

situation. This analogy defines an effective temperature or "noise temperature" for the nonequilibrium circuit. We can also define an entropy for the nonequilibrium system via its distribution function, which will be identical to the entropy of the equivalent equilibrium circuit. We have then shown that  $dS = dQ/T$ , where  $-dQ$  is the energy given up by the circuit elements to the reservoir, to the ex-

tent that it exceeds the amount predicted from the macroscopic circuit equations.

The attempt to proceed to circuits containing more than one reactance has been handled successfully only in cases where either all of the time constants for the relaxation of the degrees of freedom are fast compared to one, or else the case where all the effective temperatures are equal.

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## Atomic Coherent States in Quantum Optics

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For the description of an assembly of two-level atoms, atomic coherent states can be defined which have properties analogous to those of the field coherent states. The analogy is not fortuitous, but is shown to be related to the group contraction of exponential operators based on the angular momentum algebra to exponential operators based on the harmonic-oscillator algebra. The derivation of the properties of the atomic coherent states is made easier by the use of a powerful disentangling theorem for exponential angular momentum operators. A complete labeling of the atomic states is developed and many of their properties are studied. In particular it is shown that the atomic coherent states are the quantum analogs of classical dipoles, and that they can be produced by classical fields.

### I. INTRODUCTION

Many problems in quantum optics can be dealt with in terms of the interaction of an assembly of two-level atoms with a transverse electromagnetic field. In these problems a particular set of quan-

tum states has to be selected for the description of both field and atoms. The choice of a particular representation is always motivated by convenience rather than by necessity. A good example is given by the free field. Early treatments have made large use of Fock states, i. e., photon number

states which are eigenstates of the free-field Hamiltonian. Although they form a perfectly valid basis for the corresponding Hilbert space, these states are poorly suited for the description of laser fields which contain a large and intrinsically uncertain number of photons. In this case the reaction of the field on the radiating atoms can be approximated by a mean field, as in classical radiation problems. The field states generated from vacuum by classical currents are well known,<sup>1</sup> and happen to be eigenstates of the annihilation operator. The coordinate representation is the minimum-uncertainty packet of harmonic oscillators.<sup>2</sup> These so-called coherent states, whose usage in atom-field interaction problems was introduced by Senitzky,<sup>3</sup> have now been extensively studied and applied to quantum-optical problems,<sup>4</sup> and will be called here Glauber states.

The coherent states of the radiation field have attractive properties. They are obtained from the vacuum state by a unitary shift operator, and are minimum-uncertainty states, i. e., products of mean-square deviations of conjugated variables are minimum in these states, e. g.,  $\langle \Delta p^2 \rangle \langle \Delta q^2 \rangle = \frac{1}{4} \hbar^2$ . Though not orthogonal, they obey a completeness relation, and hence form a good set of basis states. In fact, the overcompleteness of coherent states allows the expansion of many important field operators as a single integral over projectors on these states. Finally, these states correspond to the field radiated by classical currents, i. e., currents produced by moving charges for which the field reaction is neglected. In this sense these states provide a quantum description of classical fields.

One purpose of the present paper is to show that states with completely analogous properties can be defined for the free-atom assembly. In fact, to each property of the atomic coherent states there exists a corresponding property of the field coherent states. This duality, far from being accidental, will be shown to be deeply rooted and related to the contraction of the rotation group describing motions on a sphere, onto a translation group describing motions in the harmonic-oscillator phase space.

For a single two-level system, that of atom  $n$ , the ground-state ket will be labeled  $|\psi_2^n\rangle$  and the upper-state ket  $|\psi_1^n\rangle$ . Any operator acting on this system can be expanded in the set of Pauli matrices  $\sigma_x^n$ ,  $\sigma_y^n$ ,  $\sigma_z^n$ , plus the identity matrix  $I_2^n$ , associated with this particular atom. The two-level system is thus identical to a spin- $\frac{1}{2}$  system for which spin-up and spin-down operators are defined by

$$\sigma_{\pm}^n \equiv \frac{1}{2}(\sigma_x^n \pm i\sigma_y^n). \quad (1.1)$$

For recollection, the commutation rules of these

operators are

$$[\sigma_z, \sigma_{\pm}] = \pm 2\sigma_{\pm}, \quad [\sigma_+, \sigma_-] = \sigma_z. \quad (1.2)$$

The states  $|\psi_1^n\rangle$  and  $|\psi_2^n\rangle$  are eigenstates of  $\sigma_z^n$ . Such a choice of basis is convenient but by no means unique. Any other linear combination

$$|\psi_i'^n\rangle \equiv \sum_{j=1}^2 U_{ij} |\psi_j^n\rangle \quad (i=1, 2) \quad (1.3)$$

may be chosen which preserves orthogonality and normalization. The most general transformations  $U_{ij}$  with these properties are the collection of  $2 \times 2$  unitary matrices which form the group  $U(2)$ .<sup>5,6</sup> The subgroup of transformations with determinant +1 forms the group  $SU(2)$ , familiar from angular momentum analysis.  $U(2)$  and  $SU(2)$  differ by a trivial phase factor.

Turning to the assembly of  $N$  atoms, the corresponding Hilbert space is spanned by the set of  $2^N$  product states

$$|\phi_{i_1 i_2 \dots i_N}\rangle \equiv \prod_{n=1}^N |\psi_{i_n}^n\rangle \quad (i_n=1, 2). \quad (1.4)$$

Collective angular momentum operators are defined by

$$J_{\mu} = \frac{1}{2} \sum_n \sigma_{\mu}^n \quad (\mu = x, y, z), \quad (1.5a)$$

$$J_{\pm} = \sum_n \sigma_{\pm}^n, \quad (1.5b)$$

$$J^2 = J_x^2 + J_y^2 + J_z^2. \quad (1.5c)$$

For the moment the effect of the different spatial positions of atoms 1, 2, ...,  $N$  is ignored. This effect is important for the atom-field interaction and will be discussed in Sec. VI where more appropriate collective angular momentum operators are defined.

Following the historical development of quantum mechanics one could choose as another suitable basis, in place of (1.4), the set of eigenstates of the energy operator  $J_z$ . In this case, symmetry requirements actually indicate an appropriate complete set of commuting observables to which  $J_z$  belongs and whose simultaneous eigenstates form the basis. In analogy to angular momentum eigenstates these orthonormal states will be labeled

$$|J_M; \bar{\lambda}; i\rangle, \quad (1.6)$$

where  $J(J+1)$  and  $M$  are the eigenvalues of  $J^2$  and  $J_z$ , respectively. The quantum numbers  $\bar{\lambda}$  and  $i$  are those additional eigenvalues which are required to provide a complete set of labels. They are related to the permutation properties of the free-atom Hamiltonian and will be explained in Sec. V. The energy eigenstates (1.6) have been used in the study of superradiance<sup>7</sup> and will be called Dicke states. They will be shown to have a close relationship to the Fock states of the free-field problem.

