau\}$, and say that τ is *surjective* if $b\tau^{-1} \neq \emptyset$ for all $b \in B$, that is, $B = \bigcup_{a\in A} a\tau$. In this case, $\mathrm{gr}(\tau)$ is a subdirect product of A and B. Moreover, $\tau^{-1} : B \mathrel{-\!\!\!\circ\!\!\!\rightarrow} A$ is also a surjective relational morphism, and $(\tau^{-1})^{-1} = \tau$.

In the case of *inverse* semigroups A and B, we also require

(3) $(a\tau)^{-1} = a^{-1}\tau$ for all $a \in A$

(where $X^{-1} = \{x^{-1} : x \in X\}$ for $X \subseteq A$). In this case, $\mathrm{gr}(\tau)$ is an *inverse* subsemigroup of $A \times B$, and we call τ an *inverse relational morphism.*

For monoids A and B, we impose the condition

(4) $1 \in 1\tau$

so that $\mathrm{gr}(\tau)$ is a submonoid of $A \times B$.

Examples of relational morphisms between semigroups are provided by homomorphisms, and inverses of surjective homomorphisms. It is easy to verify that composing relational morphisms gives a relational morphism. Thus, given semigroups A, B, C and homomorphisms $\alpha : C \to A$, $\beta : C \to B$ with α surjective, the composite $\alpha^{-1}\beta$ is a relational morphism from A to B. In fact, all relational morphisms arise in this way, for, given $\tau : A \mathrel{-\!\!\!\circ\!\!\!\rightarrow} B$, we may take C to be $\mathrm{gr}(\tau)$, and α and β to be the projections to A and B respectively; then $\tau = \alpha^{-1}\beta$. Moreover, if α and β are both surjective, then both $\alpha^{-1}\beta$ and $\beta^{-1}\alpha$ are surjective relational morphisms.

If T is a subsemigroup of a semigroup S, we say that a relational morphism $\tau : S \mathrel{-\!\!\!\circ\!\!\!\rightarrow} G$ from S to a group G is T-*pure* if $T = 1\tau^{-1}$. Thus if $\widehat{S}$ is a T-cover of S over G, so that we have surjective homomorphisms α and β with

$$\begin{tikzcd}
\widehat{T} \arrow[r] \arrow[d, "\alpha"'] & \widehat{S} \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & \\
T \arrow[r] & S & G
\end{tikzcd}$$

then there are surjective relational morphisms $\tau : S \mathrel{\circ\!\!\!\!\longrightarrow} G$ and $\tau^{-1} : G \mathrel{\circ\!\!\!\!\longrightarrow} S$ ($\tau = \alpha^{-1}\beta$ and $\tau^{-1} = \beta^{-1}\alpha$), and τ is T-pure since $T = \widehat{T}\alpha = (1\beta^{-1})\alpha = 1\tau^{-1}$.

Conversely, if $\tau : S \mathrel{\circ\!\!\!\!\longrightarrow} G$ is a T-pure surjective relational morphism, then $\mathrm{gr}(\tau)$ is a T-cover of S over G. For, we may take α and β to be the projections of $\mathrm{gr}(\tau)$ onto S and G respectively, so that $\tau = \alpha^{-1}\beta$. Then $T = 1\tau^{-1} = 1\beta^{-1}\alpha$ and so α maps $\widehat{T} = 1\beta^{-1}$ onto T. If $x, y \in \widehat{T}$ and $x\alpha = y\alpha$, then $x = (s_1, 1), y = (s_2, 1)$ for some $s_1, s_2 \in S$ so that $s_1 = x\alpha = y\alpha = s_2$ and hence $x = y$.

Thus we have the following.

Proposition 1.2. *Let T be a subsemigroup of a semigroup S, and G be a group. Any T-cover of S over G gives rise to a T-pure surjective relational morphism $\tau : S \mathrel{\circ\!\!\!\!\longrightarrow} G$.*

Conversely, if $\tau : S \mathrel{\circ\!\!\!\!\longrightarrow} G$ is a T-pure surjective relational morphism, then $\mathrm{gr}(\tau)$ is a T-cover of S over G.

We emphasise that, in general, not every T-cover of S over a group arises as the graph of a T-pure surjective relational morphism, as the following example from [5] shows.

Example 1.3. Let $\widehat{S}$ be the monogenic semigroup with generator a and $a^{2n} = a^n$, and let $S = G$ be the maximal subgroup $\{a^n, \dots, a^{2n}\}$. Put $T = \{a^n\}$, and define α and β by $a^i\alpha = a^i\beta = a^{n+i}$. Then $\widehat{S}$ is a T-cover of S, but, as it is not a group, it is not a subdirect product of S with itself.

In contrast, we have the following result of Auinger and Trotter [5] for regular semigroups which generalises an earlier result of McAlister and Reilly [30] for inverse semigroups.

Proposition 1.4. *Let S be a regular semigroup with subsemigroup T. If $\widehat{S}$ is regular and is a T-cover of S over a group G, then*

(1) *T is regular, and*

(2) *$\widehat{S}$ is a subdirect product of S and G.*

Proof. We have the following diagram

$$\begin{tikzcd}
\widehat{T} \arrow[r] \arrow[d, "\alpha"'] & \widehat{S} \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & \\
T \arrow[r] & S & G
\end{tikzcd}$$

where α and β are surjective homomorphisms, and $\alpha|_{\widehat{T}}$ is an isomorphism.

To see that $\widehat{T}$ (and hence T) is regular, let $a \in \widehat{T}$ and a' be an inverse of a in $\widehat{S}$. Then

$$a'\beta = 1(a'\beta)1 = (a\beta)(a'\beta)(a\beta) = (aa'a)\beta = a\beta = 1,$$

so that $a' \in \widehat{T}$.

Put $\rho = \beta\beta^{-1} \cap \alpha\alpha^{-1}$. We show that $\rho = \iota$, so that $\widehat{S}$ is a subdirect product of S and G.

First, we note that if $a \in \widehat{T}$ and $a \,\rho\, b$, then $a = b$. For $1 = a\beta = b\beta$ so that $b \in \widehat{T}$, and so, since $a\alpha = b\alpha$ and α is one-one on $\widehat{T}$, we have $a = b$.

Now let $a, b \in \widehat{S}$ with $a \,\rho\, b$, and let a', b' be inverses of a and b respectively. Then $aa' \,\rho\, ba'$ and $b'a \,\rho\, b'b$. Hence, since $\widehat{T}$ contains the idempotents of $\widehat{S}$, we have $aa' = ba'$ and $b'a = b'b$.

Now $b'ba'a \,\rho\, b'aa'a = b'a = b'b$, so that $b'ba'a = b'b$ since $b'b \in \widehat{T}$. Thus

$$a = aa'a = ba'a = bb'ba'a = bb'b = b,$$

and $\rho = \iota$ as required. $\square$

1.2. **Semidirect products.** We begin by describing semidirect products of semigroups by groups, and giving examples of semigroups which have covers which are such semidirect products. We then extend the notion of semidirect product to semidirect products of semigroupoids by groups and show that every T-cover arising from a T-pure surjective relational morphism can be described using this concept.

A group G *acts by automorphisms* on a semigroup T if, for all elements g, h of G and t, u of T, there is a unique element $g \cdot t$ in T such that

(1) $(gh) \cdot t = g \cdot (h \cdot t)$,
(2) $1 \cdot t = t$,
(3) $g \cdot (tu) = (g \cdot t)(g \cdot u)$.

The *semidirect product* $T \rtimes G$ of T by G is the set $T \times G$ with multiplication

$$(t, g)(u, h) = (t(g \cdot u), gh).$$

It is easily verified that $T \rtimes G$ is a semigroup (a monoid if T is a monoid), and that the projection $\beta : T \rtimes G \to G$ onto G is a surjective homomorphism with

$$1\beta^{-1} = \{(t, 1) : t \in T\} \cong T.$$

Sometimes, a semigroup has a cover which is a semidirect product, as we see in the next example.

Example 1.5. Let $S = M_n(F)$ be the multiplicative monoid of all $n \times n$ matrices over a field F, and let T be the submonoid of all singular matrices together with the identity. Now let G be the general linear group $GL_n(F)$, and note that G acts on T by conjugation. This is an action by automorphisms, and we have

$$\begin{tikzcd}
T \times \{1\} \arrow[r] \arrow[d, "\alpha"'] & T \rtimes G \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & \\
T \arrow[r] & S & G
\end{tikzcd}$$

where $(t, g)\alpha = tg$ for $(t, g) \in T \rtimes G$. Thus $T \rtimes G$ is a T-cover of S over G. The associated relational morphism $\tau = \alpha^{-1}\beta$ is given by

$$a\tau = \{g \in G : a = tg \text{ for some } t \in T\}.$$

Hence the association $(t, g) \leftrightarrow (tg, g)$ gives an isomorphism between $T \rtimes G$ and the subdirect product $\{(a, g) : g \in a\tau\}$ of S and G.

In a similar way, we have that, for a finite set X, the full transformation semigroup on X, the monoid of all partial transformations of X, and the symmetric inverse monoid all have covers which are semidirect products of a semigroup by a group. However, not all T-covers are semidirect products in this sense. But, if we use the notion of a semidirect product of a semigroupoid by a group, then we can describe T-covers arising from T-pure surjective relational morphisms as semidirect products.

First, recall that a *semigroupoid* C consists of a set of *objects* denoted by $\operatorname{Obj} C$ and a disjoint collection of sets $\operatorname{Mor}(u, v)$ (or $\operatorname{Mor}_C(u, v)$), one for each pair of objects u, v. The elements of the sets $\operatorname{Mor}(u, v)$ are called *morphisms* and the set of all morphisms of C is denoted by $\operatorname{Mor} C$. Finally, there is a partial operation on $\operatorname{Mor} C$, called *composition* and written $+$ which satisfies the following conditions:

(1) if p, q are morphisms of C, the *composite* $p + q$ of p and q is defined if and only if there exist objects u, v, w of C such that $p \in \operatorname{Mor}(u, v)$ and $q \in \operatorname{Mor}(v, w)$; in this case, $p + q \in \operatorname{Mor}(u, w)$;

(2) for all u, v, w, x in $\operatorname{Obj} C$ and all morphisms $p \in \operatorname{Mor}(u, v)$, $q \in \operatorname{Mor}(v, w)$ and $r \in \operatorname{Mor}(w, x)$,

$$(p + q) + r = p + (q + r).$$

A semigroupoid C is a *category* if, for each object u of C, there is a distinguished element 0_u of $\operatorname{Mor}(u, u)$ (called the *identity morphism at* u) such that

(3) for all u, v, w in $\operatorname{Obj} C$ and all morphisms $p \in \operatorname{Mor}(v, u)$, $q \in \operatorname{Mor}(u, w)$, we have

$$p + 0_u = p \text{ and } 0_u + q = q.$$

We use $+$ for composition in semigroupoids and categories, rather than the more conventional multiplicative notation, for increased clarity when we consider actions of groups on semigroupoids. It should be emphasised that there is no implication of commutativity.

