Department of Mathematics
Formal Languages and Automata 2003
Solutions for Exercises 7

1. (*a*) $L = (\{b\}^*\{aaa\}^*\{b\}^*)^*$.
   (*b*) $L = \{b\}^*\{a\}\{b\}^*\{a\}\{b\}^*\{a\}\{b\}^*$.
   (*c*) $L = \{b\}^*\{a\}\{b\}^*\{a\}\{b\}^* \cup \{b\}^*\{a\}\{b\}^*\{a\}\{b\}^*\{a\}\{b\}^*$.
   (*d*) $L = \{b, ab, aab\}^*$.
   (*e*) $L = \{a, b\}^*\{aa\} \cup \{a, b\}^*\{bb\}$.

2. (*a*) Let $L_1 = \{a^{2n}b^{2m} : n \geqslant 0, m \geqslant 0\}$. Then

$$L_1 = \{a^{2n} : n \geqslant 0\}\{b^{2m} : m \geqslant 0\} = \{a^2\}^*\{b^2\}^*.$$

   Hence L_1 is obtained from the finite sets $\{a^2\}$ and $\{b^2\}$ by a finite number of applications of rational operations (3 applications, in fact). Thus L_1 is rational, and so L_1 is recognisable by Kleene's theorem.

   (*b*) Let $L_2 = \{a^m ba^n : m \geqslant 0, n \geqslant 0\}$. Then

$$L_2 = \{a^m : m \geqslant 0\}\{b\}\{a^n : n \geqslant 0\} = \{a\}^*\{b\}\{a\}^*.$$

   Hence L_2 is obtained from the finite sets $\{a\}$ and $\{b\}$ by a finite number of applications of rational operations (4 applications, in fact). Thus L_2 is rational, and so L_2 is recognisable by Kleene's theorem.

   (*c*) Let $L_3 = \{w \in \{a, b\}^* : |w|_a \leqslant 2 \text{ or } |w|_b = 1\}$. Then

$$\begin{aligned} L_3 &= \{w \in \{a, b\}^* : |w|_a \leqslant 2\} \cup \{w \in \{a, b\}^* : |w|_b = 1\} \\ &= \{b\}^* \cup \{b\}^*\{a\}\{b\}^* \cup \{b\}^*\{a\}\{b\}^*\{a\}\{b\}^* \cup \{a\}^*\{b\}\{a\}^*. \end{aligned}$$

   Hence L_2 is obtained from the finite sets $\{a\}$ and $\{b\}$ by a finite number of applications of rational operations (19 applications, in fact). Thus L_3 is rational, and so L_3 is recognisable by Kleene's theorem.

3. Let $L, K \in \operatorname{Rat} A^*$. Then, by Kleene's theorem, $L, K \in \operatorname{Rec} A^*$. By the corollary of Theorem 2, $\operatorname{Rec} A^*$ is closed under complementation and intersection. Hence $(A^* \setminus K) \cap L \in \operatorname{Rec} A^*$. But $L \setminus K = (A^* \setminus K) \cap L$, so $L \setminus K \in \operatorname{Rec} A^*$, and hence by Kleene's theorem, $L \setminus K \in \operatorname{Rat} A^*$.

4. (*a*) To prove $(L \cup K)\sigma = L\sigma \cup K\sigma$.

$$\begin{aligned} v \in (L \cup K)\sigma &\Leftrightarrow v \in w\sigma \text{ for some } w \in L \cup K \\ &\Leftrightarrow v \in w\sigma \text{ for some } w \in L \text{ or some } w \in K \\ &\Leftrightarrow v \in L\sigma \text{ or } v \in K\sigma. \end{aligned}$$

   Hence $(L \cup K)\sigma = L\sigma \cup K\sigma$.

   (*b*) To prove $(LK)\sigma = (L\sigma)(K\sigma)$.

$$\begin{aligned} v \in (LK)\sigma &\Rightarrow v \in w\sigma \text{ for some } w \in LK \\ &\Rightarrow v \in (w_1w_2)\sigma \text{ for some } w_1 \in L \text{ and } w_2 \in K \\ &\Rightarrow v \in (w_1\sigma)(w_2\sigma) \text{ for some } w_1 \in L \text{ and } w_2 \in K \\ &\Rightarrow v \in (L\sigma)(K\sigma). \end{aligned}$$

Hence $(LK)\sigma \subseteq (L\sigma)(K\sigma)$.

$$\begin{aligned} v \in (L\sigma)(K\sigma) &\Rightarrow v = v_1 v_2 \text{ for some } v_1 \in L\sigma \text{ and } v_2 \in K\sigma \\ &\Rightarrow v_1 \in w_1\sigma \text{ for some } w_1 \in L \text{ and } v_2 \in w_2\sigma \text{ for some } w_2 \in K \\ &\Rightarrow v = v_1 v_2 \in (w_1\sigma)(w_2\sigma) \text{ for some } w_1 \in L \text{ and } w_2 \in K \\ &\Rightarrow v \in (w_1 w_2)\sigma \text{ for some } w_1 \in L \text{ and } w_2 \in K \\ &\Rightarrow v \in (LK)\sigma. \end{aligned}$$

Hence $(L\sigma)(K\sigma) \subseteq (LK)\sigma$ and so $(LK)\sigma = (L\sigma)(K\sigma)$.