Another natural way to describe the  $N$  atoms is through the overcomplete set of product states

$$|\phi(a_1, b_1; \dots; a_N, b_N)\rangle \equiv \prod_{n=1}^N (a_n |\psi_1^n\rangle + b_n |\psi_2^n\rangle), \quad (1.7)$$

with  $|a_n|^2 + |b_n|^2 = 1$ . These states display no correlations between different atoms.<sup>3</sup> For any normalized state  $|\psi\rangle$  of the  $N$ -atom assembly, a degree of correlation can be defined in the following manner: One forms the overlap integral  $|\langle\psi|\phi(a_1, b_1; \dots; a_N, b_N)\rangle|^2$  and maximizes the result with respect to the set  $(a_i, b_i)$ ,  $i=1$  to  $N$ . The complement to one of this maximized overlap integral is defined as the degree of atomic correlation of the state  $|\psi\rangle$ . It can easily be seen that all states of the form (1.7) have zero correlation, whereas the Dicke states (1.6) of maximum  $J$  ( $J=\frac{1}{2}N$ ) and small  $M$  ( $M \approx 0$ ) have a correlation which approaches unity for large- $N$  values.

Another set of overcomplete states can be obtained by rotating the Dicke states  $|\frac{J}{2}, \tilde{\lambda}; i\rangle$  through an angle  $(\theta, \varphi)$  in angular momentum space. These states, which can be labeled

$$|\frac{J}{2}, \theta, \varphi; i\rangle \quad (1.8)$$

are the atomic coherent states. They will be named Bloch states in view of their resemblance to the spin states common in nuclear-induction problems.<sup>8</sup> The profound difference between states of type (1.6) and those of type (1.7) has already been discussed by Senitzky.<sup>3</sup> The Bloch states (1.8) should not be mistaken for uncorrelated coherent states of type (1.7). Only for  $J=\frac{1}{2}N$  are the Bloch states a subset of (1.7).

The remainder of the paper is subdivided as follows: Sections II and III give a parallel treatment of the field and atomic states, respectively. For simplicity, Sec. II deals with a single-field mode, and Sec. III with a single member of the set  $(\tilde{\lambda}, i)$ . The notation in Sec. III is therefore simplified,  $|J, M\rangle$  or  $|M\rangle$  replacing (1.6), and  $|J, \theta, \varphi\rangle$  or  $|\theta, \varphi\rangle$  replacing (1.8). Section IV explains the group-contraction procedure which allows derivation of all the properties of the field states in Sec. II from the corresponding properties of the atomic states in Sec. III. Section V describes the symmetry properties of the atomic states and, in particular, explains the full notation (1.6) and (1.8). In Secs. VI and VII some aspects of the atom-field interaction are considered. In Sec. VI the spatial dependence of atomic states is introduced. In the case of a single-field mode, operators replacing the set (1.5) can be defined such that the interaction preserves the symmetry properties of the states. Various approximations are presented and radiation rates calculated. Some aspects of the interaction with a classical field are discussed in Sec. VII. Section VIII shows how

the disentangling and contraction procedures can be applied to the calculation of thermal averages. Appendix A gives a disentangling theorem for exponential angular momentum operators, and some resulting properties, such as formulas for the coupling of rotations. The disentangling properties should find great use in many other fields of physics where rotations are considered and expectation values have to be calculated. Appendix B shows an example of the application of the disentangling theorem to the calculation of generating functions for expectation values of any product of angular momentum operators in Bloch states. Appendix C shows that the contraction of the rotation group onto the oscillator group can also be used to derive the Hermite polynomials and their properties from the spherical harmonics and their properties. Appendix D gives some useful formulas relating Bloch states, spherical harmonics, and irreducible representations of the full rotation group.

## II. DESCRIPTION OF THE FREE FIELD

### A. Harmonic-Oscillator States

In order to point out with maximum clarity the analogies between the free-field description and the free-atom description, we start by listing here, in simple terms, the properties of the single harmonic oscillator. The equation numbering here and in Sec. III is done in parallel.

The single harmonic oscillator is described by its canonically conjugated coordinates  $(q, p)$  with the commutation relation

$$[q, p] = i\hbar. \quad (2.1)$$

One forms the usual lowering and raising operators

$$a = (2\hbar\omega m)^{-1/2} (\omega m q + ip), \quad (2.2a)$$

$$a^\dagger = (2\hbar\omega m)^{-1/2} (\omega m q - ip), \quad (2.2b)$$

where  $\omega m > 0$  is characteristic of the oscillator. These operators satisfy

$$[a, a^\dagger] = 1 \quad (2.3a)$$

from which one obtains

$$[a, a^\dagger a] = a, \quad (2.3b)$$

$$[a^\dagger, a^\dagger a] = -a^\dagger. \quad (2.3c)$$

The harmonic-oscillator states, or Fock states, are the eigenstates of

$$N = a^\dagger a \quad (2.4)$$

and are given by<sup>9</sup>

$$|n\rangle = (n!)^{-1/2} (a^\dagger)^n |0\rangle \quad (n=0, 1, 2, \dots) \quad (2.5)$$

with eigenvalue  $n$ . The vacuum state  $|0\rangle$  is the harmonic-oscillator ground state defined by

$$a|0\rangle = 0. \quad (2.6)$$

### B. Coherent States of the Field

Let us consider the translation operator which produces a shift  $\xi$  in  $q$  and  $\eta$  in  $p$ :

$$T_\alpha = e^{(-i/\hbar)(p - \eta a) + \alpha a^\dagger - \alpha^* a}, \quad (2.7a)$$

where

$$\alpha = (2\hbar\omega m)^{-1/2}(\omega m\xi + i\eta). \quad (2.7b)$$

A coherent state  $|\alpha\rangle$  is obtained by translation of the ground state

$$|\alpha\rangle \equiv T_\alpha|0\rangle. \quad (2.8)$$

We shall name these states Glauber states, since they have been used extensively by Glauber in quantum optics.<sup>4</sup> Since

$$T_\alpha a T_\alpha^{-1} = a - \alpha, \quad (2.9)$$

the state  $|\alpha\rangle$  satisfies the eigenvalue equation

$$(a - \alpha)|\alpha\rangle = 0. \quad (2.10)$$

Using a Baker-Campbell-Hausdorff formula<sup>10,11</sup> or Feynman's disentangling techniques,<sup>12</sup> the translation operator  $T_\alpha$  can be written in the following forms:

$$T_\alpha = e^{|\alpha|^2/2} e^{-\alpha^* a} e^{\alpha a^\dagger} = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a}. \quad (2.11)$$

The second of these forms, which is known as the normally ordered form, immediately gives the expansion of  $|\alpha\rangle$  in terms of Fock states,

$$|\alpha\rangle = T_\alpha|0\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger}|0\rangle, \quad (2.12)$$

whence, expanding the exponential and using (2.5),

$$\langle n|\alpha\rangle = e^{-|\alpha|^2/2} \alpha^n / (n!)^{1/2}. \quad (2.13)$$

The scalar product of Glauber states can be obtained either from (2.12), using the disentangling theorem (2.11), or from (2.13), using the completeness property of Fock states  $\sum |n\rangle\langle n| = 1$ . One gets

$$\langle \alpha|\beta\rangle = e^{-[|\alpha|^2 - 2\alpha^*\beta + |\beta|^2]^{1/2}}, \quad (2.14a)$$

whence,

$$|\langle \alpha|\beta\rangle|^2 = e^{-|\alpha - \beta|^2}. \quad (2.14b)$$

The coherent states are minimum-uncertainty packets. For three observables  $A$ ,  $B$ ,  $C$ , which obey a commutation relation  $[A, B] = iC$ , it is easy to show<sup>9</sup> that  $\langle A^2 \rangle \langle B^2 \rangle \geq \frac{1}{4} \langle C \rangle^2$ . In particular, with  $A = q - \xi$ ,  $B = p - \eta$ , and  $C = \hbar$ , one has

$$\langle (q - \xi)^2 \rangle \langle (p - \eta)^2 \rangle \geq \frac{1}{4} \hbar^2 \quad (2.15)$$

for any state. It is easy to show that the equality sign holds for the coherent state  $|\alpha\rangle$ , where  $\alpha$  is

related to  $\xi$  and  $\eta$  by (2.7b). This establishes the minimum-uncertainty property.

