FUNDAMENTAL REPRESENTATIONS FOR CLASSES OF SEMIGROUPS CONTAINING A BAND OF IDEMPOTENTS

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Abstract. The construction by Hall of a fundamental orthodox semigroup W_B from a band B provides an important tool in the study of orthodox semigroups. Hall's semigroup W_B has the property that a semigroup is fundamental and orthodox with band of idempotents isomorphic to B if and only if it is embeddable as a full subsemigroup into W_B. The aim of this paper is to extend Hall's approach to some classes of non-regular semigroups.

From a band B we construct a semigroup U_B that plays the role of W_B for a class of weakly B-abundant semigroups having a band of idempotents B. The semigroups we consider, in particular U_B, must also satisfy a weak idempotent connected condition. We show that U_B has subsemigroup V_B where V_B satisfies a stronger notion of idempotent connectedness, and is again the canonical semigroup of its kind. In turn, V_B contains W_B as its subsemigroup of regular elements. Thus we have the following inclusions as subsemigroups:

$$W_B \subseteq V_B \subseteq U_B,$$

either of which may be strict, even in the finite case.

The existence of the semigroups U_B and V_B enable us to prove a structure theorem for classes of weakly B-abundant semigroups having band of idempotents B, and satisfying either of our idempotent connected conditions, as spined products of U_B, or V_B, with a weakly $B/\mathcal{D}$-ample semigroup.

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1. Introduction

One of the significant early approaches to the structure theory of regular semigroups was via *fundamental* semigroups, that is, regular semigroups having no non-trivial idempotent separating congruences. Inspired by Munn's approach to inverse semigroups [14], Hall showed that an orthodox semigroup S with band of idempotents B is fundamental if and only if it is isomorphic to a full subsemigroup of W_B. Further, if S is an orthodox semigroup with band of idempotents B, then there exists a homomorphism $\varphi : S \to W_B$ whose kernel is μ, the maximum idempotent separating congruence on S [10] (c.f. [12] Chapter VI). The semigroup W_B is a subsemigroup of $\mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$, where for any partially ordered set X, $\mathcal{OP}(X)$ is the monoid of its order preserving selfmaps, with dual $\mathcal{OP}^*(X)$. A pair of maps $(\alpha, \beta) \in \mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$ lies in W_B if α and β are connected in a specific way via an isomorphism between principal ideals of B. The aim of this paper is to build an analogous theory to Hall's for classes of non-regular semigroups.

We consider *weakly U-abundant* semigroups, where U is a subset of idempotents of a semigroup. Such semigroups, also referred to as U-semiabundant semigroups, arise independently from a number of sources. They appear in the work of de Barros [1], in that of Ehresmann on certain small ordered categories [2] and in the thesis of the first author [3]. A systematic study of such semigroups was initiated by Lawson, who establishes in [13] the connection between Ehresmann's work and weakly E-abundant semigroups, where E is a semilattice.

A semigroup is *weakly U-abundant* if every class of the equivalence relations $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$ (defined in Section 2) contains an idempotent of U. Certainly $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_U$ and $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_U$, with equality if S is regular and $U = E(S)$. We remark that $\widetilde{\mathcal{L}}_U$ ($\widetilde{\mathcal{R}}_U$) need not be right (left) congruences; if they are we say that S satisfies the *congruence condition* (C) (with respect to U). We denote by $\widetilde{\mathcal{H}}_U$ the relation $\widetilde{\mathcal{L}}_U \cap \widetilde{\mathcal{R}}_U$ and say that S is *U-fundamental* if the greatest congruence μ_U contained in $\widetilde{\mathcal{H}}_U$ is the identity ι; it is easy to see that μ_U separates the idempotents of U. We show that for any semigroup S with $U \subseteq E(S)$, S/μ_U is $\overline{U}$-fundamental where $\overline{U}$ is the image of U under the natural morphism associated with μ_U. Moreover, S is weakly U-abundant (with (C)) if and only if S/μ_U is weakly $\overline{U}$-abundant (with (C)). This is where the notion of weakly abundant wins over that of being abundant; if S is abundant then S/μ need not be [3]. If $U = E(S)$ we drop the subscript U from $\widetilde{\mathcal{L}}_U, \widetilde{\mathcal{R}}_U, \widetilde{\mathcal{H}}_U$ and μ_U and refer to *weakly abundant* and *fundamental* semigroups.

In the case of several classes of weakly E-abundant semigroups where E is a semilattice, a theory analogous to that of Munn has been developed in [5], [7] and [9]. What then of classes of weakly B-abundant semigroups where B is a band? To date the furthest progress is a consideration by the first two authors in [3, 4] of a certain class of abundant semigroups having a band B of idempotents. Here $\mathcal{L}^* = \widetilde{\mathcal{L}}$ and $\mathcal{R}^* = \widetilde{\mathcal{R}}$, so that (C) always holds. To guarantee that S/μ is abundant, the extra condition of being *idempotent connected* (IC) is imposed in [3]. This is a condition of a standard type that gives some control over the position of idempotents in products of elements of the semigroup and, in the abundant case, gives rise naturally to isomorphisms between principal ideals of B. It is shown in [3, 4] that every fundamental idempotent connected abundant semigroup with band of idempotents B is a subsemigroup of W_B.

Here we move further away from the regular case and consider a weakly B-abundant semigroup S with (C), where B is a band. In this case we know that S/μ_B is weakly B-abundant with (C). However, to describe the largest fundamental semigroup in the class - and it is worth noting that in these theories this is where the difficulty lies - we content ourselves with imposing an idempotent connectedness condition, for which there are two natural candidates. One, introduced by the first author in his thesis, we again call (IC); the imposition of this condition guarantees the existence of order isomorphisms between certain principal ideals of B 'connected' via an element of S. We also develop the *weak idempotent connected* condition (WIC), that coincides with (IC) for abundant semigroups, but not for wider classes. Condition (WIC) gives us a very loose control over the position of idempotents, but does not impose artificially the existence of order isomorphisms.

From a band B we construct a weakly abundant subsemigroup of $\mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$, satisfying (C) and (WIC), calling this semigroup U_B. The semigroup U_B is fundamental, and is universal in the sense that any B-fundamental weakly B-abundant semigroup with (C) and (WIC) is a subsemigroup of U_B. We show that U_B contains as a full subsemigroup a semigroup V_B, which is fundamental, weakly abundant with (C) and (IC), and is the canonical semigroup of this type. Consequently, V_B contains W_B as a subsemigroup; moreover, W_B consists precisely of the regular elements of V_B. We give examples to show that, in general, $W_B \neq V_B$ and $V_B \neq U_B$.

The structure of the paper is as follows. In Section 2 we give some necesssary preliminaries on weakly U-abundant semigroups, specialising in Section 3 to the case where U is a band. Section 4 sets out the construction of U_B from a band B, and contains a discussion of its properties. In Section 5 we build and investigate the subsemigroup V_B of U_B. Section 6 is concerned with examples; we use our techniques to give examples of semigroups with small finite cardinality that distinguish between the classes under consideration.

In our final section we show how the existence of the semigroups U_B and V_B enable us to prove a structure theorem for weakly B-abundant semigroups with (C) and (WIC) (respectively (IC)), as spined products of U_B (repsectively V_B) with a weakly $B/\mathcal{D}$-ample semigroup. To find the latter we make heavy use of the congruence δ_B (see for example [8]), which is the analogue for weakly B-abundant semigroups of the least inverse congruence on an orthodox semigroup.

2. Preliminaries

For ease of reference we gather together in this section some basic definitions and elementary observations concerning weakly abundant semigroups. Further details may be found in [3] and [13]. For convenience we make the convention that B will always denote a band.

Let S be a semigroup with subset of idempotents U. The relation $\widetilde{\mathcal{L}}_U$ is defined by the rule that for any $a, b \in S$, $a \,\widetilde{\mathcal{L}}_U\, b$ if and only if for all $e \in U$,

$$ae = a \text{ if and only if } be = b.$$

The relation $\widetilde{\mathcal{R}}_U$ is defined dually; clearly $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$ are equivalence relations. We recall from the Introduction that (C) holds (with respect to U) if $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$ are right and left congruences, respectively. It is easy to see that

$$\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_U \text{ and } \mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_U.$$

Moreover for a regular element a such that $xa \in U$ ($ax \in U$) for some $x \in S$, we have that for any $e \in U$,

$$e \,\widetilde{\mathcal{L}}_U\, a \text{ if and only if } e \,\mathcal{L}\, a \;\; (e \,\widetilde{\mathcal{R}}_U\, a \text{ if and only if } e \,\mathcal{R}\, a).$$

It follows that for $e, f \in U$,

$$e \,\widetilde{\mathcal{L}}_U\, f \text{ if and only if } e \,\mathcal{L}\, f \;\; (e \,\widetilde{\mathcal{R}}_U\, f \text{ if and only if } e \,\mathcal{R}\, f)$$

and if S is regular and $U = E(S)$, then $\widetilde{\mathcal{L}}_U = \mathcal{L}$ and $\widetilde{\mathcal{R}}_U = \mathcal{R}$. Another useful observation is that if $a \in S$ and $e \in U$, then $a \,\widetilde{\mathcal{L}}_U\, e$ if and only if $ae = a$ and for any $f \in U$, $af = a$ implies that $ef = e$.

The semigroup S is weakly U-abundant if every $\widetilde{\mathcal{L}}_U$-class and every $\widetilde{\mathcal{R}}_U$-class contain an idempotent. If a is an element of such a semigroup, then we commonly denote idempotents in the $\widetilde{\mathcal{L}}_U$-class and $\widetilde{\mathcal{R}}_U$-class of a by a^* and a^+ respectively. Beware however, that there may not be a unique choice for a^* or a^+. The following lemma is immediate.

Lemma 2.1. *Let S be a weakly U-abundant semigroup. Then for any $a, b \in S$,*

$$(ab)^* \leq_{\mathcal{L}} b^* \text{ and } (ab)^+ \leq_{\mathcal{R}} a^+.$$

A word on notation. In the case when, for a semigroup S, we are considering $U = E(S)$, we commonly drop the 'U' from notation and terminology. For example, $\widetilde{\mathcal{L}}_{E(S)}$ and $\widetilde{\mathcal{R}}_{E(S)}$ are denoted more simply by $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$, and we say that S is *weakly abundant* if it is weakly $E(S)$-abundant. Regular semigroups are clearly weakly abundant but the latter class is much wider. Trivially, a unipotent monoid M (a monoid with one idempotent) is weakly abundant, as is any Rees matrix semigroup $\mathcal{M}^0(M; I, \Lambda; P)$ where each row and column of P contains a unit; indeed these semigroups satisfy (C) [6]. The Ehresmann semigroups of [13] are weakly E-abundant with (C) for a semilattice E. Further examples abound. A number (including some without (C)) are given in [6]; we present new ones arising from our current work at the end of this article.

Morphic images of regular and inverse semigroups are regular and inverse respectively. The same is not true even for abundant semigroups with semilattice of idempotents [5]. With this in mind we make the following definition. Let S be a semigroup with subset of idempotents U and let $\varphi : S \to T$ be a morphism. Then φ is *U-admissible* if for any $a, b \in S$,

$$a \,\widetilde{\mathcal{L}}_U\, b \text{ implies that } a\varphi \,\widetilde{\mathcal{L}}_{U\varphi}\, b\varphi$$

and

$$a \,\widetilde{\mathcal{R}}_U\, b \text{ implies that } a\varphi \,\widetilde{\mathcal{R}}_{U\varphi}\, b\varphi.$$

If, in addition, the reverse implications hold we say that φ is *strongly U-admissible.*

The following lemma is clear.

Lemma 2.2. *Let S be a semigroup, let $U \subseteq E(S)$ and let $\varphi : S \to T$ be a U-admissible surjective morphism. If S is weakly U-abundant, then T is weakly $U\varphi$-abundant.*

Lemma 2.3. *Let S be a semigroup with $U \subseteq E(S)$, and let $\varphi : S \to T$ be a surjective morphism. Then φ is strongly U-admissible if and only* *if the kernel of φ is contained in $\widetilde{\mathcal{H}}_U$. In this case, S is weakly U-abundant if and only if T is weakly $U\varphi$-abundant, and S satisfies (C) with respect to U if and only if T satisfies (C) with respect to $U\varphi$.*

Proof. Suppose that φ is strongly U-admissible and $a\varphi = b\varphi$. Clearly $a\varphi\,\widetilde{\mathcal{H}}_{U\varphi}\,b\varphi$, whence $a\,\widetilde{\mathcal{H}}_U\,b$ by assumption.