A group G *acts* on a semigroupoid C if

(1) G acts on the two sets $\operatorname{Obj} C$ and $\operatorname{Mor} C$ in such a way that, for all objects u, v, if $p \in \operatorname{Mor}(u, v)$, then $g \cdot p \in \operatorname{Mor}(g \cdot u, g \cdot v)$ for all $g \in G$, and

(2) for all $g \in G$, and all $p, q \in \operatorname{Mor} C$ such that $p + q$ is defined,

$$g \cdot (p + q) = g \cdot p + g \cdot q.$$

If C is a category, we also require

(3) $g0_u = 0_{gu}$ for all $g \in G$ and $u \in \operatorname{Obj} C$.

Let C be a semigroupoid acted upon by a group G. The *semidirect product* $C \rtimes G$ of C by G is a semigroupoid defined as follows:

- $\operatorname{Obj}(C \rtimes G) = \operatorname{Obj} C$,
- $\operatorname{Mor}_{C \rtimes G}(u, v) = \{(f, g) : g \in G \text{ and } f \in \operatorname{Mor}_C(u, gv)\}$

and composition is given by the rule:

$$(f, g)(f', g') = (f + g \cdot f', gg').$$

It is straightforward to check that $C \rtimes G$ is a semigroupoid. Our definition is a very special case of a construction which dates back to the late 1950s, and is often known as the

Grothendieck construction. A detailed account of the general construction can be found in [39].

'Choosing a basepoint' means choosing an object u of $C \rtimes G$ and taking the full sub-semigroupoid on this one object, that is, the 'local semigroup' at u,

$$L_u(C \rtimes G) = \mathrm{Mor}_{C\rtimes G}(u, u) = \{(f, g) : g \in G \text{ and } f \in \mathrm{Mor}_C(u, gu)\}.$$

This is a special case of a semidirect product with basepoints as described, for example, in [38]. Our special case was also considered in [25] where it was observed that, if the action of G is free and transitive, the local semigroup at any object u can be realised as the collection of orbits of the action on the set of morphisms of the semigroupoid. This approach is generalised in [12].

We now show that for a subsemigroup T of a semigroup S, and a T-pure surjective relational morphism $\tau : S \multimap G$ from S to a group G, the T-cover of S determined by τ, that is, $\mathrm{gr}(\tau)$ can be described as a semidirect product. We use the *weak (or unfactored) derived semigroupoid* W_τ of τ which is defined as follows:

- $\mathrm{Obj}\, W_\tau = G$,
- for $g, h \in G$,

$$\mathrm{Mor}_{W_\tau}(g, h) = \{(g, s, h) \in G \times S \times G : g^{-1}h \in s\tau\}$$

and composition is given by the rule:

$$(g, s_1, h) + (h, s_2, k) = (g, s_1s_2, h).$$

It is readily verified that W_τ is a semigroupoid, and we note that, for any $g \in G$,

$$\mathrm{Mor}(g, g) = \{(g, s, g) : 1 \in s\tau\} = \{(g, s, g) : s \in T\}$$

which is clearly isomorphic to T.

The group G acts on W_τ as follows. First, it acts on the set of objects, that is, G, by multiplication on the left. The action on $\mathrm{Mor}\, W_\tau$ is given by

$$a \cdot (g, s, h) = (ag, s, ah)$$

where $a \in G$ and $(g, s, h) \in \mathrm{Mor}\, W_\tau$. It is easy to verify that this is an action. Using this action, we can form $W_\tau \rtimes G$, and we obtain the diagram

$$\begin{tikzcd}
\{((1,t,1),1) : t \in T\} \arrow[r] \arrow[d, "\alpha"'] & L_1(W_\tau \rtimes G) \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & \\
T \arrow[r] & S & G
\end{tikzcd}$$

where $((1, s, g), g)\beta = g$ and $((1, s, g), g)\alpha = s$. We see that $L_1(W_\tau \rtimes G)$ is a T-cover of S over G; an isomorphism θ with a subdirect product of S and G is given by $((1, s, g), g)\theta = (s, g)$.

2. Existence

Let T be a subsemigroup of a semigroup S. We find a necessary and sufficient condition on T for S to have a T-cover over some group. To demonstrate sufficiency we give an explicit construction of a T-cover $\widehat{S}$ over the free group on the set S.

2.1. **A necessary condition.** If a T-cover over a group G exists, we know that there is a T-pure surjective relational morphism $\tau : S \multimap G$ for some group G. Let x be an element of S, and $g \in x\tau$. Since τ is surjective, $g^{-1} \in y\tau$ for some $y \in S$. Hence, for any $a \in T$, we have

$$1 = g1g^{-1} \in (x\tau)(a\tau)(y\tau) \subseteq (xay)\tau,$$

so that $xay \in T$. Similarly, $yax \in T$, and also $xy, yx \in T$. Thus T is strongly dense in S, in the sense of the following definition.

A subsemigroup T of a semigroup S is *strongly dense* in S if, for all $x \in S$, there exists $y \in S$ such that $xay, yax \in T$ for all $a \in T^1$.

We say that T is *dense* in S if, for all $x \in S$, there exists $y \in S$ such that $xy, yx \in T$.

We introduce the following useful notation. For $x \in S$, put

$$W_T(x) = \{y \in S : xay, yax \in T \text{ for all } a \in T^1\},$$

so that T is strongly dense in S if and only if $W_T(x) \neq \emptyset$ for all $x \in S$.

Example 2.1. Let S be an inverse semigroup and $E(S)$ be its semilattice of idempotents. For every $x \in S$ and idempotent e, the elements xex^{-1} and $x^{-1}ex$ are idempotent, so that $x^{-1} \in W_{E(S)}(x)$ and $E(S)$ is strongly dense in S.

Example 2.2. Let T be a strongly dense subgroup of a group K. For an element $x \in K$, let $y \in W_T(x)$. Then $xy \in T$ so that $y^{-1}x^{-1} \in T$, and hence $xax^{-1} = (xay)y^{-1}x^{-1} \in T$ for all $a \in T$, that is, T is a normal subgroup of K. Conversely, certainly every normal subgroup of K is strongly dense in K.

On the other hand, every subgroup of a group is dense in the group.

It follows from the fact that strongly dense subgroups of a group are normal that if a group K has a T-cover over a group, then it is a T-cover of itself over some group. This is not the case for semigroups in general: if S is a semigroup and $\beta : S \to G$ is a surjective homomorphism onto a group G, then certainly $T = 1\beta^{-1}$ is strongly dense in S, but it has additional properties.

A subset U of a semigroup S is *unitary* in S if for all elements s of S and u of U,

$$su \in U \text{ implies } s \in U, \quad \text{and} \quad us \in U \text{ implies } s \in U.$$

A subset U of a semigroup S is *reflexive* if for all $x, y \in S$,

$$xy \in U \text{ implies } yx \in U.$$

Lemma 2.3. *If $\beta : S \to G$ is a surjective homomorphism from a semigroup S onto a group G, then $T = 1\beta^{-1}$ is unitary and reflexive in S.*

Proof. If $s \in S$ and $a, sa \in T$, then $s\beta = (s\beta)1 = (s\beta)(a\beta) = (sa)\beta = 1$. Thus $s \in T$, and similarly, if $as \in T$, then $s \in T$ so that T is unitary.

If $x, y \in S$ and $xy \in T$, then $(x\beta)(y\beta) = (xy)\beta = 1$ so that $y\beta = (x\beta)^{-1}$. Hence $(yx)\beta = (y\beta)(x\beta) = 1$, and $yx \in T$. $\square$

In general, if T is a strongly dense subsemigroup of a semigroup S, it will not be unitary and reflexive, for example, if S is inverse with a zero and $S \neq E(S)$, then clearly, $E(S)$ is not unitary in S. But there is the potential for S to have an $E(S)$-cover over a group.

If $\widehat{S}$ is a T-cover of S over a group, we say that $\widehat{S}$ is a T-*unitary* semigroup because $\widehat{T}$ is a unitary subsemigroup of $\widehat{S}$ and $\widehat{T} \cong T$.

2.2. **A construction.** We have seen that if $\beta : S \to G$ is a surjective homomorphism from a semigroup S onto a group G, then $T = 1\beta^{-1}$ is a unitary, reflexive and dense subsemigroup of S. Conversely, if T is such a subsemigroup of G, then there is a group G and a surjective homomorphism $\beta : S \to G$ with $T = 1\beta^{-1}$. See, for example, [7], [23], [24] or [13]. From [13] we have the following result.

Theorem 2.4. *Let T be a unitary, reflexive, dense subsemigroup of a semigroup S. Then*

$$\rho_T = \{(a, b) \in S \times S : au = vb \text{ for some } u, v \in T\}$$

is a group congruence on S, and $T = 1\beta^{-1}$ where $\beta : S \to S/\rho_T$ is the natural homomorphism.

We have also seen that if T is a subsemigroup of a semigroup S and there is a T-pure surjective relational morphism from S to a group, then T must be strongly dense in S. We now describe a construction which shows that the converse is true. A special case of the construction was given in [10], and the general version is from [12]. Of course, in view of Proposition 1.2, the existence of a T-pure surjective relational morphism from S to a group ensures the existence of a T-cover of S over the group.

Proposition 2.5. *Let S be a semigroup with a strongly dense subsemigroup T. Then S has a T-cover over a group.*

Proof. Let G be the free group on the set S. We construct a surjective relational morphism $\tau : G \multimap S$ with $1\tau = T$. Then τ^{-1} is the required T-pure relational morphism.

The elements of G are equivalences classes of words over X where $X = S \cup \overline{S}$ with $S \cap \overline{S} = \emptyset$ and such that $s \leftrightarrow \overline{s}$ is a bijection between S and $\overline{S}$. We let $\theta : X^* \to G$ be the homomorphism onto G given by $w\theta = [w]$. Then $\theta^{-1} : G \multimap X^*$ is a surjective relational morphism. We find a surjective relational morphism $\varphi : X^* \multimap S$, and put $\tau = \theta^{-1}\varphi$.

Let $x \in S$. Then, since T is strongly dense in S,

$$W_T(x) = \{y \in S : xay, yax \in T \text{ for all } a \in T^1\}$$

is not empty. Choosing a non-empty subset $\gamma_T(x)$ of $W_T(x)$ for each x, we define φ inductively as follows:

(1) $\epsilon\varphi = T$,
(2) $s\varphi = T^1 s T^1$ for $s \in S$,
(3) $\overline{s}\varphi = T^1 \gamma_T(s) T^1$ for $s \in S$,

and for $v = x_1 \dots x_n$ where $x_i \in X$, put $v\varphi = (x_1\varphi)\dots(x_n\varphi)$.

Clearly, $v\varphi \neq \emptyset$ for all $v \in X^*$, and $(v\varphi)(w\varphi) = (vw)\varphi$ for nonempty words v, w. Also, since $T^2 \subseteq T$ and $TT^1 \subseteq T^1 \supseteq T^1 T$, we have

$$(\epsilon\varphi)(v\varphi) \subseteq v\varphi \supseteq (v\varphi)(\epsilon\varphi)$$

for all $v \in X^*$. Hence φ is a relational morphism, and it is clearly surjective.