### C. Coherent States as Basis

We now consider the completeness properties of the coherent states. Using (2.13), and the completeness of Fock states  $\sum_n |n\rangle\langle n| = 1$ , one obtains straightforwardly

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1. \quad (2.16)$$

The expansion of an arbitrary state in Glauber states follows:

$$\begin{aligned} |c\rangle &\equiv \sum_n c_n |n\rangle = \int \frac{d^2\alpha}{\pi} \sum_n c_n |\alpha\rangle\langle\alpha|n\rangle \\ &= \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2/2} f(\alpha^*) |\alpha\rangle, \end{aligned} \quad (2.17a)$$

where

$$f(\alpha^*) \equiv \sum_n c_n (\alpha^*)^n / (n!)^{1/2} = e^{|\alpha|^2/2} \langle\alpha|c\rangle. \quad (2.17b)$$

Using (2.5), it is seen that  $|c\rangle$  can also be written as

$$|c\rangle = f(a^\dagger)|0\rangle, \quad (2.18)$$

where  $f(a^\dagger)$  is defined by its expansion (2.17b).

The scalar product of any two states  $|c'\rangle$  and  $|c\rangle$  is obtained from (2.16) and (2.17b):

$$\begin{aligned} \langle c'|c\rangle &= \int \frac{d^2\alpha}{\pi} \langle c'|\alpha\rangle \langle\alpha|c\rangle \\ &= \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} [f'(\alpha^*)]^* f(\alpha^*). \end{aligned} \quad (2.19)$$

In view of the completeness relations, operators  $F$  acting on this Hilbert space can be expanded as

$$F = \sum_{m,n} |m\rangle\langle m| F |n\rangle\langle n| \quad (2.20a)$$

or

$$F = \iint \frac{d^2\alpha d^2\beta}{\pi^2} |\beta\rangle\langle\beta| F |\alpha\rangle\langle\alpha|. \quad (2.20b)$$

Owing to the overcompleteness of the  $|\alpha\rangle$  states, the expansion (2.20b) is in general not unique.

This expansion is especially useful if it can be written in the diagonal form

$$F = \int d^2\alpha f(\alpha) |\alpha\rangle\langle\alpha|. \quad (2.20c)$$

This will be discussed further for the case of the density matrix.

### D. Statistical Operator for the Field

Up to here we have considered pure quantum states. Since a field in thermal equilibrium with matter at ordinary temperatures is essentially in the ground state ( $\hbar\omega \gg kT$ ), this is an adequate description for any field obtained from the thermal

equilibrium in response to a classical current. However, the field radiated by an incoherently pumped medium is a statistical mixture described by a statistical operator  $\rho$ , which we assume normalized to unity,

$$\text{Tr}\rho = 1. \quad (2.21)$$

With the help of this operator, the statistical average of any observable  $F(a, a^\dagger)$  is obtained as

$$\langle F \rangle = \text{Tr}\rho F. \quad (2.22)$$

Of particular interest are statistical ensembles described by a statistical operator which is diagonal in the Glauber representation,<sup>4</sup>

$$\rho = \int P(\alpha) |\alpha\rangle \langle \alpha| d^2\alpha, \quad (2.23)$$

where the normalization (2.21) requires

$$\int P(\alpha) d^2\alpha = 1. \quad (2.24)$$

The statistical average of an observable  $F$  is then given by an average over the diagonal elements  $\langle \alpha | F | \alpha \rangle$ :

$$\langle F \rangle = \int P(\alpha) \langle \alpha | F | \alpha \rangle d^2\alpha. \quad (2.25)$$

The weight function  $P(\alpha)$  has thus the properties of a distribution function in  $\alpha$  space, except that it is not necessarily positive.

Let us define a set of operators  $\hat{X}(\lambda)$  such that their expectation values for coherent states

$$b^\alpha(\lambda) = \langle \alpha | \hat{X}(\lambda) | \alpha \rangle \quad (2.26)$$

form a basis in the function space of functions of  $\alpha$ . If the statistical ensemble has a diagonal representation (2.23), then the statistical averages of the operators  $\hat{X}(\lambda)$  form a kind of characteristic function of  $P(\alpha)$ :

$$X(\lambda) \equiv \langle \hat{X}(\lambda) \rangle = \int d^2\alpha P(\alpha) b^\alpha(\lambda). \quad (2.27)$$

The weight function  $P(\alpha)$  can be expressed in terms of  $X(\lambda)$  with the help of the reciprocal basis  $\bar{b}^\lambda(\alpha)$ ,

$$P(\alpha) = \int d^2\lambda X(\lambda) \bar{b}^\lambda(\alpha). \quad (2.28)$$

A convenient basis is the Fourier basis

$$b^\alpha(\lambda) = e^{\lambda\alpha^* - \lambda^*\alpha}, \quad (2.29a)$$

$$\bar{b}^\lambda(\alpha) = \frac{1}{(2\pi)^2} e^{-\lambda\alpha^* + \lambda^*\alpha}, \quad (2.29b)$$

which is generated by the normally ordered operators

$$\hat{X}_N(\lambda) = e^{\lambda a^\dagger} e^{-\lambda^* a}. \quad (2.29c)$$

The question of the existence of the  $P$  representation is a complicated one.<sup>4,13</sup> Using the Fourier basis (2.29) it can be shown, however, that the mere existence of the inverse transformation (2.28) guarantees that the resulting function  $P(\alpha)$

can be used to calculate the statistical average of any product  $a^{\dagger m} a^n$  as if  $P(\alpha)$  were the weight function defined in (2.23). This is due to the fact that the characteristic function  $X_N(\lambda)$  plays the role of a generating function for  $\langle a^{\dagger m} a^n \rangle$ :

$$\langle a^{\dagger m} a^n \rangle = \left( \frac{\partial}{\partial \lambda} \right)^m \left( -\frac{\partial}{\partial \lambda^*} \right)^n X_N(\lambda) \Big|_{\lambda=0},$$

whence, by derivation of (2.27), one obtains

$$\langle a^{\dagger m} a^n \rangle = \int d^2\alpha P(\alpha) \langle \alpha | a^{\dagger m} a^n | \alpha \rangle,$$

which is a particular case of (2.25) and proves the above statement. One could, moreover, introduce, in addition to (2.29c), symmetrically ordered  $\hat{X}_S(\lambda)$  and antinormally ordered  $\hat{X}_A(\lambda)$  exponential operators.<sup>4,13</sup> The Fourier transform of their statistical averages are the Wigner distribution and the matrix element  $(1/\pi) \langle \alpha | \rho | \alpha \rangle$ , respectively. We shall not develop these aspects further as the corresponding expressions for atomic coherent states are rather involved, and of no clear use as yet.