Conversely, suppose that ker $\varphi \subseteq \widetilde{\mathcal{H}}_U$; let $a \in S$ and $e \in U$. If $ae = a$ then certainly $a\varphi\, e\varphi = a\varphi$. On the other hand if $a\varphi\, e\varphi = a\varphi$ then $ae\,\widetilde{\mathcal{H}}_U\,a$. Now $ae \cdot e = ae$, so that $a \cdot e = a$ as $ae\,\widetilde{\mathcal{L}}_U\,a$. Similarly, $ea = a$ if and only if $e\varphi a\varphi = a\varphi$. The result now follows easily. □

We can now justify further assertions of the Introduction.

Proposition 2.4. *Let S be a semigroup and let $U \subseteq E(S)$. The natural morphism ν_U associated with μ_U is strongly U-admissible and restricts to an injection on U. Denoting the image of U under ν_U by $\overline{U}$, we have that S/μ_U is $\overline{U}$-fundamental.*

If S is weakly U-abundant, then S/μ_U is weakly $\overline{U}$-abundant; if S satisfies (C), then so does S/μ_U.

Proof. The morphism ν_U is strongly U-admissible by Lemma 2.3; consequently, by Lemmas 2.2 and 2.3, S/μ_U is weakly $\overline{U}$-abundant if S is weakly U-abundant, and inherits (C) from S. If two idempotents of U are related by μ_U, then they are $\widetilde{\mathcal{H}}_U$-related, and so, from remarks at the beginning of this section, they are $\mathcal{H}$-related and hence equal. Thus μ_U separates idempotents of U.

It remains to show that S/μ_U is $\overline{U}$-fundamental. Suppose that $a\mu_U\,\mu_{\overline{U}}\,b\mu_U$. Since $\mu_{\overline{U}}$ is the largest congruence contained in $\widetilde{\mathcal{H}}_{\overline{U}}$, we have that $a\mu_U\,\widetilde{\mathcal{H}}_{\overline{U}}\,b\mu_U$ and for any $c\mu_U, d\mu_U \in S/\mu_U$,

$$c\mu_U a\mu_U\,\widetilde{\mathcal{H}}_{\overline{U}}\,c\mu_U b\mu_U,\ \ a\mu_U c\mu_U\,\widetilde{\mathcal{H}}_{\overline{U}}\,b\mu_U c\mu_U,$$

and

$$c\mu_U a\mu_U d\mu_U\,\widetilde{\mathcal{H}}_{\overline{U}}\,c\mu_U b\mu_U d\mu_U.$$

By Lemma 2.3,

$$a\,\widetilde{\mathcal{H}}_U\,b,\ \ ca\,\widetilde{\mathcal{H}}_U\,cb,\ \ ac\,\widetilde{\mathcal{H}}_U\,bc \text{ and } cad\,\widetilde{\mathcal{H}}_U\,cbd.$$

From Proposition I.5.13 of [12], $a\,\mu_U\,b$ so that $a\mu_U = b\mu_U$, as required. □

Example 2.5.

Let B be a rectangular band and let S be weakly B-abundant. It is easy to see that for any $a, b \in S$,

$$a\,\widetilde{\mathcal{R}}_B\,ab\,\widetilde{\mathcal{L}}_B\,b,$$

whence $\widetilde{\mathcal{L}}_B, \widetilde{\mathcal{R}}_B$ and $\widetilde{\mathcal{H}}_B$ are all congruences. Moreover, every $\widetilde{\mathcal{H}}_B$-class contains an idempotent. Thus $S/\mu_B = S/\widetilde{\mathcal{H}}_B = \overline{B}$. We deduce that in this special case the only B-fundamental weakly B-abundant semigroup is the band B.

In the case of a weakly U-abundant semigroup with (C) the congruence μ_U has a description neater than the generic one used in Proposition 2.4. The proof of the following is very similar to that in the abundant case [4], and is therefore omitted.

Lemma 2.6. [3] *Let S be weakly U-abundant with (C). Then for any $a, b \in S$,*

$$a\,\mu_U\,b \textit{ if and only if } ea\,\widetilde{\mathcal{L}}_U\,eb \textit{ and } ae\,\widetilde{\mathcal{R}}_U\,be$$

for all $e \in U$.

Let T be a subsemigroup of S and let U be a subset of idempotents of S. We say that T is *U-full* if $U \subseteq T$. The last part of the final lemma of this section employs the description of μ taken from Lemma 2.6.

Lemma 2.7. *Let T be a U-full subsemigroup of S. Then for any $a, b \in T$,*

$$a\,\widetilde{\mathcal{L}}_U\,b \textit{ in } T \textit{ if and only if } a\,\widetilde{\mathcal{L}}_U\,b \textit{ in } S$$

and

$$a\,\widetilde{\mathcal{R}}_U\,b \textit{ in } T \textit{ if and only if } a\,\widetilde{\mathcal{R}}_U\,b \textit{ in } S.$$

Consequently, if S is weakly U-abundant, then so is T; if S satisfies (C) with respect to U, then so does T.

If S is U-fundamental weakly U-abundant with (C), then so is T.

3. A band of idempotents

The remainder of this paper concentrates on weakly B-abundant semigroups with (C), where, by our convention, B is always a band. In this case we can substantially improve upon Proposition 2.4, as we show below. The idempotent connected condition is also defined and discussed in this section.

Let S be a weakly B-abundant semigroup. For any $a \in S$ we define

$$\alpha_a : B/\mathcal{L} \to B/\mathcal{L} \text{ and } \beta_a : B/\mathcal{R} \to B/\mathcal{R}$$

by

$$L_x\alpha_a = L_{(xa)^*} \text{ and } R_x\beta_a = R_{(ax)^+}.$$

It follows from Lemma 2.1 that α_a and β_a are well defined. We note that for any $e \in B$,

$$(\alpha_e, \beta_e) = (\rho_e, \lambda_e)$$

where for any $x \in B$,

$$L_x\rho_e = L_{xe}, \;\; R_x\lambda_e = R_{ex}.$$

The band B admits the quasi-orders $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ associated with $\mathcal{L}$ and $\mathcal{R}$; we consider $B/\mathcal{L}$ and $B/\mathcal{R}$ as partially ordered sets under the induced orderings.

Lemma 3.1. *Let S be a weakly B-abundant semigroup. For any $a \in S$, $\alpha_a \in \mathcal{OP}(B/\mathcal{L})$ and $\beta_a \in \mathcal{OP}^*(B/\mathcal{R})$.*

Let

$$\theta : S \to \mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$$

be given by

$$a\theta = (\alpha_a, \beta_a).$$

If condition (C) holds, then θ is a strongly B-admissible morphism with kernel μ_B. Moreover, putting $\overline{B} = \{(\rho_e, \lambda_e) : e \in B\}$, we have that $\theta|_B : B \to \overline{B}$ is an isomorphism.

Proof. To justify the first assertion, notice that if $e, f \in B$ and $L_e \leq L_f$, then $e \leq_{\mathcal{L}} f$ in B and hence in S. Since $\leq_{\mathcal{L}}$ is right compatible, $ea \leq_{\mathcal{L}} fa$ in S so that as $fa(fa)^* = fa$, we also have $ea(fa)^* = ea$ and hence $(ea)^*(fa)^* = (ea)^*$. Thus $(ea)^* \leq_{\mathcal{L}} (fa)^*$ in B and so $L_e\alpha_a \leq L_f\alpha_a$. The argument that β_a is order preserving is dual.

Suppose now that (C) holds. For any $a, b \in S$, and $e \in B$,

$$(eab)^* \,\widetilde{\mathcal{L}}_B\, eab \,\widetilde{\mathcal{L}}_B (ea)^*b \,\widetilde{\mathcal{L}}_B\, ((ea)^*b)^*,$$

so that in B,

$$L_{(eab)^*} = L_{((ea)^*b)^*}$$

and consequently,

$$L_e\alpha_{ab} = L_e\alpha_a\alpha_b.$$

We have shown that $\alpha_{ab} = \alpha_a\alpha_b$; the dual argument gives that $\beta_{ab} = \beta_b\beta_a$, whence it follows easily that θ is a morphism.

To see that the kernel of θ is μ_B, notice first that if $a\theta = b\theta$, then $(\alpha_a, \beta_a) = (\alpha_b, \beta_b)$ so that in particular,

$$L_{a^+}\alpha_a = L_{a^+}\alpha_b \text{ and } L_{b^+}\alpha_a = L_{b^+}\alpha_b.$$

Thus

$$L_{a^*} = L_{(a^+b)^*} \text{ and } L_{(b^+a)^*} = L_{b^*}.$$

It follows from Lemma 2.1 that $a\,\widetilde{\mathcal{L}}_B\,b$ and dually, $a\,\widetilde{\mathcal{R}}_B\,b$. Hence the kernel of θ is contained in $\widetilde{\mathcal{H}}_B$ and therefore also in μ_B.

Conversely, if $a\,\mu_B\,b$, then for any $e \in B$, Lemma 2.6 gives that $ea\,\widetilde{\mathcal{L}}_B\,eb$ and so

$$L_e\alpha_a = L_{(ea)^*} = L_{(eb)^*} = L_e\alpha_b.$$

Thus $\alpha_a = \alpha_b$ and dually, $\beta_a = \beta_b$. We deduce that $a\theta = b\theta$ and hence the kernel of θ is μ_B. From Lemma 2.3, θ is therefore strongly B-admissible.

We remarked above that $e\theta = (\rho_e, \lambda_e)$, for any $e \in B$, and hence $\theta|_B : B \to \overline{B}$ is a surjective morphism. Now the kernel of θ is μ_B, and so θ separates the idempotents of B, giving that $\theta|_B : B \to \overline{B}$ is an isomorphism. □

It remains in this section to discuss the idempotent connected condition. A fuller version of some of the ideas we present here is contained in [15]. Essentially, all of the idempotent connected and ample (formerly, type A) conditions extant give some control over the position of idempotents in products, usually facilitating results for abundant or weakly abundant semigroups reminiscent of those in the regular case.

For a band B and element e of B we denote by $\langle e\rangle$ the principal order ideal generated by e; so that

$$\langle e\rangle = \{x \in B : x \leq e\} = \{x \in B : ex = xe = x\}.$$

Clearly $\langle e\rangle$ is a subsemigroup with identity e. Let S be a weakly B-abundant semigroup where B is a band. We say that S satisfies the *weak idempotent connected* condition (WIC) (with respect to B) if for any $a \in S$ and some a^*, a^+, if $x \in \langle a^+\rangle$ then there exists $y \in B$ with $xa = ay$; and dually, if $z \in \langle a^*\rangle$ then there exists $t \in B$ with $ta = az$.

Some observations concerning this definition are in order. First, it is easy to see that a regular semigroup satisfies (WIC) with respect to $E(S)$. Second, we can replace 'some' in (WIC) by 'any'. For suppose that S has (WIC), $a \in S$, a^+ is the chosen idempotent of B in the $\widetilde{\mathcal{R}}_B$-class of a, and $a^\dagger$ is another element of B in the same $\widetilde{\mathcal{R}}_B$-class. If $x \in \langle a^\dagger\rangle$, we certainly have that $xa^+ = a^+xa^+ \in \langle a^+\rangle$ and so by (WIC),

$$xa = (xa^+)a = ay$$

for some $y \in B$. Similarly, we can take z to lie in $\langle a^\circ\rangle$ for any $a^\circ \in B$ lying in the $\widetilde{\mathcal{L}}_B$-class of a. Finally, if $a \in S$, and $x, y \in B$ with $xa = ay$, then for any a^* we have that $xa = a(a^*ya^*)$. Thus in the definition of (WIC) we may choose the y to lie in any given $\langle a^*\rangle$, and dually, the t to lie in any given $\langle a^+\rangle$.

We now introduce certain relations which will be crucial in later constructions. Let S be weakly B-abundant, let $a \in S$ and choose a^+ and a^*. It is easy to see that

$$I^{a^+,a^*} = \{(x,y) \in \langle a^+ \rangle \times \langle a^* \rangle : xa = ay\}$$

is a subsemigroup of $\langle a^+ \rangle \times \langle a^* \rangle$. Moreover, S satisfies (WIC) if and only if every such I^{a^+,a^*} is a full relation according to the following definition.

Definition Let A, B be sets and $R \subseteq A \times B$ be a relation. Then R is *full* if both projection maps are both onto.

If S is abundant, so that $B = E(S)$ and $\widetilde{\mathcal{L}} = \mathcal{L}^*$, $\widetilde{\mathcal{R}} = \mathcal{R}^*$, then it is easy to see that if I^{a^+,a^*} is full, then it is the graph of an isomorphism. Thus an abundant semigroup satisfies (WIC) if and only if it satisfies the idempotent connected condition (IC) introduced by the first author in [3]. Consequently, an orthodox semigroup always satisfies (IC).

