Analytic solution to the time-dependent Schrödinger equation for the one-dimensional quantum harmonic oscillator with an applied uniform field

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***Abstract*—I find the analytic solutions to the time-dependent Schrödinger equation for the one-dimensional quantum harmonic oscillator which is perturbed by a uniform electric field.**

Few analytic solutions to the Schrödinger equation [1, 2] exist [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. I derive an analytic solution to the single-particle time-dependent Schrödinger equation for the quantum harmonic oscillator (QHO) perturbed by a uniform electric field in one dimension – a system relevant in many areas of physics [13, 14, 15, 16, 17, 18, 19].

In one-dimension the single-particle time-independent Schrödinger equation is

$$\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+v(x)\right)\psi_k(x)=E_\mathrm{k}\psi_k(x), \quad (1)$$

where $v(x)$ is the external potential, $\psi_k(x)$ is the k^{th} solution ("wavefunction") and E_k is the corresponding eigenenergy. I employ atomic units, hence $\hbar = m = 1$ where m is the mass of an electron ($e^2 = 4\pi\varepsilon_0 = 1$).

The one-dimensional single-particle time-dependent Schrödinger equation is

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}+v(x,t)\right)\psi_k(x,t)=i\frac{\partial}{\partial t}\psi_k(x,t), \quad (2)$$

where the wavefunctions and external potential are time dependent.

A one-dimensional QHO with an applied uniform field can be described by the following potential:

$$v(x,t)=\begin{cases}\frac{1}{2}\omega^2x^2+\varepsilon x & \text{if } t\leq 0\\ \frac{1}{2}\omega^2x^2 & \text{if } t>0,\end{cases} \quad (3)$$

where for $t > 0$ the perturbing field $-\varepsilon x$ is applied. ε is a constant which dictates the strength of the perturbing field and ω is a constant which determines the degree to which the wavefunctions are confined; see Fig. 1.

$v(x, t \leq 0) = \frac{1}{2}\omega^2\left(x+\frac{\varepsilon}{\omega^2}\right)^2-\frac{\varepsilon^2}{2\omega^2} = \frac{1}{2}y^2-\frac{\varepsilon^2}{2\omega^2}$. Hence, the *static* solutions to the time-independent Schrödinger equation, Eq. (1), for the potential given by Eq. (3) when $t \leq 0$ are known analytically: $\varphi_k(y) = \frac{1}{\sqrt{2^k k!}}\left(\frac{\omega}{\pi}\right)^{\frac{1}{4}}H_k(y)e^{-\frac{1}{2}y^2}$, where $y=\sqrt{\omega}\left(x+\frac{\varepsilon}{\omega^2}\right)$ and $\{H_k(y)\}$ are the set of Hermite polynomials. $\{\varphi_k(y)\}$ correspond to the initial states of the system, i.e., $\psi_k(x, t \leq 0) = \varphi_k(y)\ \forall\ k$.

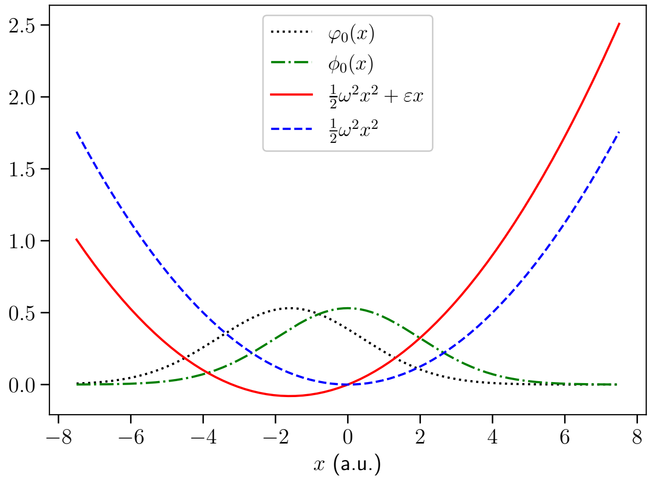

Fig. 1. Quantum harmonic oscillator (QHO) (solid red) and the perturbed harmonic oscillator (dashed blue), given by Eq. (3); in this case $\omega = 0.25$ and $\varepsilon = 0.1$ (a.u.), which have been chosen as an example. The ground-state ($k = 0$) wavefunction that corresponds to the QHO is shown in dotted black, and the dotted-dashed green is the ground-state wavefunction for the perturbed QHO.