### III. DESCRIPTION OF FREE ATOMS

#### A. Angular Momentum States

As shown in the Introduction, angular momentum operators can be defined which act on the  $N$ -atom Hilbert space. In particular we can consider a subspace of degenerate eigenstates of  $J^2$  with eigenvalues  $J(J+1)$ . Since  $J^2$  commutes with  $J_x$ ,  $J_y$ ,  $J_z$ , these operators only connect states within the same subspace. In general  $J^2$  and  $J_z$  do not form a complete set of commuting observables. As explained in Sec. V, such a complete set is formed by adding to  $J^2$  and  $J_z$  some operators of the permutation group of  $N$  objects  $P_N$ . These operators play, with respect to  $P_N$ , the same role as  $J^2$  and  $J_z$  with respect to the three-dimensional rotation group. We shall assume that the subspace considered here has also been made invariant under these permutation operations, but for simplicity we shall omit, for the time being, to indicate this in the labeling of the states. The subspace we are dealing with is identical to a constant angular momentum Hilbert space. The Dicke states, which are the analog of the Fock states (2.5), and the Bloch states, which correspond to the Glauber states (2.8), are most easily defined within such a subspace. The equation numbering is in parallel with that of Sec. II. From the angular momentum operators  $J_x$  and  $J_y$ , which satisfy the commutation relation

$$[J_x, J_y] = iJ_z, \quad (3.1)$$

the lowering and raising operators are formed,

$$J_- \equiv J_x - iJ_y, \quad (3.2a)$$

$$J_+ \equiv J_x + iJ_y, \quad (3.2b)$$

which obey

$$[J_-, J_+] = -2J_z, \quad (3.3a)$$

$$[J_-, J_z] = J_-, \quad (3.3b)$$

$$[J_+, J_z] = -J_+. \quad (3.3c)$$

The Dicke states, which are simply the usual angular momentum states,<sup>7</sup> are defined as the eigenstates of

$$J_z = \frac{1}{2}(J_+ J_- + J_- J_+). \quad (3.4)$$

They are given by<sup>9</sup>

$$|M\rangle = \frac{1}{(M+J)!} \left( \frac{2J}{M+J} \right)^{-1/2} J_+^{M+J} | -J \rangle$$

$$(M = -J, -J+1, \dots, J), \quad (3.5)$$

with eigenvalue  $M$ . They span the space of angular momentum quantum number  $J$ . The ground state  $| -J \rangle$  is defined by

$$J_- | -J \rangle = 0. \quad (3.6)$$

#### B. Coherent Atomic States

Let us consider the rotation operator which produces a rotation through an angle  $\theta$  about an axis  $\hat{n} = (\sin\varphi, -\cos\varphi, 0)$  as shown in Fig. 1:

$$R_{\theta, \varphi} = e^{-i\theta J_n} = e^{-i\theta(J_x \sin\varphi - J_y \cos\varphi)} = e^{\zeta J_+ - \zeta^* J_-}, \quad (3.7a)$$

where

$$\zeta = \frac{1}{2}\theta e^{-i\varphi}. \quad (3.7b)$$

A coherent atomic state, or Bloch state,  $|\theta, \varphi\rangle$  is obtained by rotation of the ground state  $| -J \rangle$ :

$$|\theta, \varphi\rangle \equiv R_{\theta, \varphi} | -J \rangle. \quad (3.8)$$

Referring to Fig. 1, it is seen that

$$R_{\theta, \varphi} J_n R_{\theta, \varphi}^{-1} = J_n,$$

$$R_{\theta, \varphi} J_k R_{\theta, \varphi}^{-1} = J_k \cos\theta + J_z \sin\theta,$$

$$R_{\theta, \varphi} J_z R_{\theta, \varphi}^{-1} = -J_k \cos\theta + J_z \sin\theta,$$

where

$$J_n = J_x \sin\varphi - J_y \cos\varphi, \quad J_k = J_x \cos\varphi + J_y \sin\varphi,$$

which gives

$$J_+ = (J_k - iJ_n) e^{i\varphi}, \quad J_- = (J_k + iJ_n) e^{-i\varphi}.$$

Using these relations one obtains

$$R_{\theta, \varphi} J_- R_{\theta, \varphi}^{-1} = e^{-i\varphi} [J_- e^{i\varphi} \cos^2(\frac{1}{2}\theta) - J_+ e^{-i\varphi} \sin^2(\frac{1}{2}\theta) + J_z \sin\theta], \quad (3.9a)$$

and similar relations for  $J_+$  and  $J_z$ :

$$R_{\theta, \varphi} J_+ R_{\theta, \varphi}^{-1} = e^{i\varphi} [J_+ e^{-i\varphi} \cos^2(\frac{1}{2}\theta) - J_- e^{i\varphi} \sin^2(\frac{1}{2}\theta) + J_z \sin\theta], \quad (3.9b)$$

$$R_{\theta, \varphi} J_z R_{\theta, \varphi}^{-1} = J_z \cos\theta - J_- e^{i\varphi} \sin\frac{1}{2}\theta \cos\frac{1}{2}\theta - J_+ e^{-i\varphi} \sin\frac{1}{2}\theta \cos\frac{1}{2}\theta. \quad (3.9c)$$

From (3.9a) and the definition (3.8), one obtains the eigenvalue equation

$$[J_- e^{i\varphi} \cos^2(\frac{1}{2}\theta) - J_+ e^{-i\varphi} \sin^2(\frac{1}{2}\theta) + J_z \sin\theta] |\theta, \varphi\rangle = 0. \quad (3.10a)$$

This equation, together with

$$J^2 |\theta, \varphi\rangle = J(J+1) |\theta, \varphi\rangle, \quad (3.10b)$$

specifies uniquely the Bloch state  $|\theta, \varphi\rangle$ . Note that the harmonic-oscillator analog of (3.10b) would have been the trivial relation  $(a^\dagger - \alpha^*)(a - \alpha)|\alpha\rangle = 0$ . Other forms of the eigenvalue equation can be obtained using the relation

$$R_{\theta, \varphi} J_z R_{\theta, \varphi}^{-1} |\theta, \varphi\rangle = -J |\theta, \varphi\rangle$$

and (3.9c). The resulting equation can be combined with (3.10a) to eliminate one of the operators  $J_z$ ,  $J_+$ , or  $J_-$ , giving

$$[J_- e^{i\varphi} \cos^2(\frac{1}{2}\theta) + J_+ e^{-i\varphi} \sin^2(\frac{1}{2}\theta)] |\theta, \varphi\rangle = J \sin\theta |\theta, \varphi\rangle, \quad (3.10c)$$

$$[J_- e^{i\varphi} \cos\frac{1}{2}\theta + J_z \sin\frac{1}{2}\theta] |\theta, \varphi\rangle = J \sin\frac{1}{2}\theta |\theta, \varphi\rangle, \quad (3.10d)$$

$$[J_+ e^{-i\varphi} \sin\frac{1}{2}\theta - J_z \cos\frac{1}{2}\theta] |\theta, \varphi\rangle = J \cos\frac{1}{2}\theta |\theta, \varphi\rangle. \quad (3.10e)$$

These additional relations are not independent of (3.10a) and (3.10b). It is noted that these eigenvalue equations are more complicated than their counterpart (2.10). In particular they involve at least two of the three operators  $J_-$ ,  $J_+$ ,  $J_z$ . This feature is required by the more complicated commutation relation (3.1) which applies here.