Finally, if $w\theta = 1$, we claim that $w\varphi \subseteq \epsilon\varphi = T$ so that

$$1\tau = \bigcup\{w\varphi : w\theta = 1\} = T.$$

We prove the claim by induction on $|w|$. There is nothing to prove when $|w| = 0$. If $|w| > 0$, then it must be the case that $w = ux\overline{x}v$ for some $u, v \in X^*$ and $x \in X$ (where we make the convention that $\overline{\overline{s}} = s$ for $s \in S$).

Now, for $s \in S$ and $a \in s\varphi$, $b \in \overline{s}\varphi$, we have $a = t_1 s t_2, b = t_3 y t_4$ for some $t_1, t_2, t_3, t_4 \in T^1$ and $y \in \gamma_T(s)$. Hence $ab = t_1 s t_2 t_3 y t_4$ is in T because $s t_2 t_3 y \in T$. Similarly, $ba \in T$, so that $(x\overline{x})\varphi = (x\varphi)(\overline{x}\varphi) \subseteq T$. It follows from this, together with the fact that $(uv)\theta = 1$ and the induction hypothesis, that $w\varphi \subseteq T$, and the claim is proved.

Now $1\tau = T$ so that τ^{-1} is T-pure, and we have a T-unitary cover for S over G. □

We note that the T-cover we have constructed is infinite, even if S is finite, because the group involved is infinite.

3. E-dense semigroups

The concept of an E-dense (or E-inversive) semigroup was introduced in [42] and developed by several authors including Mitsch [31]. The latter provides several examples of E-dense semigroups and notes, in particular, that regular, eventually regular and periodic semigroups, in particular, finite semigroups are all E-dense.

For the definition of E-dense semigroup, we introduce the notion of a weak inverse. A *weak inverse* of an element a of a semigroup S is an element x such that $xax = a$. We denote the set of all weak inverses of a by $W(a)$, and the set of all weak inverses of all elements in a subset A of S by $W(A)$. We say that S is *E-dense* (or *E-inversive*) if $W(a) \neq \emptyset$ for all $a \in S$.

In the next result we list some conditions equivalent to being E-dense.

Proposition 3.1. *For a semigroup S with $E = E(S)$, the following are equivalent:*

(1) *S is E-dense,*
(2) *for every $a \in S$, there are elements b, c of S such that $ba \in E$ and $ac \in E$,*
(3) *for every $a \in S$, there is an element b of S such that $ab \in E$ and $ba \in E$,*
(4) *for every $a \in S$, there is an element c of S such that $ac \in E$,*
(5) *for every $a \in S$, there is an element d of S such that $da \in E$.*

Proof. The equivalence of (2) to (5) can be found in [2] or [31] and the equivalence of (1) with the rest is in [6], but for completeness we give a short proof.

If a' is a weak inverse of a, then aa' and $a'a$ are idempotent and so (1) implies (3). Clearly, (3) implies (2) and (2) implies (4) and (5). By symmetry, it is enough to show that (4) implies (1). Let $a \in S$ and let $c \in S$ be such that $ac \in E$. Then clearly, cac is a weak inverse of a, proving (1). $\square$

Let S be an E-dense semigroup and U a subset of S; U is *weakly self conjugate* (or *closed under weak conjugation*) if, for each $a \in S$ and $a' \in W(a)$,

$$aUa' \cup a'Ua \subseteq U.$$

The set U is *full* if $E(S) \subseteq U$. Of particular interest to us is the least full weakly self conjugate subsemigroup of S, that is, the intersection of all such subsemigroups. This is called the *weakly self conjugate core* of S, and is denoted by $D(S)$. We can give a 'constructive' definition of $D(S)$ as follows. Put $D_0(S) = \langle E(S) \rangle$ (the subsemigroup generated by $E(S)$), and

$$D_{i+1}(S) = \langle axb, bxa : x \in D_i(S), a \in S^1, b \in W(a) \rangle.$$

Then it is straightforward to show that $\bigcup_{i \geqslant 0} D_i(S)$ is a weakly self conjugate subsemigroup, and hence that $D(S) = \bigcup_{i \geqslant 0} D_i(S)$.

Our next objective is to show that if S is an E-dense semigroup, then $D(S)$ is E-dense. We require some preliminary results on weak inverses valid in any semigroup. The first two are in the dissertation of Weipoltshammer [45]; they also appear in [5], the first is also given in [11] and a variant occurs in [1].

Proposition 3.2. *If $a_1, \ldots, a_n$ are elements of a semigroup S, then*

$$W(a_1 \ldots a_n) \subseteq W(a_n) \ldots W(a_1).$$

Proof. If $a' \in W(a_1 \ldots a_n)$, then, for each $i = 1, \ldots n$, define

$$a_i' = a_{i+1} \ldots a_n a' a_1 \ldots a_{i-1}.$$

It is readily verified that $a_i' \in W(a_i)$ for each i, and that $a' = a_n' \ldots a_1'$. $\square$

Proposition 3.3. *If a is an element of a semigroup S, then*

$$W(W(a)) \subseteq E(S)aE(S).$$

Proof. If $b \in W(a)$ and $c \in W(b)$, then $c = (cb)a(bc) \in E(S)aE(S)$. $\square$

Proposition 3.4. *Let S be a semigroup. Then $W(\langle E(S) \rangle) \subseteq \langle E(S) \rangle$. In particular, if S is an E-dense semigroup, then $\langle E(S) \rangle$ is E-dense.*

Proof. If $b \in W(e)$ where $e \in E(S)$, then $b = beb = (be)(eb) \in \langle E(S) \rangle$. Let $e_1, \ldots, e_n$ be idempotents and assume inductively that

$$W(e_1 \ldots e_{n-1}) \subseteq \langle E(S) \rangle.$$

Let $x \in W(e_1 \dots e_n)$. Then $e_n x \in W(e_1 \dots e_{n-1})$ and $xe_1 \dots e_n \in E(S)$ so that, by the induction hypothesis,

$$x = (xe_1 \dots e_n)(e_n x) \in \langle E(S) \rangle.$$

The result follows. $\square$

We can now prove the following stronger result.

Theorem 3.5. *Let S be a semigroup. Then $W(D(S)) \subseteq D(S)$. In particular, if S is an E-dense semigroup, then $D(S)$ is E-dense.*

Proof. In view of Proposition 3.4 and the recursive definition of $D(S)$, it is enough to prove that for all non-negative integers i, if $W(D_i(S)) \subseteq D_i(S)$, then $W(D_{i+1}(S)) \subseteq D_{i+1}(S)$. Let $c \in S$ and $q \in cD_i(S)W(c)$. Then, using Propositions 3.2 and 3.3, we have

$$\begin{aligned} W(q) &\subseteq W(W(c))W(D_i(S))W(c) \\ &\subseteq E(S)cE(S)D_i(S)W(c) \\ &\subseteq E(S)(cD_i(S)W(c)) \\ &\subseteq E(S)D_{i+1}(S) \\ &\subseteq D_{i+1}(S). \end{aligned}$$

Similarly, if $q \in W(c)D_i(S)c$, then $W(q) \subseteq D_{i+1}(S)$.

Now, if $a \in D_{i+1}(S)$, then $a = q_1 \dots q_n$ for some q_j where each q_j is in $c_j D_i(S)W(c_j)$ or $W(c_j)D_i(S)c_j$ for some $c_j \in S$. Hence, by Proposition 3.2, $W(a) \subseteq D_{i+1}(S)$, and the proof is complete. $\square$

If T is a full weakly self conjugate subsemigroup of an E-dense semigroup S, then T is strongly dense in S because $\emptyset \neq W(a) \subseteq W_T(a)$ for all $a \in S$. In particular, $D(S)$ is strongly dense in S. It follows from Proposition 2.5 that S has a T-cover.

Now, for an E-dense semigroup S, we choose T to be $D(S)$, and we want to find an E-dense $D(S)$-cover $\widehat{S}$ with $\widehat{D(S)} = D(\widehat{S})$. This is done by using the construction of Section 2 with $\gamma_T(a) = W(a)$. We continue to use the notation introduced in the proof of Proposition 2.5 so that G is the free group on S, and X^* is the free monoid on $S \cup \overline{S}$. We constructed a relational morphism $\tau : G \mathrel{-\!\!\circ\!\!\!\rightarrow} S$ by using the natural homomorphism $\theta : X^* \to G$, a relational morphism $\varphi : X^* \mathrel{-\!\!\circ\!\!\!\rightarrow} S$, and putting $\tau = \theta^{-1}\varphi$. The cover $\widehat{S}$ of S over G is $\mathrm{gr}(\tau)$, that is, $\{(a, g) \in S \times G : a \in g\tau\}$. We have

$$\begin{tikzcd} \widehat{T} \arrow[r] \arrow[d, "\alpha"'] & \widehat{S} \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & \\ D(S) \arrow[r] & S & G \end{tikzcd}$$

and $\widehat{T} = 1\beta^{-1} = \{(a, 1) : a \in 1\tau\} = \{(a, 1) : a \in D(S)\}$.

Lemma 3.6. *If $(d, g) \in \widehat{S}$ and $c \in W(d)$, then $(c, g^{-1}) \in \widehat{S}$.*

Proof. If $g = 1$, then $d \in 1\tau$, that is, $d \in D(S)$, and so, by Theorem 3.5, $c \in D(S)$. Hence $c \in 1\tau$ and $(c, 1) \in \widehat{S}$.

Now suppose that $g \neq 1$. Since

$$g\tau = \bigcup\{w\varphi : w\theta = g\},$$

we have $d \in w\varphi$ for some $w \in X^*$ with $w\theta = g$. Let $w = x_1 \dots x_k$ where $x_i \in X$. Then $w\varphi = (x_1\varphi) \dots (x_k\varphi)$ and so, by Proposition 3.2, $c \in W(x_k\varphi) \dots W(x_1\varphi)$.

If $x_j = a$ where $a \in S$, then, by definition, $x_j\varphi = D(S)^1 a D(S)^1$, and therefore, using Proposition 3.2 and Theorem 3.5, we have

$$W(x_j\varphi) \subseteq D(S)^1 W(a) D(S)^1 = \overline{x}_j\varphi.$$

On the other hand, if $x_j = \overline{a}$ where $a \in S$, then $x_j\varphi = D(S)^1 W(a) D(S)^1$ and so, by Propositions 3.2 and 3.3, and Theorem 3.5, we have

$$\begin{aligned} W(x_j\varphi) &\subseteq D(S)^1 W(W(a)) D(S)^1 \subseteq D(S)^1 E(S) a E(S) D(S)^1 \\ &\subseteq D(S)^1 a D(S)^1 = \overline{x}_j\varphi. \end{aligned}$$

Thus $c \in (\overline{x}_k\varphi) \dots (\overline{x}_1\varphi) = \overline{w}\varphi$. Now $\overline{w}\theta = (w\theta)^{-1} = g^{-1}$ so that $c \in g^{-1}\tau$ as required. □

As an immediate consequence we have the following corollary.

Corollary 3.7. *The semigroup $\widehat{S}$ is E-dense.*

Lemma 3.8. $\widehat{T} = D(\widehat{S})$.