Motivated by the abundant case, El-Qallali in [3] extended the notion of idempotent connectedness from abundant semigroups to weakly B-abundant semigroups, again calling his condition (IC). In our notation, a weakly B-abundant semigroup satisfies (IC) if for each $a \in S$ there exist a^+, a^* such that the relation I^{a^+,a^*} *contains the graph of* an order isomorphism from $\langle a^+ \rangle$ to $\langle a^* \rangle$. We expand upon this in Section 5 and show in Section 6 that a weakly B-abundant semigroup can have (WIC) without (IC).

The following lemma is an easy extension of Lemma 2.7.

Lemma 3.2. *Let T be a B-full subsemigroup of a weakly B-abundant semigroup S. If S satisfies (WIC), then so does T; if S satisfies (IC), then so does T.*

We end this section by showing that (WIC) and (IC) are respected by strongly admissible morphisms.

Lemma 3.3. *If S is a weakly B-abundant semigroup and $\theta : S \to T$ is a strongly admissible morphism from S onto a semigroup T, then S has (WIC) with respect to B if and only if T has (WIC) with respect to $B\theta$; similarly for (IC).*

Proof. As in Lemma 2.3 we can show that for any $x, y \in B$ and $a \in S$, $xa = ay$ if and only if $x\theta\, a\theta = a\theta\, y\theta$. We have that $x\,\widetilde{\mathcal{R}}_B\, a\,\widetilde{\mathcal{L}}_B\, y$ if and only if $x\theta\,\widetilde{\mathcal{R}}_{B\theta}\, a\theta\,\widetilde{\mathcal{L}}_{B\theta}\, y\theta$, and θ induces an isomorphism from B to $B\theta$. The result follows.

□

4. The semigroup U_B

Our aim in this section is to construct from B a semigroup U_B that is B-fundamental weakly B-abundant with (C) and (WIC), containing as a B-full subsemigroup any semigroup with these properties. Consequently, the semigroup W_B of [10], that is, the canonical fundamental orthodox semigroup, is embeddable into U_B. In our final section we give examples to show that this embedding may be proper. Underlying the construction of W_B is the idea of a 'connecting isomorphism' between principal ideals of B; that concept is too strong for our purposes. With this in mind we introduce certain relations between principal ideals of B.

Let $e, f \in B$; we commonly denote a relation from $\langle e \rangle$ to $\langle f \rangle$, that is, a subset of $\langle e \rangle \times \langle f \rangle$, by $I^{e,f}$. We say that $I^{e,f}$ is *connecting* if $I^{e,f}$ is a subsemigroup of $\langle e \rangle \times \langle f \rangle$ and for every $(x, x'), (y, y') \in I^{e,f}$ we have that

$$x \leq_{\mathcal{L}} y \text{ implies that } x' \leq_{\mathcal{L}} y'$$

and

$$x' \leq_{\mathcal{R}} y' \text{ implies that } x \leq_{\mathcal{R}} y.$$

Lemma 4.1. *Let $I^{e,f}$ be connecting. Then for any $(x, y), (z, t) \in I^{e,f}$,*

$$x \leq_{\mathcal{D}} z \textit{ if and only if } y \leq_{\mathcal{D}} t.$$

Proof. If $x \leq_{\mathcal{D}} z$, then

$$xzx = x(xzx)x = x$$

so that $x \,\mathcal{L}\, zx$. As $I^{e,f}$ is a semigroup, $(zx, ty) \in I^{e,f}$, so that as also $(x, y) \in I^{e,f}$, we have that $y \,\mathcal{L}\, ty$. Consequently, $y \leq_{\mathcal{D}} t$. The proof of the remainder of the lemma is dual. □

Connecting relations are of immediate importance to us due to the following observation. Let S be weakly B-abundant with (C), let $a \in S$, and let I^{a^+,a^*} be the relation defined in Section 3.

Lemma 4.2. *The relation I^{a^+,a^*} is connecting.*

Proof. First, we have already observed that I^{a^+,a^*} is a subsemigroup of $\langle a^+ \rangle \times \langle a^* \rangle$. Suppose now that $(x, x'), (y, y') \in I^{a^+,a^*}$ and $x \leq_{\mathcal{L}} y$. Then

$$xa = ax', \; ya = ay'$$

and

$$x' = a^*x' \,\widetilde{\mathcal{L}}_B\, ax' = xa \leq_{\mathcal{L}} ya = ay' \,\widetilde{\mathcal{L}}_B\, a^*y' = y'$$

whence $x'y' = x'$ and so $x' \leq_{\mathcal{L}} y'$. Dually, I^{a^+,a^*} preserves the $\leq_{\mathcal{R}}$-order from right to left, and is therefore connecting.

□

Clearly a weakly B-abundant semigroup with (C) has (WIC) if and only if all the connecting relations I^{a^+,a^*} are full.

Observe that, as a consequence of the definitions, if $I^{e,f} \subseteq \langle e\rangle \times \langle f\rangle$ is full and connecting, then

$$(e,x) \in I^{e,f} \text{ if and only if } x = f$$

and dually,

$$(x,f) \in I^{e,f} \text{ if and only if } x = e.$$

For we know that there exist $(e,u),(v,f) \in I^{e,f}$, and so, as $I^{e,f}$ is a semigroup,

$$(ev,uf) = (v,u) \in I^{e,f}.$$

Since $(v,f),(v,u) \in I^{e,f}$ and certainly $v\,\mathcal{L}\,v$, we have that $f\,\mathcal{L}\,u$; as $u \leq f$ we obtain that $f = u$. Similarly, $v = e$.

We denote $\mathcal{OP}(B/\mathcal{L}) \times \mathcal{OP}^*(B/\mathcal{R})$ by $\mathcal{O}(B)$ and use full connecting relations to define the elements of a subsemigroup of $\mathcal{O}(B)$. Let $I^{e,f}$ be full connecting; we begin by defining partial maps $I^{e,f}_\ell$ of $B/\mathcal{L}$ and $I^{e,f}_r$ of $B/\mathcal{R}$ by setting

$$L_x I^{e,f}_\ell = L_y \text{ where } (x,y) \in I^{e,f}$$

and

$$R_y I^{e,f}_r = R_x \text{ for } (x,y) \in I^{e,f}.$$

The fact that $I^{e,f}$ is full connecting gives immediately that $I^{e,f}_\ell$ and $I^{e,f}_r$ have domains $\{L_x : x \leq e\}$ and $\{R_y : y \leq f\}$ respectively, and that they are well defined and order preserving on these domains. Consider now the element $\rho_e \in \mathcal{O}(B/\mathcal{L})$; the image of ρ_e is $\{L_{ye} : y \in B\}$. Since $eye\,\mathcal{L}\,ye$, we have that the image of ρ_e is $\{L_x : x \leq e\}$, that is, the image of ρ_e is the domain of $I^{e,f}_\ell$. Thus we may compose the order preserving maps ρ_e and $I^{e,f}_\ell$ to obtain an element of $\mathcal{O}(B/\mathcal{L})$. Similarly, $\lambda_f I^{e,f}_r \in \mathcal{O}^*(B/\mathcal{R})$. We have shown that

$$U_B = \{(\rho_e I^{e,f}_\ell, \lambda_f I^{e,f}_r) : e,f \in B, I^{e,f} \subseteq \langle e\rangle \times \langle f\rangle \text{ is full connecting}\}$$

is a subset of $\mathcal{O}(B)$. We claim that U_B is a subsemigroup of $\mathcal{O}(B)$ and is the canonical B-fundamental weakly B-abundant semigroup with (C) and (WIC) for which we seek. Indeed rather more than this, for we show that the idempotents of U_B are *precisely* the elements of a band isomorphic to B.

Notice that for any $e \in B$,

$$\iota^{e,e} = \{(x,x) : x \leq e\}$$

is full connecting, and

$$(\rho_e \iota_\ell^{e,e}, \lambda_e \iota_r^{e,e}) = (\rho_e, \lambda_e),$$

so that $\overline{B} \subseteq U_B$. We show below that every idempotent of U_B belongs to $\overline{B}$. In the following, for $\mathcal{D}$-related elements e, f of B we use the notation θ_f to denote the map from $\langle e \rangle$ to $\langle f \rangle$ given by $x\theta_f = fxf$; from VI.2.13 of [12], θ_f is an isomorphism with inverse θ_e.

Lemma 4.3. *The set U_B is a subsemigroup of $\mathcal{O}(B)$ with $E(U_B) = \overline{B}$.*

Proof. For any $f, g \in B$, $fgf \,\mathcal{D}\, gfg$ so that

$$\theta_{fgf} : \langle gfg \rangle \to \langle fgf \rangle \text{ and } \theta_{gfg} : \langle fgf \rangle \to \langle gfg \rangle$$

are mutually inverse isomorphisms. As such, therefore, they preserve the order of B. Moreover,

$$x\theta_{fgf} = (fgf)x(fgf) = fxf$$

for $x \in \langle gfg \rangle$ and

$$y\theta_{gfg} = (gfg)y(gfg) = gyg$$

for $y \in \langle fgf \rangle$.

Suppose now that $e, f, g, h \in B$ and $I^{e,f}, J^{g,h}$ are full connecting relations. Since

$$fgf \leq f \text{ and } gfg \leq g$$

and $I^{e,f}, J^{g,h}$ are full connecting, there exist

$$(z, fgf) \in I^{e,f} \text{ and } (gfg, w) \in J^{g,h}.$$

We claim that $K^{z,w}$ is full connecting, where

$$K^{z,w} = (I^{e,f}\theta_{gfg}J^{g,h}) \cap (\langle z \rangle \times \langle w \rangle),$$

the composition being composition of relations from B to B.

To show that the projection maps to $\langle z \rangle$ and $\langle w \rangle$ are onto, let $u \in B$ with $u \leq z$; since $z \leq e$ and $I^{e,f}$ is full connecting, there exists an element $(u, t) \in I^{e,f}$. Now $u = zuz$ and $I^{e,f}$ is a semigroup, so that

$$(u, fgftfgf) = (z, fgf)(u, t)(z, fgf) \in I^{e,f}.$$

Clearly $fgftfgf \in \langle fgf \rangle$, so that

$$(fgftfgf, g(fgftfgf)g) \in \theta_{gfg},$$

that is,

$$(fgftfgf, (gfg)gftfg(gfg)) \in \theta_{gfg}.$$

Now $gftfg \in \langle g \rangle$; as $J^{g,h}$ is full connecting, there exists an element $(gftfg, k) \in J^{g,h}$. Consequently,

$$(gfg, w)(gftfg, k)(gfg, w) = ((gfg)gftfg(gfg), wkw) \in J^{g,h}.$$

It follows that

$$(u, wkw) \in K^{z,w}.$$

Dually, the projection of $K^{z,w}$ to the second coordinate is onto.

Since each of $I^{e,f}, \theta_{gfg}, J^{g,h}$ is a subsemigroup of $B \times B$, it follows easily that the same is true of the composition, hence of $K^{z,w}$. Finally, since each of the relations concerned preserves the $\leq_{\mathcal{L}}$-order ($\leq_{\mathcal{R}}$-order) from left to right (right to left), the same is clearly true of the composition. Thus $K^{z,w}$ is full connecting.

Consider the elements $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}), (\rho_g J_\ell^{g,h}, \lambda_h J_r^{g,h}) \in U_B$ and let $K^{z,w}$ be constructed as above. We claim that

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})(\rho_g J_\ell^{g,h}, \lambda_h J_r^{g,h}) = (\rho_z K_\ell^{z,w}, \lambda_w K_r^{z,w}).$$

To see this, let $x \in B$. A straightforward calculation gives

$$\begin{array}{rcll} L_x \rho_e I_\ell^{e,f} \rho_g J_\ell^{g,h} & = & L_{exe} I_\ell^{e,f} \rho_g J_\ell^{g,h} & \\ & = & L_u \rho_g J_\ell^{g,h} & \text{where } (exe, u) \in I^{e,f} \\ & = & L_{gug} J_\ell^{g,h} & \\ & = & L_v & \text{where } (gug, v) \in J^{g,h}. \end{array}$$

On the other hand,

$$L_x \rho_z K_\ell^{z,w} = L_{zxz} K_\ell^{z,w}$$

and

$$(zxz, (fgf)u(fgf)) = (z(exe)z, (fgf)u(fgf)) = (z, fgf)(exe, u)(z, fgf) \in I^{e,f}.$$

Hence

$$(zxz, (gfg)gug(gfg)) = (zxz, g(fgfufgf)g) \in I^{e,f}\theta_{gfg},$$

since $u = fuf$. Also,

$$((gfg)gug(gfg), wvw) = (gfg, w)(gug, v)(gfg, w) \in J^{g,h}$$

and so we conclude

$$(zxz, wvw) \in I^{e,f}\theta_{gfg}J^{g,h}$$

and hence $(zxz, wvw) \in K^{z,w}$, giving that

$$L_x \rho_z K_\ell^{z,w} = L_{wvw}.$$

Further,

$$(gfg)gug(gfg) = (gfg)(fuf)(gfg) = (gf)^2 u (fg)^2 = (gf)u(fg) = gug,$$

so that as $J^{g,h}$ is full connecting and (gug, v), $(gug, wvw) \in J^{g,h}$, we must have that $L_v = L_{wvw}$ and it follows that

$$\rho_z K_\ell^{z,w} = \rho_e I_\ell^{e,f} \rho_g J_\ell^{g,h}.$$

Dually, we obtain that

$$\lambda_w K_r^{z,w} = \lambda_h J_r^{g,h} \lambda_f I_r^{e,f},$$

so that

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})(\rho_g J_\ell^{g,h}, \lambda_h J_r^{g,h}) = (\rho_z K_\ell^{z,w}, \lambda_w K_r^{z,w}),$$

allowing us to deduce that U_B is a subsemigroup of $\mathcal{O}_B$.