Using the disentangling theorem for angular momentum operators (Appendix A), the rotation  $R_{\theta, \varphi}$  given by (3.7a) becomes

$$R_{\theta, \varphi} = e^{-\tau^* J_-} e^{-1n(1+|\tau|^2)J_z} e^{\tau J_+}.$$

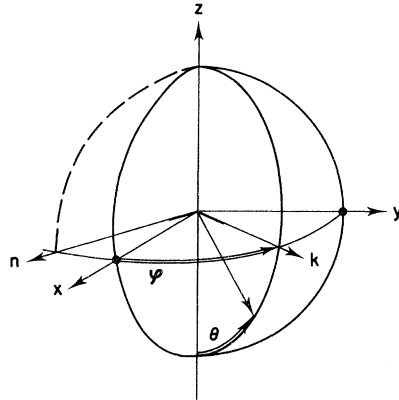


FIG. 1. Definition of the rotation  $R_{\theta, \varphi}$  in angular momentum space.

$$= e^{\tau J_+} e^{1/2(1+|\tau|^2)J_z} e^{-\tau^* J_-}, \quad (3.11a)$$

where

$$\tau = e^{-i\varphi} \tan \frac{1}{2}\theta. \quad (3.11b)$$

Let us point out that these expressions are singular for  $\theta = \pi$ , i. e., for the uppermost state. We may have to exclude from some of the following consideration the states contained within an infinitesimally small circle around  $\theta = \pi$ . The validity of expressions such as (3.13) for  $\theta = \pi$  is usually not affected and can be checked directly. The last form of (3.11a), which we call the normally ordered form, immediately gives the expansion of  $|\theta, \varphi\rangle$  in terms of Dicke states:

$$|\theta, \varphi\rangle = R_{\theta, \varphi} | -J \rangle = \left( \frac{1}{1+|\tau|^2} \right)^J e^{\tau J_+} | -J \rangle, \quad (3.12)$$

whence, expanding the exponential and using (3.5),

$$\begin{aligned} \langle M | \theta, \varphi \rangle &= \binom{2J}{M+J}^{1/2} \frac{\tau^{M+J}}{[1+|\tau|^2]^J} \\ &= \binom{2J}{M+J}^{1/2} \sin^{J+M}(\frac{1}{2}\theta) \cos^{J-M}(\frac{1}{2}\theta) e^{-i(J+M)\varphi}. \end{aligned} \quad (3.13)$$

Since the Dicke states form a basis for a well-known irreducible representation of the rotation group, these results could have been derived using the appropriate Wigner  $\mathcal{D}^{(J)}$  matrix.<sup>14</sup> The same remark applies to Eqs. (2.12) and (2.13): These could have been obtained without using a Baker-Campbell-Hausdorff formula, from the transformation properties of an irreducible representation of the group of operations  $T_\alpha$ .<sup>15-17</sup>

The overlap of two Bloch states is obtained either from (3.12), using the disentangling theorem for exponential angular momentum operators, or from

$$\begin{aligned} (2J+1) \int \frac{d\Omega}{4\pi} |\theta, \varphi\rangle \langle \theta, \varphi| \\ = (2J+1) \int \frac{d\Omega}{4\pi} \sum_{M, M'} \binom{2J}{M+J}^{1/2} \binom{2J}{M'+J}^{1/2} e^{i(M'-M)\varphi} (\cos \frac{1}{2}\theta)^{2J-M-M'} (\sin \frac{1}{2}\theta)^{2J+M+M'} |M\rangle \langle M'| \\ = (2J+1) \int_0^\pi \frac{1}{2} (\sin \theta d\theta) \sum_M \binom{2J}{M+J} (\cos \frac{1}{2}\theta)^{2J-2M} (\sin \frac{1}{2}\theta)^{2J+2M} |M\rangle \langle M| = \sum_M |M\rangle \langle M| = 1. \end{aligned} \quad (3.16)$$

The expansion of an arbitrary state in Bloch states follows:

$$\begin{aligned} |c\rangle = \sum_M c_M |M\rangle &= (2J+1) \int \frac{d\Omega}{4\pi} \sum_M c_M |\theta, \varphi\rangle \langle \theta, \varphi| M \rangle \\ &= (2J+1) \int \frac{d\Omega}{4\pi} \frac{f(\tau^*)}{[1+|\tau|^2]^J} |\theta, \varphi\rangle, \end{aligned} \quad (3.17a)$$

where

(3.13), using the completeness property of Dicke states  $\sum_M |M\rangle \langle M| = 1$ . One obtains

$$\begin{aligned} \langle \theta, \varphi | \theta', \varphi' \rangle &= \left( \frac{(1+\tau^*\tau')^2}{(1+|\tau|^2)(1+|\tau'|^2)} \right)^J \\ &= e^{iJ(\varphi-\varphi')} [\cos \frac{1}{2}(\theta-\theta') \cos \frac{1}{2}(\varphi-\varphi') \\ &\quad - i \cos \frac{1}{2}(\theta+\theta') \sin \frac{1}{2}(\varphi-\varphi')]^{2J}, \end{aligned} \quad (3.14a)$$

whence

$$|\langle \theta, \varphi | \theta', \varphi' \rangle|^2 = \cos^{4J} \frac{1}{2}\Theta, \quad (3.14b)$$

where  $\tau$  is given by (3.11b),  $\tau'$  is given by the same equation written with the primed quantities, and  $\Theta$  is the angle between the  $(\theta, \varphi)$  and  $(\theta', \varphi')$  directions, as given by

$$\cos \Theta = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi').$$

The Bloch states form minimum-uncertainty packets. The uncertainty relation can be defined in terms of the set of rotated operators  $(J_x, J_y, J_z)$  =  $R_{\theta, \varphi} (J_x, J_y, J_z) R_{\theta, \varphi}^{-1}$ . These three observables obey a commutation relation of the type  $[A, B] = iC$  with  $A = J_x$ ,  $B = J_y$ ,  $C = J_z$ , whence they have the uncertainty property

$$\langle J_x^2 \rangle \langle J_y^2 \rangle \geq \frac{1}{4} \langle J_z \rangle^2 \quad (3.15)$$

for any states. It is easy to show that the equality sign holds for the Bloch state  $|\theta, \varphi\rangle$ , which is therefore a minimum-uncertainty state.