Proof. Let $(a, g) \in \widehat{S}$. Then any weak inverse of (a, g) has the form (b, g^{-1}) where $b \in W(a)$. Since $D(S)$ is weakly self conjugate in S, it follows that $\widehat{T} = \{(a, 1) : a \in D(S)\}$ is weakly self conjugate in $\widehat{S}$. As $\widehat{T}$ is clearly full, we have $D(\widehat{S}) \subseteq \widehat{T}$.

Now let $(a, 1) \in \widehat{T}$. Then $a \in D(S)$ so that $a \in D_i(S)$ for some i. Hence it is enough to show that, for all non-negative integers i,

$$a \in D_i(S) \text{ implies } (a, 1) \in D(\widehat{S}). \qquad (\dagger)$$

If $i = 0$, then $a = e_1 \dots e_k$ for some idempotents $e_1, \dots, e_k$. Certainly, $(e_j, 1) \in D(\widehat{S})$ for each j since $D(\widehat{S})$ is full, and so $(a, 1) \in D(\widehat{S})$.

Assume inductively that (†) holds for i, and let $a \in D_{i+1}(S)$. Then $a = a_1 \dots a_m$ for some $a_j = c_j b_j d_j$ where $b_j \in D_i(S)$, $c_j, d_j \in S$ and one of $c_j \in W(d_j)$ or $d_j \in W(c_j)$ holds. In either case, by Lemma 3.6, there is an element g of G such that $(c_j, g), (d_j, g^{-1}) \in \widehat{S}$. By the induction assumption, we have $(b_j, 1) \in D(\widehat{S})$ and so $(a_j, 1) = (c_j, g)(b_j, 1)(d_j, g^{-1}) \in D(\widehat{S})$. Hence $(a, 1) \in D(\widehat{S})$.

It follows that $\widehat{T} = D(\widehat{S})$, as required. □

An E-dense semigroup S is said to be *D-unitary* if $D(S)$ is a unitary subset of S. Putting together the preceding results and Lemma 2.3, we have now proved the following theorem.

Theorem 3.9. *Every E-dense semigroup has a D-unitary E-dense cover over a group.*

The theorem gives an alternative way of looking at $D(S)$ in an E-dense semigroup S. Let $\tau : S \multimap G$ be a relational morphism into a group G. If $e \in E(S)$, then $e\tau$ is a subsemigroup of G, but, in general, it need not be a submonoid. We say that τ is *full* if $E(S) \subseteq 1\tau^{-1}$, so that $e\tau$ is a submonoid for every idempotent e of S. Note that, since subsemigroups of finite groups are subgroups, every relational morphism into a finite group is full. Let $K(S)$ be the intersection of all subsemigroups K of S such that $K = 1\tau^{-1}$ for some full relational morphism τ into a group. It is easy to see that $K(S)$ is a full weakly self conjugate subsemigroup of S, so that $D(S) \subseteq K(S)$. On the other hand, by the theorem, $D(S) = 1\tau^{-1}$ for the relational morphism used in the proof of the theorem, so that $K(S) \subseteq D(S)$. Hence we have the following.

Corollary 3.10. *If S is an E-dense semigroup, then $D(S) = K(S)$.*

On any E-dense semigroup there is a minimum group congruence σ. The existence of σ was noted by Hall and Munn [18], and an explicit description was given by Mitsch [31]. We simply require that σ exists, and that $D(S)$ is contained in a σ-class. This follows from Corollary 3.10 since the natural homomorphism associated with σ is a full relational morphism. We now use these properties to give alternative criteria for an E-dense semigroup to be D-unitary.

Lemma 3.11. *For an E-dense semigroup S, the following conditions are equivalent:*
(1) *S is D-unitary,*
(2) *there is a surjective homomorphism $\beta : S \to G$ onto a group G with $D(S) = 1\beta^{-1}$,*
(3) *$D(S)$ is a σ-class.*

Proof. Suppose that $D(S)$ is unitary in S. As S is E-dense, $D(S)$ is certainly dense in S. Suppose that $x, y \in S$ with $xy \in D(S)$. Since S is E-dense, $W(x) \neq \emptyset$. Now, $D(S)$ is weakly self conjugate, and so if $b \in W(x)$, then $bxyx \in D(S)$. Since $bx \in E(S) \subseteq D(S)$ and $D(S)$ is unitary in S, we have $yx \in D(S)$ so that $D(S)$ is reflexive. Condition (2) now follows from Theorem 2.4.

If condition (2) holds, then $\beta \circ \beta^{-1}$ is a group congruence on S, and so $\sigma \subseteq \beta \circ \beta^{-1}$. Hence, if $a \in b\sigma$ where $b \in D(S)$, then $a\beta = 1$, and so $a \in D(S)$. Since $D(S)$ is contained in a σ-class, condition (3) follows.

If condition (3) holds, then clearly, $D(S)$ is the identity of S/σ. Let $a \in S, b \in D(S)$ with $ab \in D(S)$. Then

$$a\sigma = (a\sigma)(b\sigma) = (ab)\sigma = 1,$$

and hence $a \in D(S)$. Thus S is D-unitary. □

It is worth noting the next result which is an immediate consequence of the lemma.

Corollary 3.12. *Let P, S be E-dense semigroups, and $\alpha : P \to S$ be a surjective homomorphism such that $\alpha|_{D(P)}$ maps $D(P)$ isomorphically onto $D(S)$. If P is D-unitary, then P is a D-unitary cover of S over some group.*

We comment briefly on two special cases: regular semigroups and semigroups in which the idempotents form a subsemigroup. First, let S be a regular semigroup. Then every element a of S has an inverse, that is, an element a' such that $aa'a = a$ and $a'aa' = a'$. In particular, a' is a weak inverse of a. As usual, we denote the set of all inverses of a in S by $V(a)$.

A subsemigroup T of a regular semigroup S is said to be *self conjugate* if $aTa' \subseteq T$ for all $a \in S$ and $a' \in V(a)$. Obviously, a weakly self conjugate subsemigroup of S is self conjugate. The converse is also true if T is full, as we show in the next lemma.

Lemma 3.13. *Let T be a full subsemigroup of a regular semigroup S. Then T is self conjugate if and only if it is weakly self conjugate.*

Proof. Suppose that T is self conjugate, and let $t \in T$, $a \in S$, $b \in W(a)$ and $a' \in V(a)$. Then

$$atb = atbab = atbaa'ab = a(tba)a'(ab).$$

Now $E(S) \subseteq T$, so $ab \in T$ and $tba \in T$. But T is self conjugate, and so it follows that $atb \in T$. Similarly, $bta \in T$, and so T is weakly self conjugate. $\square$

Thus $D(S)$ is the least self conjugate full subsemigroup of a regular semigroup S. We also note that it is immediate from Theorem 3.5 that $D(S)$ is regular.

Given a regular semigroup S, we can use the construction above to obtain a D-unitary E-dense cover $\widehat{S}$ of S over a group. It follows from Lemma 3.6 that the regularity of S implies that $\widehat{S}$ is regular. Thus we have the following result which was first made explicit by Trotter in [43]. Of course, our proof gives an infinite cover, but one of the two proofs in [43] (based on results in [29]) gives a finite regular cover for a finite regular semigroup S.

Corollary 3.14. *A regular semigroup has a D-unitary regular cover over a group.*

It is perhaps worth mentioning that, in the regular case, our construction of a cover can be modified by using the set $V(a)$ of inverses of a rather than the set $W(a)$ of weak inverses of a.

We now consider the case where the set of idempotents is a subsemigroup of S.

Proposition 3.15. *Let S be an E-dense semigroup. If $E(S)$ is a subsemigroup of S, then $E(S) = D(S)$.*

Proof. Certainly, $E(S) \subseteq D(S)$ so that it is enough to prove that $E(S)$ is closed under weak conjugation. Let $a \in S$, $b \in W(a)$ and $e \in E$. Then

$$(aeb)^2 = aebaeb = aebae(bab) = a(eba)(eba)b = aebab = aeb,$$

and, similarly, $bea \in E(S)$. ☐

We say that an E-dense semigroup S is *E-unitary* if $E(S)$ is a unitary subset of S.

Lemma 3.16. *If S is an E-unitary E-dense semigroup, then $E(S)$ is a subsemigroup of S.*

Proof. If $e, f \in E(S)$, then $efb \in E(S)$ for some $b \in S$ since S is E-dense. Since S is E-unitary, $fb \in E(S)$, and consequently, $b \in E(S)$. From $efb \in E(S)$ and $b \in E(S)$ we get $ef \in E(S)$. ☐

It follows from the lemma and Proposition 3.15 that if S is an E-dense semigroup, then it is E-unitary if and only if it is D-unitary and $E(S)$ is a subsemigroup. In view of this and Proposition 3.15, we have the following results as corollaries of Theorem 3.9 and Corollary 3.14 respectively.

An E-dense semigroup in which the idempotents form a commutative subsemigroup is said to be *E-commutative dense.*

Corollary 3.17. *Let S be an E-dense semigroup in which $E(S)$ is a subsemigroup. Then S has an E-unitary E-dense cover $\widehat{S}$ over a group. Moreover, if S is E-commutative dense, then so is $\widehat{S}$.*

Recall that an *orthodox* semigroup is a regular semigroup in which the idempotents form a subsemigroup.

Corollary 3.18. *An orthodox semigroup S has an E-unitary orthodox cover $\widehat{S}$ over a group. Moreover, if S is inverse, then so is $\widehat{S}$.*

The general result in the first of these corollaries was proved independently by Almeida, Pin and Weil in [2] and Zhonghao in [46]. The E-commutative dense result is due to the author [10].

The result on covers for orthodox semigroups was proved independently by three authors: McAlister, Szendrei and Takizawa in [28], [40] and [41] respectively. Their proofs give finite covers for finite orthodox semigroups whereas we have already pointed out that our construction always yields an infinite cover. As mentioned in the introductory remarks, the inverse result was the first of the covering theorems and goes back to unpublished work of Tilson in the finite case and the seminal papers [26], [27] of McAlister in the general case. We give alternative proofs of the orthodox and inverse cases in the following sections because of the desirability of having finite covers of finite semigroups.

4. Inverse semigroups

We use factorisable inverse monoids, to give an alternative proof that every inverse semigroup S has an E-unitary inverse cover which is finite if S is finite, and describe the cover constructed as P-semigroup. This is followed by an account of results of McAlister and

Reilly [30] in which E-unitary covers are characterised using idempotent pure prehomomorphisms, and certain dual prehomomorphisms. We conclude the section by illustrating the use of the covering theorem and the P-theorem to obtain Reilly's structure theorem for bisimple inverse ω-semigroups.

4.1. **Factorisable monoids.** An element a of a monoid M is *unit regular* if $a = aua$ for some unit u of M.

Lemma 4.1. *Let a be an element of a monoid M. Then the following are equivalent:*

(1) *a is unit regular,*
(2) *$a = eg$ for some idempotent e and unit g,*
(3) *$a = hf$ for some idempotent f and unit h.*

Proof. If (1) holds, let u be a unit such that $aua = a$. Then au and ua are idempotents, u^{-1} is a unit, and $a = auu^{-1} = u^{-1}ua$.