We now identify the idempotents of U_B. We have remarked that $\overline{B} \subseteq U_B$ and $\overline{B}$ forms a band; it remains to show that *every* idempotent of U_B lies in $\overline{B}$. To this end, suppose that $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})$ is idempotent. Notice that the image of $\rho_e I_\ell^{e,f}$ is $\{L_x : x \leq f\}$ and as $\rho_e I_\ell^{e,f}$ is idempotent, we must have that $\rho_e I^{e,f}$ is the identity on this set. Similarly, $\lambda_f I_r^{e,f}$ is the identity on $\{R_y : y \leq e\}$. This gives in particular that

$$L_f = L_f \rho_e I_\ell^{e,f} = L_{efe} I_\ell^{e,f} = L_g$$

where $(efe, g) \in I^{e,f}$. Since also $(e,f) \in I^{e,f}$, and $f \,\mathcal{L}\, g$, Lemma 4.1 gives that $efe \,\mathcal{D}\, e$. Dually, $fef \,\mathcal{D}\, f$ and we deduce that $e \,\mathcal{D}\, f$.

Consequently, for any $x \in B$,

$$L_x \rho_e I_\ell^{e,f} = L_{exe} I_\ell^{e,f} = L_{efexefe} I_\ell^{e,f} = L_{fexef} \rho_e I_\ell^{e,f} = L_{fexef}$$

since L_{fexef} is in the image of $\rho_e I_\ell^{e,f}$. But

$$L_{fexef} = L_{(fex)(xef)} = L_{xef} = L_x \rho_{ef}.$$

Dually, $\lambda_f I_\ell^{e,f} = \lambda_{ef}$ and so

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) = (\rho_{ef}, \lambda_{ef}) \in \overline{B}$$

as required.

□

Theorem 4.4. *The semigroup U_B is fundamental, weakly abundant with (C) and (WIC).*

Proof. We begin by showing that, for any $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})$, we have

$$(\rho_f, \lambda_f) \,\widetilde{\mathcal{L}}\, (\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \,\widetilde{\mathcal{R}}\, (\rho_e, \lambda_e).$$

First,

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})(\rho_f, \lambda_f) = (\rho_e I_\ell^{e,f} \rho_f, \lambda_f \lambda_f I_r^{e,f}).$$

Clearly the second coordinate is $\lambda_f I_\ell^{e,f}$. Considering the first coordinate, we have that for any $x \in B$,

$$L_x \rho_e I_\ell^{e,f} \rho_f = L_{exe} I_\ell^{e,f} \rho_f = L_u \rho_f,$$

where $(exe, u) \in I^{e,f}$. By definition of $I^{e,f}$, we have that $u \leq f$ and so

$$L_u \rho_f = L_{uf} = L_u = L_x \rho_e I_\ell^{e,f}.$$

Thus

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})(\rho_f, \lambda_f) = (\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}).$$

On the other hand, suppose that $g \in B$ and

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})(\rho_g, \lambda_g) = (\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}).$$

We then have that

$$L_e \rho_e I_\ell^{e,f} \rho_g = L_e \rho_e I_\ell^{e,f}$$

and so, in view of the comments following the definition of full connecting relation,

$$L_{fg} = L_f \rho_g = L_f.$$

Hence $f \leq_{\mathcal{L}} g$ in B so that $(\rho_f, \lambda_f) \leq_{\mathcal{L}} (\rho_g, \lambda_g)$ in $\overline{B}$. Consequently,

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \,\widetilde{\mathcal{L}}\, (\rho_f, \lambda_f).$$

It follows that for $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}), (\rho_x M_\ell^{x,y}, \lambda_y M_r^{x,y})$ in U_B,

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \,\widetilde{\mathcal{L}}\, (\rho_x M_\ell^{x,y}, \lambda_y M_r^{x,y}) \text{ if and only if } f \,\mathcal{L}\, y \text{ in } B.$$

We now show that $\widetilde{\mathcal{L}}$ is a left congruence. Suppose that $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})$ and $(\rho_x M_\ell^{x,y}, \lambda_y M_r^{x,y})$ are $\widetilde{\mathcal{L}}$-related elements of U_B, and that $(\rho_g J_\ell^{g,h}, \lambda_h J_r^{g,h})$ is a further element of U_B. Then

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})(\rho_g J_\ell^{g,h}, \lambda_h J_r^{g,h}) = (\rho_z K_\ell^{z,w}, \lambda_w K_r^{z,w})$$

and

$$(\rho_x M_\ell^{x,y}, \lambda_y M_r^{x,y})(\rho_g J_\ell^{g,h}, \lambda_h J_r^{g,h}) = (\rho_{z'} K\prime_\ell^{z',w'}, \lambda_{w'} K\prime_r^{z',w'})$$

where

$$(z, fgf) \in I^{e,f}, (gfg, w) \in J^{g,h}, (z', ygy) \in M^{x,y} \text{ and } (gyg, w') \in J^{g,h},$$

the relations and $K^{z,w}$ and $K\prime^{z',w'}$ being constructed as in Lemma 4.3. Since B is a band we have that

$$gfg \,\mathcal{L}\, fg \,\mathcal{L}\, yg \,\mathcal{L}\, gyg$$

so that as $J^{g,h}$ is full connecting, $w \,\mathcal{L}\, w'$, giving that

$$(\rho_z K_\ell^{z,w}, \lambda_w K_r^{z,w}) \,\widetilde{\mathcal{L}}\, (\rho_{z'} K\prime_\ell^{z',w'}, \lambda_{w'} K\prime_r^{z',w'})$$

and $\widetilde{\mathcal{L}}$ is a right congruence as required.

An argument that is completely dual gives that

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \,\widetilde{\mathcal{R}}\, (\rho_e, \lambda_e)$$

for any $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \in U_B$, and that $\widetilde{\mathcal{R}}$ is a left congruence.

To show that (WIC) holds, let $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \in U_B$ and choose (ρ_g, λ_g) with $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \,\widetilde{\mathcal{R}}\, (\rho_g, \lambda_g)$, so that $e \,\mathcal{R}\, g$ in B. Suppose that $x \in B$ and $(\rho_x, \lambda_x) \leq (\rho_g, \lambda_g)$, so that $x \leq g \,\mathcal{R}\, e$ in B. Now $x \,\mathcal{R}\, xe \leq e$ so there exists $(xe, t) \in I^{e,f}$. We claim that

$$(\rho_x, \lambda_x)(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) = (\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})(\rho_t, \lambda_t),$$

that is,

$$(\rho_{xe} I_\ell^{e,f}, \lambda_f I_r^{e,f} \lambda_x) = (\rho_e I_\ell^{e,f} \rho_t, \lambda_t I_r^{e,f}).$$

We have that for any $y \in B$,

$$L_y \rho_{xe} I_\ell^{e,f} = L_{xeyxe} I_\ell^{e,f} = L_{xe(eye)xe} I_\ell^{e,f} = L_{tut}$$

where $(eye, u) \in I^{e,f}$. On the other hand,

$$L_y \rho_e I_\ell^{e,f} \rho_t = L_{eye} I_\ell^{e,f} \rho_t = L_u \rho_t = L_{tut}.$$

Considering the second coordinate, for any $z \in B$,

$$R_z \lambda_f I_r^{e,f} \lambda_x = R_{fzf} I_r^{e,f} \lambda_x = R_v \lambda_x = R_{xvx}$$

where $(v, fzf) \in I^{e,f}$. Now

$$R_z \lambda_t I_r^{e,f} = R_{tzt} I_r^{e,f} = R_{tfzft} I_r^{e,f} = R_{xevxe}$$

since $(xe, t), (v, fzf) \in I_r^{e,f}$. But

$$xevxe \,\mathcal{R}\, xev = xv \,\mathcal{R}\, xvx.$$

We have established that

$$(\rho_x, \lambda_x)(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) = (\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})(\rho_t, \lambda_t).$$

The dual argument completes the verification that (WIC) holds.

Finally we must argue that U_B is fundamental. To this end suppose that $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}), (\rho_g J_\ell^{g,h}, \lambda_h J_r^{g,h}) \in U_B$ and

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \,\mu_{\overline{B}}\, (\rho_g J_\ell^{g,h}, \lambda_h J_r^{g,h}).$$

Then for any $b \in B$,

$$(\rho_b, \lambda_b)(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \,\widetilde{\mathcal{H}}_{\overline{B}}\, (\rho_b, \lambda_b)(\rho_g J_\ell^{g,h}, \lambda_h J_r^{g,h}).$$

Our formula for composition, together with the fact that $(\rho_b, \lambda_b) = (\rho_b \iota^{b,b}, \lambda_b \iota^{b,b})$, gives that

$$(\rho_z K_\ell^{z,w}, \lambda_w K_r^{z,w}) \,\widetilde{\mathcal{H}}_{\overline{B}}\, (\rho_{z'} M^{z',w'}, \lambda_{w'} M_r^{z',w'})$$

where

$$z = beb, (ebe, w) \in I^{e,f}, z' = bgb \text{ and } (gbg, w') \in J^{g,h}$$

and where $K^{z,w}, M^{z',w'}$ are full connecting relations. This gives in particular that $w \,\mathcal{L}\, w'$. Now

$$L_b \rho_e I_\ell^{e,f} = L_{ebe} I_\ell^{e,f} = L_w = L_{w'} = L_{gbg} J_\ell^{g,h} = L_b \rho_g J_\ell^{g,h}.$$

Thus $\rho_e I_\ell^{e,f} = \rho_g J_\ell^{g,h}$ and dually, $\lambda_f I_r^{e,f} = \lambda_h J_r^{g,h}$. We conclude that U_B is fundamental.

□

Finally in this section we prove that U_B contains a copy of every $\overline{B}$-fundamental, weakly $\overline{B}$-abundant semigroup having (C) and (WIC).

Theorem 4.5. *Let S be a weakly B-abundant semigroup with (C) and (WIC). The map $\theta : S \to U_B$ given by*

$$a\theta = (\alpha_a, \beta_a)$$

where for all $x \in B$, $L_x\alpha_a = L_{(xa)^*}$ and $R_x\beta_a = R_{(ax)^+}$, is a strongly B-admissible morphism with kernel μ_B. Moreover, $\theta|_B : B \to \overline{B}$ is an isomorphism.

Proof. In view of Lemma 3.1, it remains only to show that the image of θ is contained in U_B.

Let $a \in S$ and choose $a^+, a^* \in B$ with $a^* \,\widetilde{\mathcal{L}}_B\, a \,\widetilde{\mathcal{R}}_B\, a^+$. From Lemma 4.2 we have that I^{a^+,a^*} is connecting and is full since S has (WIC). We claim that

$$a\theta = (\alpha_a, \beta_a) = (\rho_{a^+} I_\ell^{a^+,a^*}, \lambda_{a^*} I_r^{a^+,a^*}).$$

To see this, take any $x \in B$. Then

$$L_x \rho_{a^+} I_\ell^{a^+,a^*} = L_{a^+xa^+} I_\ell^{a^+,a^*} = L_y$$

where $(a^+xa^+, y) \in I^{a^+,a^*}$, that is, $y \leq a^*$ and $a^+xa^+a = ay$. Now

$$y = a^*y \,\widetilde{\mathcal{L}}_B\, ay = a^+xa^+a \,\widetilde{\mathcal{L}}_B\, xa^+a = xa \,\widetilde{\mathcal{L}}_B\, (xa)^*,$$

giving that

$$L_x\alpha_a = L_{(xa)^*} = L_y = L_x \rho_{a^+} I_\ell^{a+,a^*}$$

and hence $\alpha_a = \rho_{a^+} I_\ell^{a^+,a^*}$. Dually, $\beta_a = \lambda_{a^*} I_r^{a^+,a^*}$ so that $a\theta \in U_B$ as required.