### C. Bloch States as Basis

Let us now consider the completeness properties of the Bloch states. Using (3.13) and the completeness of Dicke states  $\sum_M |M\rangle \langle M| = 1$ , one obtains

$$f(\tau^*) \equiv \sum_M c_M \binom{2J}{J+M}^{1/2} (\tau^*)^{J+M} = [1+|\tau|^2]^J \langle \theta, \varphi | c \rangle. \quad (3.17b)$$

Using (3.5) it is seen that  $|c\rangle$  can also be written as

$$|c\rangle = f \left( \frac{1}{J+1-J_z} J_+ \right) | -J \rangle. \quad (3.18)$$

The amplitude function  $f(\tau^*)$  is, by its definition

(3.17b), a polynomial of degree  $2J$ . However any function which has a Maclaurin expansion can be taken as a suitable amplitude function in (3.17a) or (3.18). Indeed the powers of  $\tau^*$  higher than  $2J$  give zero contribution in (3.17a) and (3.18). The coefficients  $c_M$  are then obtained from the first  $(2J+1)$  terms of the Maclaurin series, using (3.17b).

The scalar product of two states characterized by their amplitude function is, from (3.16) and (3.17b),

$$\begin{aligned} \langle c' | c \rangle &= (2J+1) \int \frac{d\Omega}{4\pi} \langle c' | \theta, \varphi \rangle \langle \theta, \varphi | c \rangle \\ &= (2J+1) \int \frac{d\Omega}{4\pi} \frac{1}{(1+|\tau|^2)^{2J}} [f'(\tau^*)]^* f(\tau^*) . \end{aligned} \quad (3.19)$$

Since (3.17b) was used to derive this equation, its validity is restricted to amplitude functions which are polynomials of degree  $2J$ .

In view of the completeness relations, operators  $G$  acting on this Hilbert space can be expanded as

$$G = \sum_{M, M'} |M\rangle \langle M| G |M'\rangle \langle M'| \quad (3.20a)$$

or

$$G = \frac{(2J+1)^2}{(4\pi)^2} \iint d\Omega d\Omega' |\theta, \varphi\rangle \langle \theta, \varphi| G |\theta', \varphi'\rangle \langle \theta', \varphi'| . \quad (3.20b)$$

However,  $G$  is completely defined by the  $(2J+1)^2$  matrix elements  $\langle M|G|M'\rangle$ , with the result that, except for pathological cases, an operator can always be written in the diagonal form

$$G = \int d\Omega g(\theta, \varphi) |\theta, \varphi\rangle \langle \theta, \varphi| , \quad (3.20c)$$

where  $g(\theta, \varphi)$  is given by a series expansion

$$g(\theta, \varphi) = \sum_{l, m} G_{lm} Y_{lm}(\theta, \varphi) . \quad (3.20d)$$

In accordance with the property (D21) only the  $(2J+1)^2$  first terms of this sum contribute to (3.20c). These are the terms for which  $0 \leq l \leq 2J$ . The corresponding coefficients  $G_{lm}$  are expressed as a function of the matrix elements  $\langle M|G|M'\rangle$  by Eq. (D26).

#### D. Statistical Operators for Atoms

In order to describe an incoherently pumped system of atoms we introduce a statistical operator  $\rho$  with the properties

$$\text{Tr}\rho = 1 , \quad (3.21)$$

$$\langle G \rangle = \text{Tr}\rho G . \quad (3.22)$$

As before, the considerations are restricted to states belonging to a single constant-angular-momentum subspace and, therefore, the statistical operator described here does not allow for the most general mixing of atomic states. Of particu-

lar interest is the expression of  $\rho$  in a diagonal Bloch representation

$$\rho = \int P(\theta, \varphi) |\theta, \varphi\rangle \langle \theta, \varphi| d\Omega , \quad (3.23)$$

with the normalization

$$\int P(\theta, \varphi) d\Omega = 1 . \quad (3.24)$$

The statistical average of an observable  $G$  is then given by

$$\langle G \rangle = \int P(\theta, \varphi) \langle \theta, \varphi | G | \theta, \varphi \rangle d\Omega . \quad (3.25)$$

The weight function  $P(\theta, \varphi)$  has thus the properties of a distribution function on the unit sphere, except that it is not necessarily positive.

Let us define a set of operators  $\hat{X}_\lambda$  such that their expectation values for Bloch states

$$b_\lambda^{(\theta, \varphi)} = \langle \theta, \varphi | \hat{X}_\lambda | \theta, \varphi \rangle \quad (3.26)$$

form a basis in the space of functions on the unit sphere. Since in this space a discrete basis can be chosen, the parameter  $\lambda$  can be restricted to discrete values  $\lambda = 1, 2, \dots$ . For a statistical ensemble described by (3.23), the statistical averages of the operators  $\hat{X}_\lambda$  form a set of characteristic coefficients of  $P(\theta, \varphi)$ :

$$X_\lambda \equiv \text{Tr}(\rho \hat{X}_\lambda) = \int P(\theta, \varphi) b_\lambda^{(\theta, \varphi)} d\Omega . \quad (3.27)$$

The weight function can be expressed as a series with the help of the reciprocal basis  $\bar{b}^\lambda(\theta, \varphi)$ ,

$$P(\theta, \varphi) = \sum_\lambda X_\lambda \bar{b}^\lambda(\theta, \varphi) . \quad (3.28)$$

A convenient basis is given by the spherical harmonics

$$b_\lambda^{(\theta, \varphi)} = Y_l^m(\theta, \varphi) , \quad \lambda \equiv (l, m) \quad (3.29a)$$

$$\bar{b}^\lambda(\theta, \varphi) = Y_l^{-m}(\theta, \varphi) , \quad (3.29b)$$

which are generated by the spherical harmonic operators<sup>18</sup>

$$\hat{X}_\lambda = y_l^m(\hat{J}) . \quad (3.29c)$$

The fact already mentioned that a diagonal representation always exists in the atomic case also corresponds to the fact that for a given  $J$  only the  $(2J+1)^2$  operators  $y_l^m$  with  $l \leq 2J$  are different from zero. The finite dimensionality of the basis is required, since  $\rho$  is completely determined by its  $(2J+1)^2$  matrix elements  $\langle M|\rho|M'\rangle$  in the Dicke representation. An illustration is given in Appendix D, where the statistical operator corresponding to a pure Bloch state is derived [Eq. (D29)].

Other differences with the field case should also be noted. Firstly, the spherical harmonic operators are usually written in a fully symmetrized form, whereas the operators (2.29c) are normally ordered. This is only a formal difficulty as it should be possible to write normally ordered and antinormally ordered multipole operators<sup>19</sup> with



properties similar to the  $\mathcal{Y}_i^m(\vec{J})$ . A second, and more fundamental difference, is that the expectation values  $X_\lambda$  are not generating functions for products of the type  $\langle J_+^m J_-^n J_z^p \rangle$ , in view of the discreteness of the set. This does not cause much difficulty, as generating functions can be defined from exponential operators (Appendix B) whose expectation values can be calculated with the help of the disentangling theorem (Appendix A). It is tempting to use for the  $\hat{X}_\lambda$ 's of Eq. (3.29c) these exponential operators themselves. Though the parallel with the field case then seems more transparent, the use of the discrete set  $\hat{X}_\lambda$  may be of more fundamental significance as it takes into account symmetry properties of the states.