The *static* solutions to the time-independent Schrödinger equation where $v(x) = \frac{1}{2}\omega^2x^2$, $\{\phi_n(x)\}$, are

$$\phi_n(x)=\frac{1}{\sqrt{2^n n!}}\left(\frac{\omega}{\pi}\right)^{\frac{1}{4}}H_n(\sqrt{\omega}x)e^{-\frac{1}{2}\omega x^2}. \quad (4)$$

The n^{th} eigenenergy is also known analytically: $E_n = \left(n+\frac{1}{2}\right)\omega$. The wavefunction $\psi_k(x,t)$ can be expressed as a superposition of the states $\{\phi_n(x)\}$, as such

$$\psi_k(x,t)=\sum_{n=0}^{\infty}c_{k,n}\phi_n(x)e^{-iE_nt}. \quad (5)$$

Each $c_{k,n}$ is given by the overlap between the initial state $\psi_k(x, t \leq 0)$ and $\phi_n(x)$:

$$c_{k,n} = \int_{-\infty}^{\infty} \psi_k\left(x, t \leq 0\right) \phi_n^*\left(x\right) \mathrm{d}x = \sqrt{\frac{\omega}{2^{n+k} n! k! \pi}} \int_{-\infty}^{\infty} H_k\left(y\right) e^{-\frac{1}{2}y^2} H_n\left(\sqrt{\omega}x\right) e^{-\frac{1}{2}\omega x^2} \mathrm{d}x. \tag{6}$$

Note that because one can equally think of starting the system in this initial state as the system beginning in an excited state of the *static* perturbed QHO ($\frac{1}{2}\omega^2 x^2$), the coefficients $\{c_{k,n}\}$ are time independent.

I begin by determining $c_{k=0,n}$ ($k = 0$ corresponds to the solution which begins in the ground-state state of the QHO for $t \leq 0$):

$$\begin{aligned} c_{0,n} &= \int_{-\infty}^{\infty} \psi_0\left(x, t \leq 0\right) \phi_n^*\left(x\right) \mathrm{d}x \\ &= \sqrt{\frac{\omega}{2^n n! \pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} H_n\left(\sqrt{\omega}x\right) e^{-\frac{1}{2}\omega x^2} \mathrm{d}x. \end{aligned}$$

I make the following substitution: let $z = \sqrt{\omega}x$. Therefore, $y = z + \frac{\varepsilon}{\sqrt{\omega^3}}$, and

$$\begin{aligned} c_{0,n} &= \frac{1}{\sqrt{2^n n! \pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z + \frac{\varepsilon}{\sqrt{\omega^3}}\right)^2} H_n\left(z\right) e^{-\frac{1}{2}z^2} \mathrm{d}z \\ &= \frac{e^{-\frac{\varepsilon^2}{4\omega^3}}}{\sqrt{2^n n! \pi}} \int_{-\infty}^{\infty} e^{-\left(z + \frac{\varepsilon}{2\sqrt{\omega^3}}\right)^2} H_n\left(z\right) \mathrm{d}z. \end{aligned}$$

To evaluate this integral I employ the known result

$$\int_{-\infty}^{\infty} e^{-(\alpha - z)^2} H_n\left(z\right) \mathrm{d}z = \left(2\alpha\right)^n \sqrt{\pi}, \tag{7}$$

where, in this case, $\alpha = -\frac{\varepsilon}{2\sqrt{\omega^3}}$. Hence

$$c_{0,n} = \sqrt{\frac{2^n}{n!}}\ \alpha^n e^{-\alpha^2} = \sqrt{\frac{2^n}{n!}} \left(-\frac{\varepsilon}{2\sqrt{\omega^3}}\right)^n e^{-\frac{\varepsilon^2}{4\omega^3}}. \tag{8}$$

Substituting this expression for $c_{0,n}$ and Eq. (4) into Eq. (5) I find that

$$\psi_0(x, t \geq 0) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\omega(x^2 + it) - \frac{\varepsilon^2}{4\omega^3}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\varepsilon}{2\sqrt{\omega^3}}\right)^n H_n(\sqrt{\omega}x) e^{-in\omega t}. \tag{9}$$

In order to find the other solutions to Eq. (2), i.e., for $k = 1, 2, \ldots \infty$, I evaluate the following integral which is contained within Eq. (6):

$$I_{k,n} \equiv \int_{-\infty}^{\infty} e^{-(\alpha - z)^2} H_k\left(z - 2\alpha\right) H_n\left(z\right) \mathrm{d}z. \tag{10}$$

I begin with the recurrence relation for the Hermite polynomials:

$$H_{n+1}\left(z\right) = 2zH_n\left(z\right) - 2nH_{n-1}\left(z\right). \tag{11}$$