□

The following corollary is immediate.

Corollary 4.6. *If S is a weakly B-abundant semigroup with (C) and (WIC), then any idempotent of S is $\widetilde{\mathcal{H}}_B$-related to an idempotent of B. In particular, if S is, in addition, B-fundamental, we have that $B = E(S)$.*

5. The semigroup V_B

The aim of this section is to construct a full subsemigroup V_B of U_B that satisfies the stronger version of (WIC), namely the *idempotent connected* condition (IC) as introduced by El Qallali in [3]. It follows from Lemmas 2.7 and 3.2 that V_B is a fundamental, weakly abundant semigroup with (C). In addition we show that every B-fundamental weakly B-abundant semigroup with (C) and (IC) embeds into V_B. Many of the results and techniques of this section appear in their original form in [3].

We begin by reminding the reader that a weakly B-abundant semigroup S satisfies (IC) if for all $a \in S$ and for some a^+, a^*, there is an order isomorphism $\alpha : \langle a^+\rangle \to \langle a^*\rangle$ such that for all $x \in \langle a^+\rangle$,

$$xa = a(x\alpha).$$

The graph $\mathrm{gr}(\alpha)$ of such an α is clearly contained in I^{a^+,a^*}, which must therefore be a full relation. We see in Section 6 that I^{a^+,a^*} can be full without containing the graph of an order isomorphism, so that S can have (WIC) without having (IC).

The order isomorphism α given above is said to be a *connecting order isomorphism*. As with the definition of (WIC), we can replace 'some' by 'any', but now we have to be slightly more careful. If a, a^+ and a^* are chosen as above, and $a^\dagger, a^\circ$ are idempotents of B with

$$a \,\widetilde{\mathcal{R}}_B\, a^+ \,\widetilde{\mathcal{R}}_B\, a^\dagger \text{ and } a \,\widetilde{\mathcal{L}}_B\, a^* \,\widetilde{\mathcal{L}}_B\, a^\circ,$$

then in B we have that $a^+ \,\mathcal{R}\, a^\dagger$ and $a^* \,\mathcal{L}\, a^\circ$. Thus $\beta : \langle a^\dagger\rangle \to \langle a^+\rangle$ and $\gamma : \langle a^*\rangle \to \langle a^\circ\rangle$ given by

$$x\beta = a^+xa^+ = xa^+ \text{ and } y\gamma = a^\circ ya^\circ = a^\circ y$$

are isomorphisms. Thus

$$\beta\alpha\gamma : \langle a^\dagger\rangle \to \langle a^\circ\rangle$$

is an order isomorphism. Moreover, for any $x \in a^\dagger$,

$$xa = xa^+a = a(x\beta\alpha) = a(a^\circ(x\beta\alpha)) = a(x\beta\alpha\gamma),$$

so that $\beta\alpha\gamma$ is connecting.

The subset V_B of $\mathcal{O}(B)$ is constructed in a manner analogous to the Hall semigroup, beginning as follows. For any $e, f \in B$ we define $V_{e,f}$ to be the set of all order isomorphisms from $\langle e\rangle$ to $\langle f\rangle$ such that

$$x\alpha \, y\alpha \,\mathcal{L}\, (xy)\alpha \text{ and } u\alpha^{-1} \, v\alpha^{-1} \,\mathcal{R}\, (uv)\alpha^{-1}$$

for all $x, y \in \langle e \rangle$ and $u, v \in \langle f \rangle$. For any $\alpha \in V_{e,f}$ we can define partial maps of $B/\mathcal{L}$ and $B/\mathcal{R}$ by

$$L_x \alpha_\ell = L_{x\alpha} \text{ and } R_y \alpha_r^{-1} = R_{y\alpha^{-1}}.$$

That α_ℓ and α_r are well defined and order preserving is a consequence of the next lemma.

Lemma 5.1. *Let $e, f \in B$ and let $\alpha : \langle e \rangle \to \langle f \rangle$ be an order isomorphism. Then $\alpha \in V_{e,f}$ if and only if the graph $\mathrm{gr}(\alpha)$ of α is contained in a (necessarily full) connecting relation $I^{e,f}$. If this is the case, then in particular, for all $x, x' \in \langle e \rangle$ and $y, y' \in \langle f \rangle$,*

$$x \leq_{\mathcal{L}} x' \textit{ implies that } x\alpha \leq_{\mathcal{L}} x'\alpha,$$

$$y \leq_{\mathcal{R}} y' \textit{ implies that } y\alpha^{-1} \leq_{\mathcal{R}} y'\alpha^{-1},$$

$$\alpha_\ell = I_\ell^{e,f} \textit{ and } \alpha_r = I_r^{e,f}.$$

Proof. Suppose that $e, f \in B$ and $\alpha \in V_{e,f}$. Notice that if $x, x' \in \langle e \rangle$ and $x \leq_{\mathcal{L}} x'$, then $x = xx'$ so that

$$x\alpha = (xx')\alpha \,\mathcal{L}\, x\alpha \, x'\alpha$$

and consequently, $x\alpha \leq_{\mathcal{L}} x'\alpha$. Dually, if $y, y' \in \langle f \rangle$ and $y \leq_{\mathcal{R}} y'$, then $y\alpha^{-1} \leq_{\mathcal{R}} y'\alpha^{-1}$.

Consider the graph $\mathrm{gr}(\alpha)$ of α. We have that $\mathrm{gr}(\alpha) \subseteq \langle e \rangle \times \langle f \rangle$ is a full relation such that for any $x \in \langle e \rangle$ and $y \in \langle f \rangle$, $(x, x\alpha), (y\alpha^{-1}, y) \in \mathrm{gr}(\alpha)$.

Let $\overline{\alpha}^{e,f}$ be the subsemigroup of $\langle e \rangle \times \langle f \rangle$ generated by $\mathrm{gr}(\alpha)$; since $\mathrm{gr}(\alpha) \subseteq \overline{\alpha}^{e,f}$ and $\mathrm{gr}(\alpha)$ is full, certainly $\overline{\alpha}^{e,f}$ is full.

Let

$$(x_1, y_1)(x_2, y_2) \ldots (x_m, y_m), (u_1, v_1)(u_2, v_2) \ldots (u_n, v_n) \in \overline{\alpha}^{e,f},$$

where $(x_i, y_i) = (x_i, x_i\alpha), (u_j, v_j) = (u_j, u_j\alpha) \in \mathrm{gr}(\alpha)$ for $1 \leq i \leq m$, $1 \leq j \leq n$, be such that

$$x_1 x_2 \ldots x_m \leq_{\mathcal{L}} u_1 u_2 \ldots u_n.$$

Then

$$y_1 y_2 \ldots y_m = x_1\alpha \, x_2\alpha \, \ldots x_m\alpha \,\mathcal{L}\, (x_1x_2)\alpha \, x_3\alpha \ldots x_m\alpha \,\mathcal{L}\, \ldots \,\mathcal{L}\, (x_1x_2 \ldots x_m)\alpha$$

and similarly,

$$v_1 v_2 \ldots v_n \,\mathcal{L}\, (u_1 u_2 \ldots u_n)\alpha.$$

From remarks above, since $x_1 \ldots x_m \leq_{\mathcal{L}} u_1 \ldots u_n$, we have that

$$y_1 \ldots y_m \,\mathcal{L}\, (x_1 \ldots x_m)\alpha \leq_{\mathcal{L}} (u_1 \ldots u_n)\alpha \,\mathcal{L}\, v_1 \ldots v_n.$$

Thus $\overline{\alpha}^{e,f}$ preserves the $\leq_{\mathcal{L}}$-order from left to right, and dually, it preserves the $\leq_{\mathcal{R}}$-order from right to left. Consequently, $\overline{\alpha}^{e,f}$ is a connecting relation.

Conversely, suppose that $\mathrm{gr}(\alpha) \subseteq I^{e,f}$ where $I^{e,f}$ is connecting. Let $x, x' \in \langle e \rangle$. Then as

$$(x, x\alpha), (x', x'\alpha) \in I^{e,f}$$

and the latter is a subsemigroup, we have that $(xx', x\alpha x'\alpha) \in I^{e,f}$. But also $(xx', (xx')\alpha) \in I^{e,f}$ and since $I^{e,f}$ preserves the $\leq_{\mathcal{L}}$-order from left to right we have that $(xx')\alpha \,\mathcal{L}\, (x\alpha)(x'\alpha)$. Together with the dual argument we have shown that $\alpha \in V_{e,f}$. The lemma follows. ☐

We remark that if S is a weakly B-abundant semigroup with (C) and (IC), $a \in S$ and $\alpha : \langle a^+ \rangle \to \langle a^* \rangle$ is a connecting order isomorphism, then since the graph $\mathrm{gr}(\alpha)$ of α is contained in I^{a^+,a^*}, we have from Lemmas 4.2 and 5.1 that $\alpha \in V_{a^+,a^*}$.

From Lemma 5.1 it is clear that

$$V_B = \{(\rho_e \alpha_\ell, \lambda_f \alpha_r^{-1}) : e, f \in B, \alpha \in V_{e,f}\}$$

is a subset of U_B.

Theorem 5.2. *(c.f.[3]) The set V_B is a full subsemigroup of U_B. Consequently, V_B is fundamental weakly abundant with (C) and (WIC). Further, V_B has (IC).*

Proof. For any $e \in B$ we have that

$$(\rho_e, \lambda_e) = (\rho_e \iota^e_\ell, \lambda_e (\iota^e)_r^{-1})$$

where ι^e is the identity relation on $\langle e \rangle$. Clearly $\iota^e \in V_{e,e}$ so that $\overline{B} \subseteq V_B$.

To see that V_B is a subsemigroup of U_B, let $e, f, g, h \in B$, $\alpha \in V_{e,f}$ and $\beta \in V_{g,h}$. According to the proof of Lemma 4.3,

$$(\rho_e \alpha_\ell, \lambda_f \alpha_r^{-1})(\rho_g \beta_\ell, \lambda_h \beta_r^{-1}) = (\rho_z K_\ell^{z,w}, \lambda_w K_r^{z,w})$$

where $(z, fgf) \in \overline{\alpha}^{e,f}$, $(gfg, w) \in \overline{\beta}^{g,h}$ and

$$K^{z,w} = (\overline{\alpha}^{e,f} \theta_{gfg} \overline{\beta}^{g,h}) \cap (\langle z \rangle \times \langle w \rangle).$$

Clearly we can take $z = (fgf)\alpha^{-1}$ and $w = (gfg)\beta$; $K^{z,w}$ then contains the graph of the order isomorphism $\gamma = \alpha|_{\langle z \rangle} \theta_{gfg} \beta$. Moreover, since $K^{z,w}$ is connecting, Lemma 5.1 gives that $\gamma \in V_{z,w}$. It follows that

$$(\rho_z K_\ell^{z,w}, \lambda_w K_r^{z,w}) = (\rho_z \gamma_\ell, \lambda_w \gamma_r^{-1}) \in V_B,$$

as required.

It remains only to show that V_B has (IC). To this end, let $e, f \in B$, $\alpha \in V_{e,f}$ and consider $(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1}) \in V_B$. From the proof of Theorem 4.4 we have that

$$(\rho_e, \lambda_e)\,\widetilde{\mathcal{R}}\,(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1})\,\widetilde{\mathcal{L}}\,(\rho_f, \lambda_f).$$

Further, for any $x \in B$ with $(\rho_x, \lambda_x) \leq (\rho_e, \lambda_e)$ (so that $x \leq e$ in B) and any $(x, t) \in \overline{\alpha}^{e,f}$

$$(\rho_x, \lambda_x)(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1}) = (\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1})(\rho_t, \lambda_t).$$

In particular, we can take $t = x\alpha$. Since $\alpha : \langle e \rangle \to \langle f \rangle$ is an order isomorphism, we can clearly define an order isomorphism

$$\underline{\alpha} : \langle (\rho_e, \lambda_e) \rangle \to \langle (\rho_f, \lambda_f) \rangle$$

by $(\rho_x, \lambda_x)\underline{\alpha} = (\rho_{x\alpha}, \lambda_{x\alpha})$. It follows that V_B has (IC).

□

We now show that V_B is the canonical $\overline{B}$-fundamental weakly $\overline{B}$-abundant semigroup with (C) and (IC) which we seek.