A final comment should be made about the difficulty of dealing with creation and annihilation operations in a finite Hilbert space. The existence of two terminal states,  $|J\rangle$  and  $|-J\rangle$ , requires the presence of a third operator with the properties of  $J_z$ , and prevents the writing of an eigenvalue equation in terms of one compound operator alone. For instance, comparison of (3.18) and (2.18) suggests that  $J_-(J+1-J_z)^{-1}$  could be a "good" annihilation operator. Using (3.13) one finds immediately

$$J_- \frac{1}{J+1-J_z} |\theta, \varphi\rangle = \tau |\theta, \varphi\rangle - \tau \sin^{2J}(\frac{1}{2}\theta) e^{-2iJ\varphi} |J\rangle,$$

which for small  $\theta$  and large  $J$  is "almost" an eigenvalue equation, i. e., the application of the operator reproduces  $|\theta, \varphi\rangle$  except for the uppermost Dicke state. There is no doubt that a theory could be developed in terms of more complicated annihilation and creation operators of such type, but the advantage is not clear.

#### IV. CONTRACTION, OR THE RELATION BETWEEN ATOMIC STATES AND FIELD STATES

The extreme similarity between the treatments of Secs. III and II suggests a close connection between atomic and field states. This connection is made here through a process known as group contraction.<sup>20-22</sup>

The time evolution of a single two-level atom is governed by a  $2 \times 2$  unitary transformation matrix. The commutation relation for the generators of the group  $U(2)$  are rewritten here:

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z, \quad [\vec{J}, J_0] = 0, \quad (4.1)$$

where  $J_0$  in the third relation is essentially the identity. An arbitrary  $2 \times 2$  unitary transformation matrix is given by

$$U(2) = \exp(i \sum_\mu \lambda_\mu J_\mu), \quad (4.2)$$

where the summation is over all four indices and the  $\lambda_\mu$ 's are  $c$ -number parameters which characterize the group operation.

If another set of generators  $h_+, h_-, h_z, h_0$  is related to  $J_+, J_-, J_z, J_0$  by a nonsingular transformation  $A_{\nu\mu}$ ,

$$h_\nu = \sum_\mu A_{\nu\mu} J_\mu, \quad (4.3)$$

then the group operation (4.2) may be written

$$\exp(i \sum_\mu \lambda_\mu J_\mu) = \exp(i \sum_\nu \alpha_\nu h_\nu), \quad (4.4)$$

with

$$\alpha_\nu = \sum_\mu \lambda_\mu (A^{-1})_{\mu\nu}. \quad (4.3')$$

We select the following transformation  $A$ , which depends on a real parameter  $c$ :

$$\begin{bmatrix} h_+ \\ h_- \\ h_z \\ h_0 \end{bmatrix} = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 1/2c^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} J_+ \\ J_- \\ J_z \\ J_0 \end{bmatrix}. \quad (4.5)$$

It is easily verified that the  $h_\nu$ 's satisfy the commutation relations

$$\begin{aligned} [h_z, h_\pm] &= \pm h_\pm, & [h_+, h_-] &= 2c^2 h_z - h_0, \\ [\vec{h}, h_0] &= 0. \end{aligned} \quad (4.6)$$

In the limit  $c \rightarrow 0$  the transformation  $A$  becomes singular, and  $A^{-1}$  fails to exist. Nevertheless, the commutation relations (4.6) are well defined and, in fact, identical to the commutation relations (2.3) under the identification

$$\lim_{c \rightarrow 0} h_z = n = a^\dagger a, \quad \lim_{c \rightarrow 0} h_+ = a^\dagger, \quad \lim_{c \rightarrow 0} h_- = a. \quad (4.7)$$

Although the inverse  $A^{-1}$  (4.3b) does not exist as  $c \rightarrow 0$ , the parameter  $\alpha_\nu$  may approach a well-defined limit if we demand all the parameters  $\lambda_\mu$  to shrink ("contract") to zero in the limit  $c \rightarrow 0$ , in such a way that the following ratios are well defined:

$$\begin{aligned} \lim_{c \rightarrow 0} \frac{i\lambda_+}{c} &= \lim_{c \rightarrow 0} \frac{e^{-i\varphi}\theta}{2c} = \alpha, \\ \lim_{c \rightarrow 0} \frac{i\lambda_-}{c} &= \lim_{c \rightarrow 0} -\frac{e^{i\varphi}\theta}{2c} = -\alpha^*, \\ \lim_{c \rightarrow 0} \frac{\lambda_z}{c} &= \lim_{c \rightarrow 0} c \frac{\lambda_z}{c^2} = 0. \end{aligned} \quad (4.8)$$

Within any  $(2J+1)$ -dimensional representation of the group  $U(2)$  the eigenvalue of the diagonal operator  $h_z$  is

$$h_z |J, M\rangle = (J_z + 1/2c^2) |J, M\rangle = |J, M\rangle (M + 1/2c^2). \quad (4.9)$$

We demand that this have a definite limit as  $c \rightarrow 0$ . Physically, for both Fock and Dicke states we progress upwards from the ground or vacuum state. It is convenient to demand that the (energy) eigenvalue in (4.9) be zero for the ground state  $M = -J$ ;

$$\lim_{c \rightarrow 0} (-J + 1/2c^2) = 0. \quad (4.10)$$

In the limit  $c \rightarrow 0$ ,  $2Jc^2 = 1$ , the unitary irreducible representations  $\mathfrak{D}^J [U(2)]$  go over into unitary irreducible representations for the contracted group with generators (4.6). In simple words, the contraction procedure amounts to letting the radius of the Bloch sphere tend to infinity as  $1/c^2$ , while considering smaller and smaller rotations on the sphere. The motion on the sphere then becomes identical to the motion on the bottom tangent plane which goes over into the phase plane of the harmonic oscillator.

This procedure for contracting groups, commutation relations, and representations will now be used to show the similarity between Dicke and Fock states. We define

$$|\infty, n\rangle \equiv \lim_{c \rightarrow 0} |J, M\rangle \quad (J+M=n, \text{ fixed}). \quad (4.11)$$

Then

$$\begin{aligned} a^\dagger a |\infty, n\rangle &= \lim_{c \rightarrow 0} (J_z + 1/c^2) |J, M\rangle \\ &= \lim_{c \rightarrow 0} |J, M\rangle [(J+M) + (-J + 1/2c^2)] \\ &= n |\infty, n\rangle. \end{aligned} \quad (4.12a)$$

The computations for  $a^\dagger$  and  $a$  are handled in an entirely analogous way:

$$a^\dagger |\infty, n\rangle = \lim_{c \rightarrow 0} h_+ |J, M\rangle = (n+1)^{1/2} |\infty, n+1\rangle, \quad (4.12b)$$

$$a |\infty, n\rangle = \lim_{c \rightarrow 0} h_- |J, M\rangle = n^{1/2} |\infty, n-1\rangle. \quad (4.12c)$$

These equations provide a straightforward connection between Dicke and Fock states. The operators  $h_+$ ,  $h_-$ , and  $h_z$  contract to  $a^\dagger$ ,  $a$ , and  $a^\dagger a$  with the proper commutation relations (2.3), and with the proper matrix elements between contracted Dicke states as shown in (4.12). The contracted Dicke states (4.11) can thus be identified with the Fock states, and we conclude that every property of Dicke and Bloch states listed in Sec. III must contract to a corresponding property of Fock and Glauber states listed in Sec. II. The contraction procedure is summarized in Table I.