If (2) holds, then $a = ea = ag^{-1}a$ so that a is unit regular. Similarly, (3) implies (1). □

A monoid M is *factorisable* (or *unit regular*) if $M = GE$ where G is a subgroup of M and $E = E(M)$ is the set of idempotents of M.

Lemma 4.2. *A monoid M is factorisable if and only if every element of M is unit regular.*

Proof. If $M = GE$ for some subgroup G, then $1 = ge$ for some $g \in G$ and idempotent e, so $e = (ge)e = ge = 1$. Hence $1 = g \in G$. Thus 1 is the identity of G and so G consists of units. (In fact, G is the group of units.) If $a \in M$, then $a = ue$ for some unit u and idempotent e, so that a is unit regular by Lemma 4.1.

The converse is immediate by Lemma 4.1. □

Examples of factorisable monoids include the following. For a finite set X, the symmetric inverse monoid $I(X)$ on X; the full transformation monoid $T(X)$ on X; the monoid $PT(X)$ of all partial transformations on X; and the multiplicative monoid $M_n(F)$ of all $n \times n$ matrices over a field F.

The significance of factorisable monoids in the inverse case arises from the following two results.

Proposition 4.3. *If S is an inverse semigroup, then S can be embedded in a factorisable inverse monoid.*

Proof. By the Vagner-Preston theorem (see [21], [22] or [32]), S can be embedded in the symmetric inverse monoid $I(S)$ on S. If S is finite, $I(S)$ is factorisable and we have the desired embedding.

If S is infinite, let S' be a set disjoint from S and having the same cardinality as S. Clearly, $I(S)$ (and hence S) can be embedded in $I(S \cup S')$. Moreover, if $\alpha \in I(S)$, then $\alpha = \theta|_{\operatorname{dom}\alpha}$ for some θ in the group of units of $I(S \cup S')$. Hence $\alpha \in F$ where

$$F = \{\gamma \in I(S \cup S') : \gamma \leqslant \theta \text{ for some unit } \theta\}.$$

Thus we see that S is embedded in F and that F is a factorisable inverse submonoid of $I(S \cup S')$. $\square$

Proposition 4.4. *Let F be a factorisable inverse monoid with set of idempotents E and group of units G. Then G acts (by automorphisms) on E by conjugation, and $E \rtimes G$ is an E-unitary inverse cover of F.*

Proof. The claim about the action is clear, so that we can form the semidirect product $E \rtimes G$ with multiplication $(e, g)(f, h) = (e(gfg^{-1}), gh)$. It is straightforward to verify that $E \rtimes G$ is E-unitary inverse, and that $E(E \rtimes G) = E \times \{1\}$. Now $\alpha : E \rtimes G \to F$ defined by $(e, g)\alpha = eg$ is a surjective homomorphism which restricts to an isomorphism from $E(E \rtimes G)$ to $E(F)$. $\square$

The finite versions of these two results tell us about the complexity of finite inverse semigroups. A *pseudovariety* of semigroups is a class of semigroups closed under finite direct product, homomorphic images and subsemigroups. Let $\mathbf{A}$ denote the pseudovariety of all finite *aperiodic* semigroups, that is, finite semigroups on which Green's relation $\mathscr{H}$ is trivial, and let $\mathbf{G}$ denote the pseudovariety of all finite groups. Let $\mathbf{V}_0 = \mathbf{A}$ and, for $n \geqslant 0$, let $\mathbf{V}_{n+1} = \mathbf{A} * \mathbf{G} * \mathbf{V}_n$. Here $\mathbf{V} * \mathbf{W}$ is the least pseudovariety which contains all semidirect products $S \rtimes T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$. It follows from the Krohn-Rhodes Decomposition Theorem that every semigroup S is in $\mathbf{V}_n$ for some n. The least such n is called the *(group) complexity* of S. If S is an inverse semigroup, then it follows from Propositions 4.3 and 4.4 that $S \in \mathbf{V}_1$ (more precisely, it is in $\mathbf{A} * \mathbf{G}$), and we have the following result of Tilson.

Proposition 4.5. *An inverse semigroup has complexity at most* 1.

Several results on the complexity of various classes of regular semigroups were given by Trotter [43] using the finite version of Corollary 3.14.

4.2. **E-unitary covers and the P-theorem.** We use Propositions 4.3 and 4.4 to give another proof of the covering theorem, and then show that the cover is a P-semigroup. First, we note some elementary variations on the condition that the covering map restricts to an isomorphism between the semilattices of idempotents. In proving the equivalence of the various conditions, we make use of Lallement's lemma. As we will use it several times, we record it here.

Lemma 4.6 (Lallement's Lemma)**.** *Let S, T be regular semigroups and let $\theta : S \to T$ be a surjective homomorphism. If $f \in E(T)$, then there is an idempotent e in S such that $e\theta = f$.*

Equivalently, if ρ is a congruence on a regular semigroup S, and if $a\rho$ is an idempotent in S/ρ, then there is an idempotent e in S such that $a\rho = e\rho$.

A proof of the lemma can be found, for example, in [21, Chapter 2].

Recall that a homomorphism $\alpha : S \to T$ of semigroups is *idempotent separating* if its restriction to $E(S)$ is one-one.

Lemma 4.7. *Let $\alpha : P \to S$ be a surjective homomorphism of inverse semigroups, and $a, b \in P$. Then the following are equivalent:*

(1) *$\alpha|_{E(P)}$ is an isomorphism from $E(P)$ onto $E(S)$,*
(2) *α is idempotent separating,*
(3) *$a\alpha \mathscr{L} b\alpha$ implies $a\mathscr{L}b$,*
(4) *$a\alpha \mathscr{R} b\alpha$ implies $a\mathscr{R}b$.*

Proof. If (1) holds, then *a fortiori*, (2) holds. If (2) holds and $a\alpha\mathscr{L}b\alpha$, then $(a^{-1}a)\alpha = (a\alpha)^{-1}(a\alpha) = (b\alpha)^{-1}(b\alpha) = (b^{-1}b)\alpha$. Hence $a^{-1}a = b^{-1}b$ since α is idempotent separating, and so $a\mathscr{L}b$. Thus (3) holds. Similarly, (2) implies (4).

If (3) holds, and $e, f \in E(P)$ with $e\alpha = f\alpha$, then, by assumption, $e\mathscr{L}f$ so that $e = f$ and $\alpha|_{E(P)}$ is one-one. Moreover, by Lallement's lemma (Lemma 4.6), we have that $\alpha|_{E(P)}$ maps $E(P)$ onto $E(S)$. Similarly, (4) implies (1). $\square$

We now give a second proof of the existence of E-unitary inverse covers for an inverse semigroup.

Proposition 4.8. *An inverse semigroup S has an E-unitary inverse cover. Moreover, if S is finite, then the cover can be chosen to be finite.*

Proof. By Lemmas 3.12 and 4.7, it is enough to find an E-unitary inverse semigroup P and an idempotent separating homomorphism from P onto S.

By Proposition 4.3, we can embed S in a factorisable inverse monoid F with group of units G and $E = E(F)$. It follows from Proposition 4.4 that the semidirect product $E \rtimes G$ is an E-unitary inverse cover for F over G. Let $\alpha : E \rtimes G \to F$ be the covering homomorphism ($(e, g)\alpha = eg$).

Then $S\alpha^{-1}$ is an inverse subsemigroup of $E \rtimes G$, and hence it is E-unitary. Also, α restricted to $S\alpha^{-1}$ must be idempotent separating, and so $S\alpha^{-1}$ is the desired cover. $\square$

As before, we let β be the projection of $E \rtimes G$ onto G. Then the group over which $S\alpha^{-1}$ is a cover is the image H of β restricted to $S\alpha^{-1}$. Thus

$$H = \{h \in G : eh \in S \text{ for some } e \in E(F)\}.$$

We also note that if $e \in E(F)$ and $eh \in S$ for some $h \in G$, then $e \in S$. For, $(e, h) \in S\alpha^{-1}$, and $S\alpha^{-1}$ is an inverse subsemigroup of $E \rtimes G$, so that $(e, 1) = (e, h)(h^{-1}eh, h^{-1}) = (e, h)(e, h)^{-1} \in S\alpha^{-1}$ and hence $e = (e, 1)\alpha \in S$.

The cover $S\alpha^{-1}$ can be described as another type of "semidirect product" known as a *P-semigroup*, a concept we now define.

Let G be a group, X a poset such that G acts on X by order automorphisms, and Y be a subset of X. Suppose that

(1) Y is an order ideal of X and a meet semilattice under the induced ordering,
(2) $G \cdot Y = X$,
(3) $g \cdot Y \cap Y \neq \emptyset$ for all $g \in G$.

Then we put

$$P = P(G, X, Y) = \{(y, g) \in Y \times G : g^{-1}y \in Y\}$$

with multiplication

$$(y,g)(y',g') = (y \wedge g \cdot y', gg').$$

The proof of the following is straightforward.

Proposition 4.9. *With the above notation, P is an E-unitary inverse semigroup with $E(P) = \{(y,1) : y \in Y\} \cong Y$, and maximum group homomorphic image G. Moreover, for $(y,g),(z,h) \in P$,*

(1) *$(y,g)\mathscr{R}(z,h)$ if and only if $y = z$,*
(2) *$(y,g)\mathscr{L}(z,h)$ if and only if $g^{-1} \cdot y = h^{-1} \cdot z$.*

The importance of P-semigroups arises from the following theorem of McAlister [27] which gives the converse of Proposition 4.9.

Theorem 4.10. *Let S be an inverse semigroup. If S is E-unitary, then S is isomorphic to a P-semigroup.*

It follows that if S is an inverse semigroup, then the E-unitary cover $S\alpha^{-1}$ in the proof of Proposition 4.8 is a P-semigroup. We could realise it as such by using the subdirect product description or the 'semigroupoid semidirect product' description, but it is easy to do it directly.

Let $Y = E(S)$ and H be the group $\{h \in G : eh \in S \text{ for some } e \in E(F)\}$ considered above. Recall that G (and hence H) acts on $E(F)$ by conjugation. Using this action, put $X = \{h \cdot e : h \in H, e \in E(S)\}$ so that X is a subset of $E(F)$ (and hence inherits the partial order of $E(F)$). It is easy to verify that H, X, Y provide the data for a P-semigroup, and that this semigroup actually is the cover $S\alpha^{-1}$.

The E-unitary cover $S\alpha^{-1}$ was constructed from an embedding of S in a factorisable inverse monoid. McAlister and Reilly proved [30] proved that every E-unitary cover arises from a certain kind of embedding. An embedding $\iota : S \to F$ of an inverse semigroup S into a factorisable inverse monoid F is said to be *strict* if, for each unit g of F, there is an element s of S such that $\iota(s) \leqslant g$.

Theorem 4.11. *Every E-unitary inverse cover of an inverse semigroup S over a group G is isomorphic to one constructed (as in Proposition* 4.8*) from a strict embedding of S into a factorisable inverse monoid with group of units G.*

A discussion of this theorem in terms of enlargements of groupoids is given in Chapter 8 of [22].

4.3. **Prehomomorphisms and dual prehomomorphisms.** In [30] a number of different ways in which E-unitary covers of inverse semigroups can arise are discussed. We explain two of these approaches using prehomomorphisms and dual prehomomorphisms.