Theorem 5.3. [3] *Let S be a weakly B-abundant semigroup with (C) and (IC). The map $\theta : S \to V_B$ given by*

$$a\theta = (\alpha_a, \beta_a)$$

where for all $x \in B$, $L_x\alpha_a = L_{(xa)^*}$ and $R_x\beta_a = R_{(ax)^+}$, is a strongly B-admissible morphism with kernel μ_B. Moreover, $\theta|_B : B \to \overline{B}$ is an isomorphism.

Proof. We need only show that the image of θ is contained in V_B. Let $a \in S$, choose a^+, a^* and let $\alpha : \langle a^+ \rangle \to \langle a^* \rangle$ be a connecting isomorphism. We know from Theorem 4.5 that

$$a\theta = (\rho_{a^+} I_\ell^{a+,a^*}, \lambda_{a^*} I_r^{a^+,a^*}).$$

From the comments following Lemma 5.1, $\alpha \in V_{a^+,a^*}$ and as the graph of α is a full relation contained in I^{a^+,a^*}, clearly

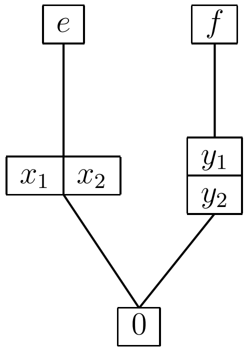
$$a\theta = (\rho_{a^+}\alpha_\ell, \lambda_{a^*}\alpha_r^{-1}) \in V_B.$$

□

We end this section by showing that the regular elements of V_B form the Hall semigroup W_B. For elements $e, f \in B$, the definition of the set $V_{e,f}$ is, of course, close to that of $W_{e,f}$, where $W_{e,f}$ is the set of isomorphisms from $\langle e \rangle$ to $\langle f \rangle$. To see that not every element of $V_{e,f}$ need lie in $W_{e,f}$ we give the following example, taken from [3].

Example 5.4.

Let B be the band with the following $\mathcal{D}$-class structure:



It is easy to see that $\langle e\rangle = \{e, x_1, x_2, 0\}$ and $\langle f\rangle = \{f, y_1, y_2, 0\}$; clearly, they are not isomorphic. However, the function $\alpha : \langle e\rangle \to \langle f\rangle$ given by

$$e\alpha = f, x_i\alpha = y_i,\ (i = 1, 2), 0\alpha = 0$$

is easily seen to be a connecting order isomorphism.

To show that W_B is the set of regular elements of V_B, we begin with some observations concerning $\alpha \in V_{e,f}$.

First, if $g\,\mathcal{R}\,e$ and $f\,\mathcal{L}\,h$ for some $e, f, g, h \in B$, then $\beta \in V_{g,h}$ where $\beta = \theta_e\alpha\theta_h$. It is not hard to see that, further, $(\rho_g\beta_\ell, \lambda_h\beta_r^{-1}) = (\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1})$.

Next, if $e = f$ and $(\rho_e\alpha_\ell, \lambda_e\alpha_r^{-1}) = (\rho_e, \lambda_e)$, then α is the identity in $\langle e\rangle$. For if $x \in \langle e\rangle$, then

$$L_x = L_x\rho_e = L_x\rho_e\alpha_\ell = L_x\alpha_\ell = L_{x\alpha}$$

so that $x\,\mathcal{L}\,x\alpha$; similarly, $x\,\mathcal{R}\,x\alpha^{-1}$. It follows that for any $y \in \langle e\rangle$, $y\alpha\,\mathcal{R}\,y\alpha\alpha^{-1} = y$. Consequently, for any $x \in \langle e\rangle$, $x\,\mathcal{H}\,x\alpha$, giving that $x = x\alpha$ as required.

Finally, if $\alpha^{-1} \in V_{f,e}$, then α is a semigroup isomorphism. For in this case, if $x, y \in \langle e\rangle$, then as $\alpha^{-1} \in V_{f,e}$ and $(\alpha^{-1})^{-1} = \alpha$,

$$x\alpha\, y\alpha\,\mathcal{R}\,(xy)\alpha.$$

Certainly $x\alpha\, y\alpha\,\mathcal{L}\,(xy)\alpha$, yielding $x\alpha\, y\alpha = (xy)\alpha$ as required.

Theorem 5.5. [3] *For a band B, the Hall semigroup*

$$W_B = \{(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1}) : e, f \in B, \alpha \in W_{e,f}\}$$

is the set of regular elements of V_B.

Proof. From [10], we know that W_B is an orthodox subsemigroup of $\mathcal{O}(B)$ which clearly is contained in V_B. It remains only to show that every regular element of V_B lies in W_B.

To this end, let $e, f \in B$ and $\alpha \in V_{e,f}$ with $(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1})$ regular. We know that

$$(\rho_e, \lambda_e)\,\widetilde{\mathcal{R}}\,(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1})\,\widetilde{\mathcal{L}}\,(\rho_f, \lambda_f)$$

so that from comments in Section 2, since $(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1})$ is regular,

$$(\rho_e, \lambda_e)\,\mathcal{R}\,(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1})\,\mathcal{L}\,(\rho_f, \lambda_f).$$

From II.3.5 of [12], there is an inverse $(\rho_g\beta_\ell, \lambda_h\beta_r^{-1})$ of $(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1})$ in V_B with

$$(\rho_e, \lambda_e) = (\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1})(\rho_g\beta_\ell, \lambda_h\beta_r^{-1})$$

and

$$(\rho_f, \lambda_f) = (\rho_g\beta_\ell, \lambda_h\beta_r^{-1})(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1}).$$

Notice however that we must have that $g\,\mathcal{R}\,f$ and $h\,\mathcal{L}\,e$, so that from comments preceding the theorem we can assume that $g = f$ and $h = e$, so that $\beta \in V_{f,e}$.

For $x \leq e$ we have that

$$L_x = L_x\rho_e = L_x\rho_e\alpha_\ell\rho_f\beta_\ell,$$

whence $x\,\mathcal{L}\,x\alpha\beta$. Similarly we can show that $x\,\mathcal{R}\,x\alpha\beta$ and so $x = x\alpha\beta$. Dually, $\beta\alpha$ is the identity in $\langle f\rangle$, so that $\beta = \alpha^{-1} \in V_{f,e}$. From remarks above, $\alpha \in W_{e,f}$ so that $(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1}) \in W_B$. □

6. Examples

We now present a number of examples, allowing us to compare semigroups of the form W_B, V_B and U_B.

In what follows we bear in mind that, consequent upon Lemma 4.1, if $I^{e,f}$ is a full connecting relation on $\langle e\rangle \times \langle f\rangle$, then $I^{e,f}$ induces an order isomorphism between $\{D_x : x \leq e\}$ and $\{D_y : y \leq f\}$. Therefore, if we are determining a full connecting relation $I^{e,f}$, we know that $I^{e,f}$ is the disjoint union of subsets of sets of the form $D^e_x \times D^f_y$, where $x \leq e, y \leq f$, $D^e_x = D_x \cap \langle e\rangle$ and $D^f_y = D_y \cap \langle f\rangle$.

We recall also that if $\alpha \in V_{e,f}$, in particular, if $\alpha \in W_{e,f}$, then the graph of α is contained in a full connecting relation $I^{e,f}$. On the other hand, if we can show that a full connecting relation $I^{e,f}$ contains the graph of an order isomorphism α, then from Lemma 5.1, we know that $\alpha \in V_{e,f}$ and

$$(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1}) = (\rho_e I^{e,f}_\ell, \lambda_f I^{e,f}_r).$$

Example 6.1.

We begin by considering a rectangular band B. If $e \in B$, then $\langle e \rangle = \{e\}$, so that for any $e, f \in B$, there is just one full relation from $\langle e \rangle$ to $\langle f \rangle$, namely $\{(e, f)\}$, which is clearly the graph of an isomorphism $\iota(e, f)$. Thus

$$W_B = V_B = U_B = \{(\rho_e \iota(e, f)_\ell, \lambda_f \iota(e, f)_r^{-1}) : e, f \in B\}.$$

But for any $x \in B$,

$$L_x \rho_e \iota(e, f)_\ell = L_e \iota(e, f)_\ell = L_f = L_{ef} = L_x \rho_{ef}$$

and dually, $R_x \lambda_f \iota(e, f)_r^{-1} = R_x \lambda_{ef}$. This gives that

$$W_B = V_B = U_B = \{(\rho_e, \lambda_e) : e \in B\} = \overline{B},$$

thus confirming the result of Example 2.

Notice that for any weakly B-abundant semigroup, if $S/\mu_B = B/\mu_B$, that is, if $S/\mu_B = \overline{B}$, then $\widetilde{\mathcal{H}}_B = \mu_B$ is a congruence on S. For if $a, b \in S$ and $a \, \widetilde{\mathcal{H}}_B \, b$, then $a\mu_B \, \widetilde{\mathcal{H}}_{\overline{B}} \, b\mu_B$, giving that $a\mu_B = b\mu_B$, since $\widetilde{\mathcal{H}}_{\overline{B}} = \mathcal{H}$ is trivial in the band $\overline{B}$.

Proposition 6.2. *A band B has the property that every weakly B-abundant semigroup having (C) and (WIC) must also have (IC), if and only if $U_B = V_B$.*

Proof. Suppose that $U_B = V_B$ and S is a weakly B-abundant semigroup with (C) and (WIC). By Theorem 4.5, $\theta : S \to U_B = V_B$ is a strongly admissible morphism onto a full subsemigroup, with kernel μ_B. By Lemma 3.2 we have that $S\theta$ has (IC), whence S has (IC) by Lemma 3.3.

Conversely, assume that U_B has (IC), and let $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \in U_B$. From Theorem 4.4 we have that

$$(\rho_e, \lambda_e) \, \widetilde{\mathcal{R}} \, (\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \, \widetilde{\mathcal{L}} \, (\rho_f, \lambda_f),$$

so that by assumption there exists an order isomorphism

$$\overline{\theta} : \langle (\rho_e, \lambda_e) \rangle \to \langle (\rho_f, \lambda_f) \rangle$$

such that for all $(\rho_z, \lambda_z) \in \langle (\rho_e, \lambda_e) \rangle$,

$$(\rho_z, \lambda_z)(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) = (\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})(\rho_z, \lambda_z)\overline{\theta}.$$

Clearly $\overline{\theta}$ induces an order isomorphism $\theta : \langle e \rangle \to \langle f \rangle$. Moreover, from the remarks preceding the statement of Theorem 5.2, $\overline{\theta} \in V_{(\rho_e, \lambda_e), (\rho_f, \lambda_f)}$ and so also $\theta \in V_{e,f}$. We claim that

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) = (\rho_e \theta_\ell, \lambda_f \theta_r).$$

Let $x \in \langle e \rangle$, and let $(x,t) \in I^{e,f}$. Then

$$\begin{aligned} L_e \rho_x \rho_e I_\ell^{e,f} &= L_{exe} I_\ell^{e,f} \\ &= L_x I_\ell^{e,f} \\ &= L_t \end{aligned}$$

and so as

$$(\rho_x, \lambda_x)(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) = (\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})(\rho_{x\theta}, \lambda_{x\theta}),$$

we have that

$$\begin{aligned} L_t &= L_e \rho_e I_\ell^{e,f} \rho_{x\theta} \\ &= L_f \rho_{x\theta} \\ &= L_{x\theta}. \end{aligned}$$

Consequently, for any $w \in B$,

$$L_w \rho_e \theta_\ell = L_{ewe} \theta_\ell = L_{(ewe)\theta} = L_{ewe} I_\ell^{e,f} = L_w \rho_e I_\ell^{e,f}.$$

Hence $\rho_e I_\ell^{e,f} = \rho_e \theta_\ell$ and dually, $\lambda_f I_r^{e,f} = \lambda_f \theta_r$, whence $(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) \in V_B$ as required.

□

Where appropriate we denote a map α from a finite set $\{x_1, x_2, \ldots, x_n\}$ to itself by

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ x_1\alpha & x_2\alpha & \ldots & x_n\alpha \end{pmatrix}$$

Example 6.3.