With some simple manipulation it follows that

$$2\int_{-\infty}^{\infty} z e^{-(\alpha - z)^2} H_k\left(y\right) H_n\left(z\right) \mathrm{d}z = \int_{-\infty}^{\infty} e^{-(\alpha - z)^2} H_k\left(y\right) \left[H_{n+1}\left(z\right) + 2nH_{n-1}\left(z\right)\right] \mathrm{d}z, \tag{12}$$

where $y = z - 2\alpha$. I now express $I_{k+1,n}$ as follows, employing Eq. (12),

$$I_{k+1,n} = \int_{-\infty}^{\infty} e^{-(\alpha - z)^2} \left(\left[H_{k+1}\left(y\right) - 2zH_k\left(y\right)\right] H_n\left(z\right) + H_k\left(y\right)\left[H_{n+1}\left(z\right) + 2nH_{n-1}\left(z\right)\right]\right) \mathrm{d}z. \tag{13}$$

Recalling Eq. (11) I arrive at

$$H_{k+1}\left(y\right) - 2zH_k\left(y\right) = H_{k+1}\left(y\right) - 2yH_k\left(y\right) - 4\alpha H_k\left(y\right) = -2kH_{k-1}\left(y\right) - 4\alpha H_k\left(y\right),$$

which I then substitute into Eq. (13):

$$I_{k+1,n} = \int_{-\infty}^{\infty} e^{-(\alpha - z)^2} \left(\left[-2kH_{k-1}\left(y\right) - 4\alpha H_k\left(y\right)\right] H_n\left(z\right) + H_k\left(y\right)\left[H_{n+1}\left(z\right) + 2nH_{n-1}\left(z\right)\right]\right) \mathrm{d}z. \tag{14}$$

I then express the integrals in Eq. (14) in terms of their corresponding Is defined by Eq. (10):

$$I_{k+1,n} = I_{k,n+1} + 2nI_{k,n-1} - 2kI_{k-1,n} - 4\alpha I_{k,n}, \tag{15}$$

which yields the recurrence relation for the integrals for $k > 0$. Next I define $\beta_{k,n} \equiv \frac{I_{k,n}}{I_{0,n}}\alpha^k$. Employing the above recurrence relation (Eq. (15)) I derive the recurrence relation for $\{\beta_{k,n}\}$, as follows

$$\begin{aligned}\beta_{k+1,n} \equiv \frac{I_{k+1,n}}{I_{0,n}}\alpha^{k+1} &= \alpha^{k+1}\frac{I_{k,n+1}}{I_{0,n}} + 2n\alpha^{k+1}\frac{I_{k,n-1}}{I_{0,n}} - 4\alpha^{k+2}\ \frac{I_{k,n}}{I_{0,n}} - 2k\alpha^{k+1}\frac{I_{k-1,n}}{I_{0,n}} \\ &= \alpha\beta_{k,n+1}\frac{I_{0,n+1}}{I_{0,n}} + 2n\alpha\beta_{k,n-1}\frac{I_{0,n-1}}{I_{0,n}} - 4\alpha^2\ \beta_{k,n} - 2k\alpha^2\beta_{k-1,n} \\ &= 2\alpha^2\left(\beta_{k,n+1} - 2\beta_{k,n} - k\beta_{k-1,n}\right) + n\beta_{k,n-1}.\end{aligned}$$

From the definition of $\beta_{k,n}$, I obtain the mathematical result

$$\int_{-\infty}^{\infty} e^{-(\alpha-z)^2} H_k(z-2\alpha) H_n\left(z\right) \mathrm{d}z = 2^n \beta_{k,n}\alpha^{n-k}\sqrt{\pi}, \tag{16}$$

where the polynomials, $\{\beta_{k,n}\}$, are

$$\begin{aligned}\beta_{0,n} &= 1 \\ \beta_{1,n} &= n - 2\alpha^2 \\ \beta_{2,n} &= n^2 - n - 4n\alpha^2 + 4\alpha^4 \\ &\vdots \\ \beta_{k+1,n} &= 2\alpha^2\left(\beta_{k,n+1} - 2\beta_{k,n} - k\beta_{k-1,n}\right) + n\beta_{k,n-1}.\end{aligned}$$

From this result I find an expression for $c_{k,n}$ by recalling Eq. (6):

$$\begin{aligned}c_{k,n} &= \int_{-\infty}^{\infty} \psi_k\left(x, t \leq 0\right)\phi_n^*\left(x\right)\mathrm{d}x \\ &= \frac{e^{-\alpha^2}}{\sqrt{2^{k-n}n!k!}}\beta_{k,n}\alpha^{n-k}.\end{aligned} \tag{17}$$

Therefore the solutions to Eq. (2) for $t \geq 0$, with external potential given by Eq. (3), are

$$\psi_k(x, t \geq 0) = \frac{1}{\sqrt{2^k k!}}\left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\omega(x^2+it)-\alpha^2}\sum_{n=0}^{\infty}\frac{1}{n!}\ \beta_{k,n}\ \alpha^{n-k} H_n(\sqrt{\omega}x)e^{-in\omega t}.$$

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