Let S, T be inverse semigroups and $\theta : S \to T$ be a function. We say that

(1) θ is a *prehomomorphism* if $(ab)\theta \leqslant (a\theta)(b\theta)$ for all $a, b \in S$;
(2) θ is a *dual prehomomorphism* if $(ab)\theta \geqslant (a\theta)(b\theta)$ for all $a, b \in S$ and $(a\theta)^{-1} = a^{-1}\theta$ for all $a \in S$.

Here we follow the terminology of [22]; we note that in [32] what we have called a dual prehomomorphism is known as a prehomomorphism.

Let S be an inverse semigroup and G be a group. If S has an E-unitary inverse cover over G, then by Propositions 1.4 and 1.2, there is an $E(S)$-pure, that is, *idempotent pure* surjective relational morphism $\tau : S \multimap G$ such that the cover is isomorphic to $\mathrm{gr}(\tau)$. Since $\mathrm{gr}(\tau)$ is inverse, so is τ. Let $s \in S$. If $g \in s\tau$, then $s\tau = gH$ where $H = g^{-1}(s\tau)$ is a subgroup of G.

Now the set $K(G)$ of all cosets is an inverse monoid under the operation $\centerdot$ where $aH \centerdot bK$ is the smallest coset in G which contains the set product $(aH)(bK)$. The set of idempotents of $K(G)$ is the set of subgroups of G, and the identity is the trivial subgroup. The natural partial order on $K(G)$ is given by $aH \leqslant bK$ if and only if $aH \supseteq bK$.

Thus τ is an idempotent pure prehomomorphism from S to $K(G)$ such that, for all $g \in G$, there is an element $s \in S$ with $g \in s\tau$. Conversely, such a prehomomorphism is clearly an idempotent pure surjective relational morphism from S to G.

If we now consider the inverse relational morphism $\tau^{-1} : G \multimap S$, then it is not difficult to verify that for each element g of G, we have:

(1) $g\tau^{-1}$ is an order ideal of S (in the natural partial order),
(2) if $a, b \in g\tau^{-1}$, then $a^{-1}b, ab^{-1} \in E(S)$.

A subset of an inverse semigroup S satisfying condition (2) is said to be *compatible*, and a compatible order ideal (in the natural partial order) is said to be *permissible*. The set of all permissible subsets is denoted by $C(S)$. Under multiplication of subsets, $C(S)$ is an inverse monoid which was studied by Schein [37] (see also [22]). We mention that if H is a permissible subset of S, then its inverse as an element of $C(S)$ is just $H^{-1} = \{h^{-1} : h \in H\}$, and H is idempotent if and only if $H \subseteq E(S)$. We also note that the natural partial order is given by $H \leqslant K$ if and only if $H \subseteq K$.

Thus τ^{-1} maps G into $C(S)$. Now $(g\tau^{-1})^{-1} = g^{-1}\tau^{-1}$, and $1\tau^{-1} = E(S)$. Hence it is clear from the definition of surjective relational morphism that τ^{-1} is a dual prehomomorphism from G to $C(S)$ satisfying $\bigcup g\tau^{-1} = S$. Conversely, any such dual prehomomorphism gives an idempotent pure surjective relational morphism $G \multimap S$, and thus we have the following result.

Proposition 4.12. *Let S be an inverse semigroup, and G be a group. Then the following are equivalent:*

(1) *there is an E-unitary inverse cover of S over G,*
(2) *there is an idempotent pure prehomomorphism $\theta : S \to K(G)$ such that, for each $g \in G$, there is an element s of S with $g \in s\theta$.*
(3) *there is a dual prehomomorphism from G to $C(S)$ such that $\bigcup g\tau^{-1} = S$.*

4.4. **Bisimple inverse ω-semigroups.** We illustrate the use of covers and the P-theorem by giving a proof of a structure theorem due to Reilly [35]. Recall that a *bisimple ω-inverse semigroup* S is an inverse semigroup on which Green's relation $\mathscr{D}$ is the universal relation, and in which the idempotents form an ω-chain, that is, $E(S)$ is order isomorphic to the chain of negative integers. We remark that such an inverse semigroup is actually a monoid.

Let K be a group, and α be an endomorphism of K. On the set $\mathbb{N} \times K \times \mathbb{N}$ we define a binary operation by the rule:

$$(m, a, n)(p, b, q) = (m - n + t, (a\alpha^{n-t})(b\alpha^{p-t}), q - p + t),$$

where $t = \max\{n, p\}$ and α^0 is the identity map on K. The set $\mathbb{N} \times K \times \mathbb{N}$ together with this operation is a monoid denoted by $B(K, \alpha)$ and called a *Reilly monoid*. It is straightforward to show that $B(K, \alpha)$ is an inverse monoid with identity $(0, 1, 0)$, and semilattice of idempotents $\{(m, 1, m) : m \in \mathbb{N}\}$ with $(m, 1, m) \leqslant (n, 1, n)$ if and only if $m \geqslant n$. Further, the elements (m, a, n) and (p, b, q) are $\mathscr{R}$-related if and only if $m = p$, and $\mathscr{L}$-related if and only if $n = q$ so that $B(K, \alpha)$ is clearly bisimple. Our aim is to prove the following theorem.

Theorem 4.13. *Every bisimple inverse ω-semigroup is isomorphic to a Reilly monoid.*

We start with the following simple lemma.

Lemma 4.14. *Let $\theta : P \to S$ be an idempotent separating surjective homomorphism of inverse semigroups. If S is bisimple, then P is bisimple.*

Proof. If $a, b \in P$, then $a\theta, b\theta$ are $\mathscr{D}$-related in S, and , since θ is onto, there is an element c in P such that $a\theta \mathscr{L} c\theta \mathscr{R} b\theta$. By Lemma 4.7, $a \mathscr{L} c \mathscr{R} b$ so that all elements of P are $\mathscr{D}$-related, and P is bisimple. □

Lemma 4.15. *If $\theta : B(K, \alpha) \to S$ is an idempotent separating homomorphism from a Reilly monoid onto an inverse monoid S, then S is isomorphic to a Reilly monoid $B(H, \beta)$ where H is the group of units of S.*

Proof. The group of units of $B(K, \alpha)$ is $U = \{(0, k, 0) : k \in K\}$. Since θ is surjective and idempotent separating, $H = U\theta$. Define $\varphi : K \to H$ by $k\varphi = (0, k, 0)\theta$. Then φ is a surjective homomorphism; moreover, if $k \in \ker\varphi$, then $(0, k, 0)\theta = 1$ and hence

$$\begin{aligned}(0, k\alpha, 0)\theta &= ((0, 1, 1)(0, k, 0)(1, 1, 0))\theta = (0, 1, 1)\theta(1, 1, 0)\theta \\ &= ((0, 1, 1)(1, 1, 0))\theta = 1,\end{aligned}$$

so $k\alpha \in \ker\varphi$. Thus the function $\beta : H \to H$, defined by $h\beta = k\alpha\varphi$ where $k\varphi = h$ is well defined. Also, β is an endomorphism so that we have a Reilly monoid $B(H, \beta)$.

If we now define $\psi : B(K, \alpha) \to B(H, \beta)$ by $(m, k, n)\psi = (m, k\varphi, n)$, then ψ is a surjective homomorphism, and $\theta \circ \theta^{-1} = \psi \circ \psi^{-1}$. Hence

$$B(H, \beta) \cong B(K, \alpha)/\psi \circ \psi^{-1} = B(K, \alpha)/\theta \circ \theta^{-1} \cong S.$$

□

It follows from Lemmas 4.14 and 4.15 that to prove Theorem 4.13, it is enough to prove it for E-unitary bisimple inverse ω-semigroups. Let Q be such a semigroup. By Theorem 4.10, Q is isomorphic to a P-semigroup, say $Q \cong P = P(G, X, Y)$. Thus

$$P = \{(e, g) \in Y \times G : g^{-1}e \in Y\},$$

and $E(P) \cong Y$, so that Y is an ω-chain, say $Y = \{e_0, e_1, e_2, \dots\}$ with the order given by $e_0 > e_1 > e_2 > \dots$. Put

$$K = \mathrm{stab}_G(e_0) = \{g \in G : g^{-1} \cdot e_0 = e_0\}.$$

Then the group of units of P is $\{(e_0, g) \in P : g \in K\}$ which is isomorphic to K.

Next, we claim that if $k \in K$, then $k \cdot e_i = e_i$ for all i. Certainly, $k \cdot e_0 = e_0$, and, if $k \cdot e_j = e_j$ for some j, then since $e_{j+1} < e_j$ and G acts by order automorphisms, $k \cdot e_{j+1} < e_j$ and $k^{-1} \cdot e_{j+1} < e_j$. As Y is an order ideal, $k \cdot e_{j+1}, k^{-1} \cdot e_{j+1} \in Y$. Hence $k \cdot e_{j+1} \leqslant e_{j+1}$ and $k^{-1} \cdot e_{j+1} \leqslant e_{j+1}$ so that

$$k \cdot e_{j+1} \leqslant e_{j+1} = 1 \cdot e_{j+1} = (kk^{-1}) \cdot e_{j+1} = k \cdot (k^{-1} \cdot e_{j+1}) \leqslant k \cdot e_{j+1}.$$

Thus $k \cdot e_{j+1} = e_{j+1}$, and, by induction, the claim is true.

For any $i, j \in \mathbb{N}$ the pairs $(e_i, 1), (e_j, 1)$ are elements of P, and so they must be $\mathscr{D}$-related. Hence there is an element (e, g) of P with $(e_i, 1)\mathscr{R}(e, g)\mathscr{L}(e_j, 1)$. By Proposition 4.9, $g^{-1} \cdot e = e_j$ and $e = e_i$, and hence $g \cdot e_j = e_i$. In particular, we can choose an element $h \in G$ such that $h \cdot e_1 = e_0$. An induction argument similar to the above shows that $h^{-i} \cdot e_0 = e_i$ for all $i \in \mathbb{N}$.

For $k \in K$, we have $hkh^{-1} \cdot e_0 = hk \cdot e_1 = h \cdot e_1 = e_0$, so we can define an endomorphism $\alpha : K \to K$ by $k\alpha = hkh^{-1}$, and form the Reilly monoid $B(K, \alpha)$.

Finally, we define a mapping $\theta : P \to B(K, \alpha)$ by

$$(e_i, g)\theta = (i, h^i g h^{-j}, j)$$

where $g^{-1} \cdot e_i = e_j$.

We claim that θ is an isomorphism. First, note that $h^i g h^{-j} \cdot e_0 = h^i g \cdot e_j = h^i \cdot e_i = e_0$ so that $(i, h^i g h^{-j}, j)$ is an element of $B(K, \alpha)$.

Clearly, θ is one-one. If $(i, k, j) \in B(K, \alpha)$, put $g = h^{-i} k h^j$. Then $(e_i, g) \in P$ and $(e_i, g)\theta = (i, k, j)$ so that θ is onto.