Let B be the band of Example 5.4; we have already shown that $V_{e,f} \neq W_{e,f}$. We show that $W_B \neq V_B = U_B$. From the remarks at the beginning of this section we need only consider full connecting relations of the form $I^{u,v}$, where (u,v) is a pair in the following set:

$$(\{e,f\} \times \{e,f\}) \cup (\{x_1,x_2,y_1,y_2\} \times \{x_1,x_2,y_1,y_2\}) \cup \{(0,0)\}.$$

Consider first relations from $\langle e \rangle$ to itself. Clearly $V_{e,e}$ consists of the identity and the isomorphism

$$\beta = \begin{pmatrix} e & x_1 & x_2 & 0 \\ e & x_2 & x_1 & 0 \end{pmatrix}.$$

Moreover,

$$L_{x_1} \rho_e = L_{x_1} \neq L_{x_2} = L_{x_1} \rho_e \beta_\ell,$$

so that $(\rho_e, \lambda_e) \neq (\rho_e \beta_\ell, \lambda_e \beta_r^{-1})$. Suppose now that $I^{e,e}$ is a full connecting relation. From comments above, we must have $(e,e), (0,0) \in I^{e,e}$ and the remaining elements form a full subset of $\{x_1,x_2\} \times \{x_1,x_2\}$. If $(x_1,x_1) \in I^{e,e}$, then we cannot have also that $(x_1,x_2) \in I^{e,e}$, since $I^{e,e}$ is $\mathcal{L}$-preserving from left to right. But $I^{e,e}$ is full, so that $(x_2,x_2) \in I^{e,e}$

and consequently, $(x_2, x_1) \notin I^{e,e}$. We deduce that in this case, $I^{e,e}$ is the graph of the identity map; a similar argument gives that if $(x_1, x_2) \in I^{e,e}$, then $I^{e,e}$ is the graph of β. The dual argument gives that the only full connecting relations from $\langle f\rangle$ to itself are the graphs of the identity map and a second isomorphism γ. Moreover, these isomorphisms give rise to distinct elements of W_B.

We have already seen in Example 5.4 that $W_{e,f}$ is empty, but $V_{e,f}$ contains α, where α has graph

$$\{(e, f), (x_1, y_1), (x_2, y_2), (0, 0)\}.$$

We know that the graph of α generates a full connecting relation from $\langle e\rangle$ to $\langle f\rangle$. On the other hand, for any full connecting relation $I^{e,f}$, we must have that $(e, f), (0, 0) \in I^{e,f}$ and that the remaining elements of $I^{e,f}$ lie in $\{x_1, x_2\} \times \{y_1, y_2\}$. However, it is easy to see that for any such $I^{e,f}$,

$$\rho_e I^{e,f}_\ell = \begin{pmatrix} L_e & L_{x_1} & L_{x_2} & L_f & L_{y_1} & L_0 \\ L_f & L_{y_1} & L_{y_1} & L_0 & L_0 & L_0 \end{pmatrix} = \rho_e \alpha_\ell$$

and

$$\lambda_f I^{e,f}_r = \begin{pmatrix} R_e & R_{x_1} & R_f & R_{y_1} & R_{y_2} & R_0 \\ R_0 & R_0 & R_e & R_{x_1} & R_{x_1} & R_0 \end{pmatrix} = \lambda_f \alpha_r^{-1}.$$

There are no connecting relations from $\langle f\rangle$ to $\langle e\rangle$. For if $I^{f,e}$ were such a relation, we would have to have $(y_i, x_1), (y_j, x_2) \in I^{f,e}$ for some $i, j \in \{1, 2\}$, since $I^{f,e}$ is full. But this is impossible since $y_i \,\mathcal{L}\, y_j$ but x_1 is not $\mathcal{L}$-related to x_2.

For $u, v \in \{x_1, x_2, y_1, y_2\}$, it is clear that $\langle u\rangle = \{u, 0\}$ and $\langle v\rangle = \{v, 0\}$ and consequently, the only full connecting relations from $\langle u\rangle$ to $\langle v\rangle$ is the graph $\{(u, v), (0, 0)\}$ of an isomorphism $\iota(u, v)$.

Clearly the only other candidate for a full connecting relation between principal ideals is $\{(0, 0)\}$ from $\langle 0\rangle$ to itself.

We conclude that $U_B = V_B$ and there are potentially 22 elements in V_B, of which at most one, $(\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1})$, does not lie in W_B. However, not all of these elements are distinct. We know from the proof of Theorem 4.4 that for elements of V_B written in standard form,

$$(\rho_u\delta_\ell, \lambda_v\delta_r^{-1}) \,\widetilde{\mathcal{R}}\, (\rho_x\eta_\ell, \lambda_y\eta_r^{-1}) \text{ if and only if } u \,\mathcal{R}\, x$$

and dually,

$$(\rho_u\delta_\ell, \lambda_v\delta_r^{-1}) \,\widetilde{\mathcal{L}}\, (\rho_x\eta_\ell, \lambda_y\eta_r^{-1}) \text{ if and only if } v \,\mathcal{L}\, y.$$

This allows us to deduce that $(\rho_e\alpha_\ell, \lambda_f\alpha_r) \notin W_B$. Moreover, straightforward checks show that for any $i, j \in \{1, 2\}$,

$$\rho_{x_i}\iota(x_i, y_j)_\ell = \begin{pmatrix} L_e & L_{x_1} & L_{x_2} & L_f & L_{y_1} & L_0 \\ L_{y_1} & L_{y_1} & L_{y_1} & L_0 & L_0 & L_0 \end{pmatrix}$$

and

$$\lambda_{y_j}\iota(x_i, y_j)_r^{-1} = \begin{pmatrix} R_e & R_{x_1} & R_f & R_{y_1} & R_{y_2} & R_0 \\ R_0 & R_0 & R_{x_1} & R_{x_1} & R_{x_1} & R_0 \end{pmatrix}.$$

Continuing is a similar manner we can argue that V_B has 15 distinct elements, consisting of the seven elements $(\rho_x, \lambda_x) \in \overline{B}$, together with

$$(\rho_e\beta_\ell, \lambda_e\beta_r^{-1}), (\rho_e\alpha_\ell, \lambda_f\alpha_r^{-1}), (\rho_f\gamma_\ell, \lambda_f\gamma_r^{-1}),$$

$$(\rho_{x_1}\iota(x_1, y_1)_\ell, \lambda_{y_1}\iota(x_1, y_1)_r^{-1}), (\rho_{y_1}\iota(y_1, x_1)_\ell, \lambda_{x_1}\iota(y_1, x_1)_r^{-1}), (\rho_{y_1}\iota(y_1, x_2)_\ell, \lambda_{x_2}\iota(y_1, x_2)_r^{-1}),$$

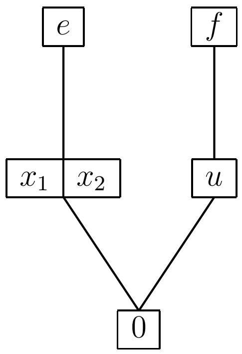
$$(\rho_{y_2}\iota(y_2, x_1)_\ell, \lambda_{x_1}\iota(y_2, x_1)_r^{-1}) \text{ and } (\rho_{y_2}\iota(y_2, x_2)_\ell, \lambda_{x_2}\iota(y_2, x_2)_r^{-1}).$$

We remark that V_B cannot be abundant; for if it were, then by the results of [4], it would be embeddable into W_B.

Our final example is of a weakly B-abundant semigroup with (C) and (WIC), but not (IC).

Example 6.4.

Let $B = \{e, f, x_1, x_2, u, 0\}$ be the band with the following $\mathcal{D}$-class structure:



Arguments very similar to those of Example 6 allow us to show that $W_B = V_B$ is a regular semigroup with 10 elements. However, the relation

$$I^{e,f} = \{(e, f), (x_1, u), (x_2, u), (0, 0)\}$$

is full and connecting and gives rise to an element

$$(\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f}) = \left(\begin{pmatrix} L_e & L_{x_1} & L_{x_2} & L_f & L_u & L_0 \\ L_f & L_u & L_u & L_0 & L_0 & L_0 \end{pmatrix}, \begin{pmatrix} R_e & R_{x_1} & R_f & R_u & R_0 \\ R_0 & R_0 & R_e & R_{x_1} & R_0 \end{pmatrix} \right)$$

which lies in U_B but not in V_B; in fact, it is the only such. We have that

$$(\rho_e, \lambda_e)\, \widetilde{\mathcal{R}}\, (\rho_e I_\ell^{e,f}, \lambda_f I_r^{e,f})\, \widetilde{\mathcal{L}}\, (\rho_f, \lambda_f)$$

but there is no order isomorphism from $\langle(\rho_e, \lambda_e)\rangle$ to $\langle(\rho_f, \lambda_f)\rangle$. We deduce that U_B is an 11 element weakly abundant semigroup with (C) and (WIC), but not (IC). From comments in Section 3, U_B cannot be abundant.

7. Structure of weakly B-abundant semigroups

We end the paper by using the existence of the semigroups U_B and V_B to determine the structure of weakly B-abundant semigroups with (C) and (WIC) (or (IC)), as spined products of U_B (or V_B) with a weakly $B/\mathcal{D}$-ample semigroup. Our approach is inspired by that of Yamada [16] and Hall [11] in the orthodox case.

We first remind the reader that if we are given semigroups S, T, H and morphisms $\varphi : S \to H, \psi : T \to H$, then the *spined product* $\mathcal{S} = \mathcal{S}(S, T, \varphi, \psi)$ of S and T with respect to H, φ and ψ is

$$\mathcal{S} = \{(s,t) \in S \times T : s\varphi = t\psi\}.$$

Clearly, if non-empty, $\mathcal{S}$ is a subsemigroup of $S \times T$.

Next, we recall some facts about the relation δ_B, which is the analogue for a weakly B-abundant semigroup S with (C) and (WIC) of the notion of the least inverse congruence on an orthodox semigroup; for convenience we cite from [8]. The relation δ_B is defined on S by the rule

$$a\, \delta_B\, b \text{ if and only if } a = ebf, b = gah \text{ for some } e, f, g, h \in B.$$

It is shown in [8] that δ_B is a congruence on S, which restricts to $\mathcal{D}$ on B, and is such that the natural morphism $\delta_B^\natural : S \to S/\delta_B$ is B-admissible. Moreover, putting $B\delta_B^\natural = \underline{B}$, we have that S/δ_B is weakly $\underline{B}$-ample.

Proposition 7.1. *Let S be a weakly B-abundant semigroup with (C) and (WIC) and let T be a weakly E-ample semigroup, where E is a semilattice isomorphic to $B/\mathcal{D}$. Suppose that there exists an admissible morphism $\psi : T \to S/\delta_B$ such that $\psi|_E : E \to \underline{B}$ is an isomorphism. Let*

$$B' = \{(b, e_b) : b \in B\}$$

where $e_b \in E$ is such that $e_b\psi = b\delta_B$. Then B' is a band isomorphic to B and the spined product $\mathcal{S} = \mathcal{S}(S, T, \delta_B^\natural, \psi)$ is weakly B'-abundant semigroup with (C) and (WIC). Moreover, if S has (IC), then so does $\mathcal{S}$.

Proof. We begin by remarking that for any $b \in B$,

$$b\delta_B \in \underline{B} = E\psi,$$

and there exists a *unique* $e_b \in E$ such that $e_b\psi = b\delta_B$. Thus $B' \subseteq \mathcal{S}$. It is easy to check that for $b, c \in B$,

$$e_{bc} = e_b e_c = e_c e_b = e_{cb}$$

and

$$b\,\mathcal{D}\,c \text{ if and only if } b\delta_B = c\delta_B \text{ if and only if } e_b = e_c.$$

Consequently, $\kappa : B \to B'$ given by $b\kappa = (b, e_b)$ is an isomorphism.

Suppose now that $(x, s) \in \mathcal{S}$. As $\delta_B^\natural$ is B-admissible, we have that for any x^+,

$$e_{x^+}\psi = x^+\delta_B = (x\delta_B)^+ = (s\psi)^+ = s^+\psi,$$

so that $e_{x^+} = s^+$. Consequently, if $(x, s), (y, t) \in \mathcal{S}$, then if $x\,\widetilde{\mathcal{R}}_B\,y$ we have that

$$s^+ = e_{x^+} = e_{y^+} = t^+,$$

so that $s\,\widetilde{\mathcal{R}}_E\,t$ in T.

Next, we show that for any $(x, s), (y, t) \in \mathcal{S}$,

$$(x, s)\,\widetilde{\mathcal{R}}_{B'}\,(y, t) \text{ if and only if } x\,\widetilde{\mathcal{R}}_B\,y.$$

If $x\,\widetilde{\mathcal{R}}_B\,y$, then by the above, $s\,\widetilde{\mathcal{R}}_E\,t$. It follows easily that for any $(b, e_b) \in B'$,

$$(b, e_b)(x, s) = (x, s) \text{ if and only if } (b, e_b)(y, t) = (y, t),$$

so that $(x, s)\,\widetilde{\mathcal{R}}_{B'}\,(y, t)$ as required.