(c) To prove $L^*\sigma = (L\sigma)^*$.

$$\begin{aligned} v \in L^*\sigma &\Rightarrow v \in w\sigma \text{ for some } w \in L^* \\ &\Rightarrow v \in \varepsilon\sigma \text{ or } v \in (w_1 \dots w_n)\sigma \text{ for some } w_1, \dots, w_n \in L \\ &\Rightarrow v \in \{\varepsilon\} \text{ or } v \in (w_1\sigma) \dots (w_n\sigma) \text{ for some } w_1, \dots, w_n \in L \\ &\Rightarrow v \in \{\varepsilon\} \text{ or } v \in (L\sigma)^n \text{ for some } n \geqslant 1 \\ &\Rightarrow v \in (L\sigma)^*. \end{aligned}$$

Hence $L^*\sigma \subseteq (L\sigma)^*$.

$$\begin{aligned} v \in (L\sigma)^* &\Rightarrow v = \varepsilon \text{ or } v = v_1 \dots v_n \text{ for some } v_1, \dots, v_n \in L\sigma \\ &\Rightarrow v = \varepsilon \text{ or } v = v_1 \dots v_n \text{ and } v_1 \in w_1\sigma, \dots, v_n \in w_n\sigma \text{ for some } w_1, \dots, w_n \in L \\ &\Rightarrow v \in \varepsilon\sigma \text{ or } v \in (w_1\sigma) \dots (w_n\sigma) \text{ for some } w_1, \dots, w_n \in L \\ &\Rightarrow v \in \varepsilon\sigma \text{ or } v \in (w_1 \dots w_n)\sigma \text{ for some } w_1, \dots, w_n \in L \\ &\Rightarrow v \in w\sigma \text{ for some } w \in L^* \\ &\Rightarrow v \in L^*\sigma. \end{aligned}$$

Hence $(L\sigma)^* \subseteq L^*\sigma$ and so $L^*\sigma = (L\sigma)^*$.

5. Let σ be a rational substitution on A^*.
   (a) If $w = \varepsilon$, then $\varepsilon\sigma = \{\varepsilon\} \in \mathrm{Rat}\, A^*$.
   Otherwise, $w = a_1 \dots a_n$ for some $a_1, \dots, a_n \in A$,
   so $w\sigma = (a_1 \dots a_n)\sigma = (a_1\sigma) \dots (a_n\sigma) \in \mathrm{Rat}\, A^*$ since each $a_i\sigma \in \mathrm{Rat}\, A^*$ and $\mathrm{Rat}\, A^*$ is closed under product.
   (b) If L is finite, then either $L = \emptyset$ or $L = \{w_1, \dots, w_n\}$ for some $w_1, \dots, w_n \in A^*$. Now $\emptyset\sigma = \emptyset \in \mathrm{Rat}\, A^*$ and using Q4,

   $$\{w_1, \dots, w_n\}\sigma = (\{w_1\} \cup \dots \cup \{w_n\})\sigma = \{w_1\}\sigma \cup \dots \cup \{w_n\}\sigma \in \mathrm{Rat}\, A^*$$

   since each $\{w_i\}\sigma \in \mathrm{Rat}\, A^*$ by (a), and $\mathrm{Rat}\, A^*$ is closed under union. Thus $L\sigma \in \mathrm{Rat}\, A^*$.
   Now let $P(n)$ be the statement "if L can be constructed from finite subsets of A^*, by applying rational operations n times, then $L \in \mathrm{Rat}\, A^*$".
   $P(0)$ is true by (b). Assume $P(h)$ is true for all h with $h < k$, and let L be a languguage that can be constructed from finite subsets of A^* by applying rational operations k times. Then one of $L = L_1 \cup L_2$, $L = L_1 L_2$ or $L = L^*$ holds for some L_1, L_2 which can be constructed from finite subsets of A^* by applying rational operations less than k times. Hence $L_1\sigma \in \mathrm{Rat}\, A^*$ and $L_2\sigma \in \mathrm{Rat}\, A^*$ so that $L_1\sigma \cup L_2\sigma$, $(L_1)\sigma(L_2)\sigma$ and $(L_1^\sigma)*$ are in $\mathrm{Rat}\, A^*$. By Q4, $(L_1 \cup L_2)\sigma = L_1\sigma \cup L_2\sigma$, $(L_1 L_2)\sigma = (L_1)\sigma(L_2)\sigma$ and $L_1^*\sigma = (L_1\sigma)^*$, so $L\sigma \in \mathrm{Rat}\, A^*$. Hence $P(k)$ is true, and so by induction, if $L \in \mathrm{Rat}\, A^*$, then $L \in \mathrm{Rat}\, A^*$.