Let $(e_i, g), (e_m, b) \in P$ with $g^{-1} \cdot e_i = e_j$. Straightforward calculations, considering the two cases $m \leqslant j$ and $m > j$ show that $((e_i, g)(e_m, b))\theta = (e_i, g)\theta(e_m, b)\theta$ so that θ is a homomorphism. Hence $P \cong B(K, \alpha)$ and the proof of Theorem 4.13 is complete.

Other inverse semigroup structure theorems can be proved using the same approach, for example, in unpublished notes, Victoria Gould has used the method to obtain new proofs of the Munn-Kochin result describing the structure of simple inverse ω-semigroups.

5. Orthodox semigroups

This section is based on McAlister's paper [28]. We show that an orthodox semigroup S has an E-unitary orthodox cover which is finite if S is finite. This result is used to obtain a characterisation of those orthodox semigroups in the pseudovariety $\mathbf{A} \vee \mathbf{G}$. Recall that $\mathbf{A}$ is the pseudovariety of all finite aperiodic semigroups, and $\mathbf{G}$ is the pseudovariety of all finite groups. In [28], McAlister proves the following result.

Theorem 5.1. *Let S be a finite orthodox semigroup. Then $S \in \boldsymbol{A} \vee \boldsymbol{G}$ if and only if $\mathscr{H}$ is a congruence on S.*

The proof of the "only if" part does not involve covers, and we do not present this. Our aim is to show how the covering theorem is used to prove the "if" part of the theorem.

For basic results on orthodox semigroups, see [21, Chapter 6]. We will need the following easy lemma.

Lemma 5.2. *If e is an idempotent in an orthodox semigroup S, then*

$$V(e) \subseteq E(S).$$

Proof. Let $x \in V(e)$. Then $xex = x$ so that $xe, ex \in E(S)$, and hence the product $(xe)(ex) \in E(S)$. But $x = xex = (xe)(ex)$. □

We also use three special congruences on an orthodox semigroup: the maximum idempotent separating congruence μ, the minimum group congruence σ and the minimum inverse semigroup congruence $\mathscr{Y}$. We have already introduced σ in the more general context of E-dense semigroups. One can give an explicit description of μ (see, for example, [20, Theorem VI.1.17]), but we simply need its existence and the fact that it is the largest congruence contained in $\mathscr{H}$ (see, for example, [21, Proposition 2.4.5] where these facts are proved for regular semigroups). However, we do want the following explicit description of $\mathscr{Y}$ due to Hall [16] (see also [21, Theorem 6.2.5]).

Proposition 5.3. *Let S be an orthodox semigroup, and let $\mathscr{Y}$ be the relation defined by*

$$a \mathscr{Y} b \text{ if and only if } V(a) = V(b).$$

Then $\mathscr{Y}$ is the minimum inverse semigroup congruence on S.

We also use a corollary of the following general result from [17], the proof of which we leave as an exercise.

Lemma 5.4. *Let ρ be a congruence on a semigroup S with $\rho \subseteq \mathscr{L}$. Then $(a, b) \in \mathscr{L}$ in S if and only if $(a\rho, b\rho) \in \mathscr{L}$ in S/ρ.*

Corollary 5.5. *If S is a regular semigroup, then the maximum idempotent separating congruence $\mu_{S/\mu}$ on S/μ is trivial.*

Proof. For some congruence ρ on S, we have $\mu_{S/\mu} = \rho/\mu$. If $a, b \in S$ and $a\rho b$, then $(a\mu, b\mu) \in \rho/\mu$. Hence $(a\mu, b\mu) \in \mathscr{H}$, and so by the lemma and its right-left dual, $a\mathscr{H}b$ in S. Thus $\rho \subseteq \mathscr{H}$ and so $\rho \subseteq \mu$. Hence $\mu_{S/\mu}$ is trivial. □

As a further corollary, we have the following.

Corollary 5.6. *Let S and T be regular semigroups with maximum idempotent separating congruences μ_S and μ_T respectively, and let $\theta : S \to T$ be a surjective idempotent separating homomorphism. Then $S/\mu_S \cong T/\mu_T$.*

Proof. Let $\varphi : T \to T/\mu_T$ be the natural homomorphism, and let $\rho = \theta\varphi(\theta\varphi)^{-1}$. Then ρ is idempotent separating, so $\rho \subseteq \mu_S$ and μ_S/ρ is idempotent separating on S/ρ. But $S/\rho \cong T/\mu_T$, so that by Corollary 5.5, μ_S/ρ is trivial. Hence $\rho = \mu$. □

Lemma 5.7. *If S is an orthodox semigroup, then $\mathscr{H} \cap \mathscr{Y} = \iota$.*

Proof. Suppose that $a, b \in S$ with $a \mathscr{H} b$ and $a \mathscr{Y} b$ and let $a' \in V(a)$. From the proof of Proposition 2.4.1 of [21], $a'a = b'b$ for some $b' \in V(b)$. But $V(a) = V(b)$, so that $b' \in V(a)$, and, by the same argument, there is an inverse b'' of b such that $ab' = bb''$. Now we have

$$a = aa'a = ab'b = bb''b = b.$$

□

If S is E-unitary, we have a stronger result [28].

Lemma 5.8. *If S is an E-unitary orthodox semigroup, then $\mathscr{H} \cap \sigma = \iota$ where σ is the minimum group congruence on S.*

Proof. Suppose that $a, b \in S$ with $a \mathscr{H} b$ and $a \sigma b$. From $a \mathscr{H} b$, it follows by Proposition 2.4.1 of [21], that there are inverses a', b' of a and b respectively such that $aa' = bb'$ and $a'a = b'b$. Hence $a' \mathscr{H} b'$, and since $L_a \cap R_{a'}$ contains an idempotent, we have $ab' \mathscr{H} aa'$.

From $a \sigma b$, we get $ab' \sigma bb'$. By Lemma 3.11, $E(S)$ is a σ-class, and so ab' is idempotent. Hence $ab' = aa'$. Similarly, $b'a = b'b$. Thus

$$a = aa'a = ab'a = ab'b = aa'b = bb'b = b.$$

□

Next, we note that if $\theta : T \to S$ is an idempotent separating surjective homomorphism of orthodox semigroups, then by Lallement's lemma, $\theta|_E(T)$ is an isomorphism from $E(T)$ onto S. Thus to prove that an E-unitary orthodox semigroup T is an E-unitary cover of an orthodox semigroup S, it is enough to show that there is an idempotent separating homomorphism from T onto S.

Theorem 5.9. *If S is a (finite) orthodox semigroup, then there is a (finite) E-unitary orthodox cover $\widehat{S}$ of S.*

Proof. Let $I = S/\mathscr{Y}$ and let $\gamma : S \to I$ be the natural homomorphism. By Proposition 4.8, I has an E-unitary inverse cover P with covering homomorphism $\alpha : P \to I$; moreover, if S is finite, then I is finite, and P can be chosen to be finite. Let

$$\widehat{S} = \{(s\mu, p) \in S/\mu \times P : s \in S \text{ and } p\alpha = s\gamma\}.$$

Clearly, if S is finite, then so is $\widehat{S}$. It is straightforward to verify that $\widehat{S}$ is a regular subsemigroup of the direct product $S/\mu \times P$.

We claim that if $(s\mu, p) \in \widehat{S}$ and $p \in E(P)$, then $s \in E(S)$.

To prove the claim, note that under the hypotheses, $s\gamma = p\alpha \in E(I)$. By Lallement's lemma, $s\gamma = e\gamma$ for some idempotent e of S. By the definition of γ, this gives $V(s) = V(e)$, and hence by Lemma 5.2, $V(s) \subseteq E(S)$. Hence s is an inverse of an idempotent, so that, again by Lemma 5.2, $s \in E(S)$.

It follows that

$$E(\widehat{S}) = \{(e\mu, f) : e \in E(S), f \in E(P) \text{ and } e\gamma = f\alpha\},$$

and hence that $\widehat{S}$ is orthodox.

Next we show that $\widehat{S}$ is E-unitary. Since P is E-unitary, there is, by Lemma 3.11, a homomorphism $\beta : P \to G$ onto a group G with $E(P) = 1\beta^{-1}$. Define $\psi : \widehat{S} \to G$ by the rule:

$$(s\mu, p)\psi = p\beta.$$

If $(s\mu, p)\psi = 1$, then $p \in 1\beta^{-1} = E(P)$, and so, by the claim, s is idempotent. Hence $E(\widehat{S}) = 1\psi^{-1}$, and so, by Lemma 3.11, $\widehat{S}$ is E-unitary.

To see that $\widehat{S}$ is a cover, define $\theta : \widehat{S} \to S$ by $(s\mu, p)\theta = s$. If $s\mu t$ and $(t\mu, p) \in \widehat{S}$, then $s\gamma = p\alpha = t\gamma$ so that $(s, t) \in \mu \cap \gamma\gamma^{-1} = \mu \cap \mathscr{Y}$. Hence, by Lemma 5.7, $s = t$, and θ is well defined. Clearly, it is a surjective homomorphism. Suppose that $(e\mu, p), (f\mu, q)$ are idempotents of $\widehat{S}$ with $(e\mu, p)\theta = (f\mu, q)\theta$. Then $e = f$, that is, $p\alpha = q\alpha$. But $p, q \in E(P)$ and α is idempotent separating, so $p = q$. Thus θ is idempotent separating, and $\widehat{S}$ is an E-unitary cover of S. $\square$

We now prove the "if" part of Theorem 5.1.

Proposition 5.10. *Let S be a finite orthodox semigroup. If $\mathscr{H}$ is a congruence on S, then $S \in \boldsymbol{A} \vee \boldsymbol{G}$.*

Proof. By Theorem 5.9, there is a finite E-unitary orthodox cover $\widehat{S}$ for S. Let σ be the minimum group congruence on $\widehat{S}$, and let $G = \widehat{S}/\sigma$. By Lemma 5.8, $\mathscr{H} \cap \sigma = \iota$ so that $\mu \cap \sigma = \iota$, and hence, by Proposition 1.1, $\widehat{S}$ can be embedded (as a subdirect product) in $\widehat{S}/\mu \times G$.

Now S is an idempotent separating homomorphic image of $\widehat{S}$, and so, by Corollary 5.6, $\widehat{S}/\mu \cong S/\mu$. By assumption, $\mu = \mathscr{H}$ on S, so $\widehat{S}/\mu \cong S/\mathscr{H}$, and thus $\widehat{S}$ can be embedded in $S/\mathscr{H} \times G$. Since $\mathscr{H}$ is a congruence on S, it follows from Lemma 5.4 that $\mathscr{H}$ is trivial on $S/\mathscr{H}$, so $S/\mathscr{H} \in \mathbf{A}$. Hence $S/\mathscr{H} \times G \in \mathbf{A} \vee \mathbf{G}$, and as pseudovarieties are closed under taking subsemigroups and homomorphic images, we have $S \in \mathbf{A} \vee \mathbf{G}$ as required. $\square$

In [29] McAlister generalised Theorem 5.1 to get the following result for regular semigroups.