Conversely, we suppose that $(x, s)\,\widetilde{\mathcal{R}}_{B'}\,(y, t)$. Choosing $x^+ \in B$, we know that $e_{x^+} = s^+$ and so

$$(x^+, e_{x^+})(x, s) = (x, s),$$

giving

$$(x^+, e_{x^+})(y, t) = (y, t).$$

In particular, $x^+y = y$ and so $x^+y^+ = y^+$; dually we can argue that $y^+x^+ = x^+$ and so $x\,\widetilde{\mathcal{R}}_B\,y$ as desired.

We now have that for any $(x, s) \in \mathcal{S}$,

$$(x, s)\,\widetilde{\mathcal{R}}_{B'}\,(x^+, s^+),$$

so that $\mathcal{S}$ is weakly B'-abundant, and condition (C) holds with respect to B'.

It remains to show that $\mathcal{S}$ has (WIC), (and (IC) if S does). To this end, suppose that $(x, s) \in \mathcal{S}$; choose x^+, so that $(x, s)\,\widetilde{\mathcal{R}}_{B'}\,(x^+, s^+)$ and suppose that $(b, e_b) \leq (x^+, s^+)$. Since κ is an isomorphism, $b \leq x^+$ in

B and from $b = x^+bx^+$ we also deduce that $e_b \leq e_{x^+} = s^+$ in E. Now S has (WIC), so that $bx = xc$ for some $c \in B$ with $c \leq x^*$, for some chosen x^*. Since both δ_B and ψ are admissible,

$$\begin{array}{rcl} (e_bs)^*\psi & = & (e_bs)\psi^* \\ & = & (e_b\psi s\psi)^* \\ & = & (b\delta_B x\delta_B)^* \\ & = & (bx)\delta_B^* \\ & = & (xc)\delta_B^* \\ & = & (xc)^*\delta_B \\ & = & (x^*c)\delta_B \\ & = & c\delta_B \\ & = & e_c\psi, \end{array}$$

whence $(e_bs)^* = e_c$. Consequently,

$$\begin{array}{rcl} (b, e_b)(x, s) & = & (bx, e_bs) \\ & = & (xc, s(e_bs)^*) \\ & = & (xc, se_c) \\ & = & (x, s)(c, e_c), \end{array}$$

using the fact that T is weakly E-ample. Thus $\mathcal{S}$ has (WIC).

Finally, we suppose that S has (IC). We must show that for any $(x, s) \in \mathcal{S}$ and for some $(x, s)^+, (x, s)^*$, there is an order isomorphism $\overline{\alpha} : \langle (x, s)^+ \rangle \to \langle (x, s)^* \rangle$ such that for all $(b, e_b) \in \langle (x, s)^+ \rangle$,

$$(b, e_b)(x, s) = (x, s)(b, e_b)\overline{\alpha}.$$

We choose x^+ and x^*, and take $(x, s)^+ = (x^+, s^+)$ and $(x, s)^* = (x^*, s^*)$. Suppose that $(b, e_b) \leq (x^+, s^+)$; then as above, $b \leq x^+, e_b \leq s^+$, and for any $c \in B$ with $bx = xc$, we have $(b, e_b)(x, s) = (x, s)(c, e_c)$. Since S has (IC), we know there is an order isomorphism $\alpha : \langle x^+ \rangle \to \langle x^* \rangle$ such that for any $d \in \langle x^+ \rangle, dx = x(d\alpha)$. We thus have that

$$(b, e_b)(x, s) = (x, s)(b\alpha, e_{b\alpha}),$$

and as $b\alpha \leq x^*$, $(b\alpha, e_{b\alpha}) \leq (x^*, s^*)$. Clearly

$$\overline{\alpha} : \langle (x^+, s^+) \rangle \to \langle (x^*, s^*) \rangle$$

given by

$$(b, e_b)\overline{\alpha} = (b\alpha, e_{b\alpha})$$

is a connecting order isomorphism. It follows that $\mathcal{S}$ has (IC).

□

Let S be a weakly B-abundant semigroup with (C) and (WIC). We know from Theorem 4.5 that $\theta : S \to U_B$ is a strongly admissible morphism, with kernel μ_B, to the weakly $\overline{B}$-abundant semigroup U_B, where U_B also has (C) and (WIC). Denoting $\overline{B}\delta^{\natural}_{\overline{B}}$ by B^*, the remarks preceding Proposition 7.1 give that $U_B/\delta_{\overline{B}}$ is weakly B^*-ample. We have the following diagram of semigroups and admissible morphisms:

$$\begin{tikzcd}
S \arrow[r, "\theta"] \arrow[d, "\delta^{\natural}_{B}"] & U_B \arrow[d, "\delta^{\natural}_{\overline{B}}"'] \\
S/\delta_B & U_B/\delta_{\overline{B}}
\end{tikzcd}$$

Let $a, b \in S$ with $a\,\delta_B\,b$, so that $a = ebf, b = gah$ for some elements $e, f, g, h \in B$. As $B\theta = \overline{B}$ it is clear that $a\theta\,\delta_{\overline{B}}\,b\theta$ in U_B and so $a\theta\delta^{\natural}_{\overline{B}} = b\theta\delta^{\natural}_{\overline{B}}$. We can therefore define a map $\psi : S/\delta_B \to U_B/\delta_{\overline{B}}$ by $(s\delta_B)\psi = s\theta\delta^{\natural}_{\overline{B}}$. Clearly the following diagram commutes

$$\begin{tikzcd}
S \arrow[r, "\theta"] \arrow[d, "\delta^{\natural}_{B}"] & U_B \arrow[d, "\delta^{\natural}_{\overline{B}}"'] \\
S/\delta_B \arrow[r, "\psi"] & U_B/\delta_{\overline{B}}
\end{tikzcd}$$

Notice that

$$B^* = \overline{B}\delta^{\natural}_{\overline{B}} = B\theta\delta^{\natural}_{\overline{B}} = B\delta^{\natural}_{B}\psi = \underline{B}\psi.$$

Lemma 7.2. *With notation as above, ψ is a $\underline{B}$-admissible morphism such that $\psi|_{\underline{B}} : \underline{B} \to B^*$ is an isomorphism.*

Proof. Suppose that $a\delta_B\,\widetilde{\mathcal{R}}_{\underline{B}}\,b\delta_B$. Since $\delta^{\natural}_{B}$ is admissible, we know that for $a^+, b^+ \in B$,

$$a^+\delta_B\,\widetilde{\mathcal{R}}_{\underline{B}}\,a\delta_B\,\widetilde{\mathcal{R}}_{\underline{B}}\,b\delta_B\,\widetilde{\mathcal{R}}_{\underline{B}}\,b^+\delta_B.$$

But $\underline{B}$ is a semilattice, and so $a^+\delta_B = b^+\delta_B$, giving that $a^+\,\mathcal{D}\,b^+$ in B. Since $\theta|_B : B \to \overline{B}$ is an isomorphism, certainly $a^+\theta\,\mathcal{D}\,b^+\theta$ in $\overline{B}$, so that by the same remarks, $a^+\theta\delta_{\overline{B}} = b^+\theta\delta_{\overline{B}}$. Consequently, since both θ and $\delta_{\overline{B}}$ are admissible

$$a\delta_B\psi = a\theta\delta_{\overline{B}}\,\widetilde{\mathcal{R}}_{B^*}\,(a\theta\delta_{\overline{B}})^+ = a^+\theta\delta_{\overline{B}}$$

$$= b^+\theta\delta_{\overline{B}} = (b\theta\delta_{\overline{B}})^+\,\widetilde{\mathcal{R}}_{B^*}\,b\theta\delta_{\overline{B}} = b\delta_B\psi,$$

so that ψ preserves $\widetilde{\mathcal{R}}_{\underline{B}}$. Dually, ψ preserves $\widetilde{\mathcal{L}}_{\underline{B}}$, so that ψ is $\underline{B}$-admissible.

We have remarked that $\psi|_{\underline{B}} : \underline{B} \to B^*$ is onto. Suppose now that $e\delta_B\psi = f\delta_B\psi$, for some $e, f \in B$. Then $e\theta\delta_{\overline{B}} = f\theta\delta_{\overline{B}}$, giving that $e\theta\,\mathcal{D}\,f\theta$ in $\overline{B}$. But $\theta|_B$ is an isomorphism from B onto $\overline{B}$, and so $e\,\mathcal{D}\,f$ in B. We deduce that $e\delta_B = f\delta_B$ and $\psi|_{\underline{B}}$ is one to one, finishing the proof of the lemma.

□

We are now in a position to prove the main result of this section.

Theorem 7.3. *Let S be a weakly B-abundant semigroup with (C) and (WIC). Then there exists a $\underline{B}$-admissible morphism $\psi : S/\delta_B \to U_B/\delta_{\overline{B}}$ such that $\psi|_{\underline{B}} : \underline{B} \to B^* = \overline{B}\delta^\natural_{\overline{B}}$ is an isomorphism. Moreover, S is isomorphic to the spined product*

$$\mathcal{S} = \mathcal{S}(U_B, S/\delta_B, \delta^\natural_{\overline{B}}, \psi).$$

Conversely, let T be a weakly E-ample semigroup, where E is a semilattice isomorphic to $B/\mathcal{D}$. Suppose that there exists an E-admissible morphism $\psi : T \to U_B/\delta_{\overline{B}}$ such that $\psi|_E : E \to B^*$ is an isomorphism. Then the spined product $\mathcal{S} = \mathcal{S}(U_B, T, \delta^\natural_{\overline{B}}, \psi)$ is a weakly B'-abundant semigroup with (C) and (WIC), for a band B' isomorphic to B.

Proof. In view of Proposition 7.1, it remains only to show that if S is weakly B-abundant with (C) and (WIC), then S is isomorphic to

$$\mathcal{S} = \mathcal{S}(U_B, S/\delta_B, \delta^\natural_{\overline{B}}, \psi),$$

where ψ is constructed as for Lemma 7.2. Clearly, $\varphi : S \to \mathcal{S}$ given by

$$s\varphi = (s\theta, s\delta_B)$$

is a morphism from S to the direct product $U_B \times S/\delta_B$. Since $s\theta\delta^\natural_{\overline{B}} = s\delta^\natural_B\psi$ for any $s \in S$, we have that the image of φ is contained in $\mathcal{S}$. If $s\varphi = t\varphi$, then $(s,t) \in \mu_B \cap \delta_B$ since the kernel of θ is μ_B. From [8] we know that $\widetilde{\mathcal{H}}_B \cap \delta_B = \iota$, and so $s = t$ and φ is one to one.

It remains only to show that φ is onto. Let $(X, s\delta_B) \in \mathcal{S}$, so that $X\delta_{\overline{B}} = s\delta_B\psi = s\theta\delta_{\overline{B}}$. From the definition of $\delta_{\overline{B}}$, we must have that

$$X = e\theta\, s\theta\, f\theta,$$

for some $e\theta, f\theta \in \overline{B} = B\theta$, where again using [8] we may take $e\theta$ in $E((s\theta)^+) = E(s^+\theta)$ and $f\theta$ in $E((s\theta)^*) = E(s^*\theta)$. But then $X =$

$(esf)\theta$, and as θ maps B isomorphically onto $B\theta$,

$$\begin{aligned}(esf)\delta_B &= e\delta_B\, s\delta_B\, f\delta_B \\ &= s^+\,\delta_B s\delta_B\, s^*\delta_B \\ &= (s^+ss^*)\delta_B \\ &= s\delta_B.\end{aligned}$$

We have shown that $(esf)\varphi = (X, s\delta_B)$, so that φ is an isomorphism as required. □

With almost no adjustment we can replace '(WIC)' by '(IC)' and U_B by V_B in Theorem 7.3 and obtain our final result.

Theorem 7.4. *Let S be a weakly B-abundant semigroup with (C) and (IC). Then there exists a $\underline{B}$-admissible morphism $\psi : S/\delta_B \to V_B/\delta_{\overline{B}}$ such that $\psi|_{\underline{B}} : \underline{B} \to B^* = \overline{B}\delta^{\natural}_{\overline{B}}$ is an isomorphism. Moreover, S is isomorphic to the spined product*

$$\mathcal{S} = \mathcal{S}(V_B, S/\delta_B, \delta^{\natural}_{\overline{B}}, \psi).$$

Conversely, let T be a weakly E-ample semigroup, where E is a semilattice isomorphic to $B/\mathcal{D}$. Suppose that there exists an E-admissible morphism $\psi : T \to V_B/\delta_{\overline{B}}$ such that $\psi|_E : E \to B^*$ is an isomorphism. Then the spined product $\mathcal{S} = \mathcal{S}(V_B, T, \delta^{\natural}_{\overline{B}}, \psi)$ is a weakly B'-abundant semigroup with (C) and (WIC), for a band B' isomorphic to B.

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