Theorem 5.11. *Let S be a finite regular semigroup. Then $S \in \boldsymbol{A} \vee \boldsymbol{G}$ if and only if $D(S) \in \boldsymbol{A}$ and $\mathscr{H}$ is a congruence on S.*

6. Generalisations and Related Topics

We conclude the paper by giving a brief account of some other work related to covers. We do not give proofs but point the reader to some of the relevant literature. We start by describing some aspects of covers over monoids other than groups. We then mention the work of Auinger and Trotter on covers over groups belonging to a given variety of groups. Finally, we discuss finite covers of finite semigroups.

6.1. **Left ample and weakly left ample semigroups.** For a set X, we define the operation $^+$ on the monoid $PT(X)$ of all partial transformations on X by taking α^+ to be the identity mapping on the domain of α. Let S be a semigroup with a unary operation $^+$ such that $e = e^+$ for every idempotent e of S. Then S is said to be *weakly left ample* if there is a $(2,1)$-algebra embedding of S into $PT(X)$ for some set X.

Note that $I(X)$, the symmetric inverse monoid on X, is a $(2,1)$-subalgebra of $PT(X)$ and that $\alpha^+ = \alpha\alpha^{-1}$ for all $\alpha \in I(X)$. If S is a semigroup with a unary operation $^+$ and there is a $(2,1)$-algebra embedding of S into $I(X)$, we say that S is *left ample*. It is easy to see that left ample semigroups are weakly left ample. On an inverse semigroup S, we can define a unary operation $^+$ by $a^+ = aa^{-1}$ for $a \in S$, and then the Vagner-Preston representation shows that S is left ample.

In these definitions we assume that the partial transformations are written on the right of their arguments. By using the dual monoids with the partial transformations written on the left of their arguments, we get definitions of weakly right ample and right ample semigroups. In this case, the unary operation is written as *.

It is clear from the definition that, in a weakly left ample semigroup, the idempotents commute with each other, and so $E(S)$ is a subsemilattice of S.

On a weakly left ample semigroup S, there is a least congruence σ such that S/σ is a unipotent monoid, that is, the identity is the only idempotent of S/σ. If S is actually left ample, then S/σ is a right cancellative monoid. In both cases (weakly left ample and left ample), S is said to be *proper* if for all elements a and b of S such that $a^+ = b^+$ and $a\sigma b$, we have $a = b$.

It is well known that an inverse semigroup is proper if and only if it is E-unitary (see, for example, [21, Proposition 5.9.1]). Example 3 of [9] shows that the corresponding statement does not hold for left ample semigroups.

Let S be a left ample semigroup and T be a right cancellative monoid. We say that a left ample semigroup P is a *proper cover* of S (*over* T) if P is proper and there is a surjective $(2,1)$-algebra homomorphism from P onto S which maps $E(P)$ isomorphically onto $E(S)$ (and is such that $P/\sigma \cong T$). A *proper cover* of a weakly left ample semigroup over a unipotent monoid is defined similarly.

The existence of a proper cover for a right ample monoid S was established in [9]; the result is easily extended to the case where S is a semigroup, and, of course, the left ample results are simply the duals. Moreover, the cover is finite if S is finite. Thus we have the following analogue for left ample semigroups of Proposition 4.8.

Proposition 6.1. *A left ample semigroup S has a proper left ample cover. Moreover, if S is finite, then the cover can be chosen to be finite.*

In the weakly left ample case, the existence of proper weakly left ample covers was proved in [14]. The fact that a finite weakly left ample semigroup has a finite cover was first shown in [15] using elementary methods, and subsequently obtained, as a consequence of a more general result, in [4] using a sophisticated result of Ash. Moreover, it was shown in [15], that a finite proper weakly left ample semigroup is actually left ample. Combining these results we have the following result.

Proposition 6.2. *A weakly left ample semigroup S has a proper weakly left ample cover. Moreover, if S is finite, then the cover can be chosen to be finite, and is left ample.*

6.2. **Covers over group varieties.** In Section 3 we showed that an E-dense semigroup S has a D-unitary E-dense cover $\widehat{S}$ over a group G; in fact, G is the maximum group homomorphic image of $\widehat{S}$ and, letting $\beta : \widehat{S} \to G$ be the natural homomorphism, we have $D(S) \cong D(\widehat{S}) = 1\beta^{-1}$. If $\mathbf{H}$ is a variety of groups, and we want our cover to be over a group in $\mathbf{H}$, what subsemigroup do we use instead of $D(S)$, and what relational morphism do we use? These questions have recently been answered by Auinger and Trotter in [5]. We restrict ourselves to describing the subsemigroup and the relational morphism, and stating some of the main theorems. We refer the reader to [5] for proofs.

Let $\mathbf{H}$ be a variety of groups, S be an E-dense semigroup and X be a countably infinite set. Let $X^{-1} = \{x^{-1} : x \in X\}$ be a set disjoint from X and such that $x \mapsto x^{-1}$ is a bijection. Let $\widetilde{X} = X \cup X^{-1}$ and let $\widetilde{X}^+$ be the free semigroup on $\widetilde{X}$.

For a function $\varphi : X \to S$ and element x of X, put

$$x\overline{\varphi} = \{x\varphi\} \quad \text{and} \quad x^{-1}\overline{\varphi} = W(x\varphi),$$

and, for $x_1, \ldots, x_n \in \widetilde{X}$, put

$$(x_1 \ldots x_n)\overline{\varphi} = x_1\overline{\varphi} \ldots x_n\overline{\varphi}.$$

For $w \in \widetilde{X}^+$, say that $w\overline{\varphi}$ is the *set of values of w under the substitution φ*. If $w \simeq 1$ is a law in $\mathbf{H}$, write $\mathbf{H} \models w \simeq 1$. Now put

$$C_{\mathbf{H}}(S) = \bigcup\{w\overline{\varphi} : w \in \widetilde{X}^+,\ \mathbf{H} \models w \simeq 1 \text{ and } \varphi : X \to S \text{ is a substitution}\},$$

that is, $C_{\mathbf{H}}(S)$ is the set of all values in S of all words w for which $w \simeq 1$ is a law in $\mathbf{H}$. It turns out that $C_{\mathbf{H}}(S)$ is a full weakly self conjugate subsemigroup of S. Moreover, the analogue of Theorem 3.5 holds, that is, $C_{\mathbf{H}}(S)$ contains every weak inverse of each of its elements, so that, in particular, $C_{\mathbf{H}}(S)$ is E-dense, and it is regular if S is regular.

We now describe the relational morphism used to get the covering result. Given an E-dense semigroup S, let $X_S = \{x_s : s \in S\}$ be a copy of S disjoint from S, and $X_S^{-1} = \{x^{-1} : x \in X_S\}$ be a copy of X_S disjoint from $X_S \cup S$. Let $\widetilde{X_S} = X_S \cup X_S^{-1}$, and $\rho_{\mathbf{H}}$ be the congruence on $\widetilde{X_S}^+$ such that $\widetilde{X_S}^+/\rho_{\mathbf{H}}$ is the relatively free group $F_{\mathbf{H}}(X_S)$ in $\mathbf{H}$ on X_S. Finally, let $\varphi_S : X_S \to S$ be the substitution given by $x_s\varphi_S = s$, and for each $g \in F_{\mathbf{H}}(X_S)$, define

$$g\tau_{\mathbf{H}} = \bigcup\{w\overline{\varphi}_S : w \in \widetilde{X_S}^+ \text{ and } w\rho_{\mathbf{H}} = g\}.$$

Then $\tau_{\mathbf{H}} : F_{\mathbf{H}}(X_S) \multimap S$ is a surjective relational morphism such that $1\tau_{\mathbf{H}} = C_{\mathbf{H}}(S)$, and so $\tau_{\mathbf{H}}^{-1} : S \multimap F_{\mathbf{H}}(X_S)$ is a $C_{\mathbf{H}}(S)$-pure surjective relational morphism, and this is the relational morphism which is used to prove the following result.

Theorem 6.3. *An E-dense semigroup S, has an E-dense $C_{\boldsymbol{H}}(S)$-unitary cover $\widehat{S}$ over the relatively free group $F_{\boldsymbol{H}}(X_S)$ with $\widehat{C_{\boldsymbol{H}}(S)} = C_{\boldsymbol{H}}(\widehat{S})$. Moreover, if S is regular, then so is $\widehat{S}$.*

For the variety $\mathbf{G}$ of all groups, we have $C_{\mathbf{G}}(S) = D(S)$, and so Theorem 3.9 is a special case of Theorem 6.3.

6.3. **Finite semigroups.** As we have repeatedly stressed, our general construction of covers always gives an infinite cover. We have, however, also shown that finite inverse and orthodox semigroups have finite E-unitary covers, and mentioned that finite regular semigroups have finite regular D-unitary covers. The existence of a finite D-unitary cover for an arbitrary finite semigroup is a much deeper result as we now explain.

First, for a *finite* semigroup S, we redefine $K(S)$ to be the intersection of all subsemigroups K of S such that $K = 1\tau^{-1}$ for some relational morphism τ into a *finite* group. It is shown in [33] that there is such a τ with $K(S) = 1\tau^{-1}$, and as $E(S) \subseteq 1\tau^{-1}$ for any such τ, we see that $K(S)$ is a full weakly self conjugate subsemigroup of S so that $D(S) \subseteq K(S)$. We now give the straightforward connection with covers, quoting from [44] and [12].

Proposition 6.4. *For a finite semigroup S, the following conditions are equivalent:*

(1) *S has a finite D-unitary cover;*
(2) *$D(S) = K(S)$.*

Proof. If (1) holds, then the cover must be over a finite group G, and by Proposition 1.2, $D(S) = 1\tau^{-1}$ for some relational morphism τ into G. Hence $K(S) \subseteq D(S)$ and (2) holds.

If (2) holds, then $D(S) = K(S) = 1\tau^{-1}$ for some relational morphism τ into a finite group, and by Proposition 1.2, $\mathrm{gr}(\tau)$ is a D-unitary cover for S, so that (1) holds. □

The conjectured truth of (2) was one of the major open problems during the 1970s and 1980s, known as the *type II conjecture*. It was finally proved by Ash [3], as a consequence of a much more general result. Shortly afterwards Ribes and Zalesskiĭ [36] proved that finite products of finitely generated subgroups of a free group are closed in the profinite topology of the group. That this result implied the type II conjecture had already been shown by Pin and Reutenauer [34]. More recently, Herwig and Lascar [19] obtained a result on the extendability of partial automorphisms of a relational structure to automorphisms of a containing structure, a consequence of which is the Ribes-Zalesskiĭ theorem. Thus this gives a third proof of the type II conjecture. The methods used in the three proofs have been significantly developed, and the articles by Almeida, Ribes and Coulbois in this volume provide excellent introductions to these topics.

Acknowledgements

I would like to thank the organisers (Gracinda Gomes, Jean-Eric Pin and Pedro Silva) of the Thematic Term on Semigroups, Algorithms, Automata and Languages for inviting me to speak at one of the summer schools at the Centro Internacional de Mátematica in Coimbra. I would also like to acknowledge the financial support of Fundação Calouste Gulbenkian (FCG), Fundação para a Ciência e a Tecnologia (FCT), Faculdade de Ciências da Universidade de Lisboa (FCUL), Reitoria da Universidade do Porto, Centro de Álgebra da Universidade de Lisboa (CAUL) and Centro Internacional de Mátematica (CIM) in Coimbra.

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