We demonstrate this correspondence in three particular cases.

*Example 1.* Just as the angular momentum eigenstates  $|J, M\rangle$  are obtained from the ground state  $|J, -J\rangle$  by  $(J+M)$  successive applications of the shift-up operator  $J_+$ , the Fock state  $|n\rangle$  is obtained by  $n$  successive applications of  $a^\dagger$ . By contraction of (3.5) we get

$$\begin{aligned} |\infty, n\rangle &= \lim_{c \rightarrow 0} \frac{(J_+)^n |J, -J\rangle}{[(2J!/(2J-n)!)n!]^{1/2}} \\ &= \lim_{c \rightarrow 0} \frac{(cJ_+)^n |J, -J\rangle}{[(2Jc^2)^n n!]^{1/2}} = \frac{(a^\dagger)^n |\infty, 0\rangle}{n!^{1/2}}, \end{aligned} \quad (4.13)$$

which is nothing but (2.5).

*Example 2.* Let us now contract Bloch states to Glauber states, using Eq. (3.12) and Table I:

$$\begin{aligned} |\alpha\rangle &= \lim_{c \rightarrow 0} |\theta, \varphi\rangle = \lim_{c \rightarrow 0} \left( \frac{1}{1+|\tau|^2} \right)^J e^{\tau J_+} | -J\rangle \\ &= \lim_{c \rightarrow 0} (1 - 2c^2 \frac{1}{2} \alpha \alpha^*)^{1/2c^2} e^{\alpha a^\dagger} |0\rangle \\ &= e^{-\alpha \alpha^*/2} e^{\alpha a^\dagger} |0\rangle, \end{aligned} \quad (4.14)$$

which is nothing but (2.12).

We leave it to the reader to verify that every equation of Sec. III goes over to the corresponding equation of Sec. II under contraction. This is true in particular of the disentangling theorem (3.11) whose contracted limit is the Baker-Campbell-Hausdorff formula (2.11) as shown in (A5) and (A6). In general, all properties related to angular momentum have a harmonic-oscillator property as counterpart. Thus, the total momentum contracts to the harmonic-oscillator Hamiltonian, while the spherical harmonics (and their properties) contract to the harmonic-oscillator eigenfunction (and corresponding properties) (Appendix C).

*Example 3.* As a final case of special interest we contract the uncertainty relations (3.15). Multiplying (3.15) by  $c^4$  on both sides, and using (3.9), we obtain the limits

$$\lim_{c \rightarrow 0} c J_z = \frac{a^\dagger + a}{2} - \frac{\alpha^* + \alpha}{2} = (\omega m / 2\hbar)^{1/2} (q - \xi),$$

TABLE I. Rules for the contraction of the angular momentum algebra to the harmonic-oscillator algebra. [The limit of the angular momentum quantities (first line) for  $c \rightarrow 0$  are the corresponding harmonic-oscillator quantities (second line).]

Operators	Coordinates	Eigenvalues	Eigenstates	Coherent states
$cJ_+, cJ_-, J_z + 1/2c^2$	$\frac{\theta}{2c} e^{-i\varphi}$	Angular momentum $2c^2 J, J+M$	$ J, M\rangle$ (Dicke)	$ \theta, \varphi\rangle$ (Bloch)
$a^\dagger, a, a^\dagger a$	$\alpha$	Harmonic oscillator $1, n$	$ \infty, n\rangle$ (Fock)	$ \alpha\rangle$ (Glauber)

$$\lim_{c \rightarrow 0} c J_\eta = \frac{a^\dagger - a}{2i} - \frac{\alpha^* - \alpha}{2i} = - (2\hbar\omega m)^{-1/2} (p - \eta),$$

$$\lim_{c \rightarrow 0} c^2 J_\xi = \frac{1}{2}. \quad (4.15)$$

Introducing these relations in (3.15) we reproduce (2.15), and since the equality sign in (3.15) holds for the coherent state  $|\theta, \varphi\rangle$  corresponding to the rotation  $R_{\theta, \varphi}$ , it follows that the equality sign in (2.15) holds for the coherent state  $\alpha$  corresponding to the translation  $(\xi, \eta)$ . This proves the minimum-uncertainty property of Glauber states, being given the minimum uncertainty of Bloch states.

## V. SYMMETRY PROPERTIES OF ATOMIC STATES

### A. Problem

We have seen in Sec. III that the symmetrized states arising in the description of  $N$  identical two-level atoms can be labeled by the quantum number  $J, M$ . These are the so-called Dicke states. The quantum numbers  $J, M$  are familiar from the study of the *angular momentum* group  $SU(2)$ .

The group  $SU(2)$  arises in the description of symmetrized atomic states in the following way: The time evolution of each single two-level atom is governed by a  $2 \times 2$  unitary transformation matrix. The evolution of  $N$  identical two-level atoms is described by a direct product of  $N$  such matrices when all atoms evolve in time in the same way. All transformation matrices are identical and in fact are a direct product representation of the group  $SU(2)$ .

The quantum numbers  $J, M$  do not provide a large enough set of quantum numbers for a complete labeling of symmetrized atomic states. That is, many distinct symmetrized atomic states may be labeled by the same quantum number  $J, M$ . Since the study of symmetries leads to a generation of quantum numbers, we are led to a study of an additional symmetry group  $P_N$ —the *permutation* group on  $N$  atoms.<sup>6</sup>

The group  $P_N$  arises in the description of symmetrized atomic states in the following way: The time evolution for  $N$  indistinguishable atoms is governed by a Hamiltonian which is left unchanged under any permutation of the atomic labels

$$\mathcal{H}(1, \dots, i, \dots, j, \dots, N)$$

$$= +\mathcal{H}(1, \dots, j, \dots, i, \dots, N). \quad (5.1)$$

This is exactly what it means for particles to be identical.

The quantum numbers of  $P_N$ , which are in exact analogy with the quantum numbers  $J, M$  of  $SU(2)$ , then provide additional labels for symmetrized atomic states. With these additional quantum numbers, all symmetrized atomic states are uniquely labeled.

### B. Subspaces Invariant under Rotation Operations

The properties of such spaces are well known from the study of angular momentum in quantum theory.<sup>23</sup> The total angular momentum operator  $J^2$  commutes with the operators  $J_x, J_z$  and therefore has the same eigenvalue  $J(J+1)$  on all bases for an irreducible representation  $\mathfrak{D}^J$ .

The number of distinct basis vectors within any irreducible representation is equal to

$$\dim \mathfrak{D}^J = 2J + 1. \quad (5.2a)$$

These bases are conveniently labeled by the eigenvalues  $M$  of the diagonal operator  $J_z$ ,

$$M = -J, -J+1, \dots, +J. \quad (5.3a)$$

Bases are therefore labeled  $|J_M^J; \alpha'\rangle$ . Here  $\alpha'$  is the set of all other quantum numbers necessary to distinguish different copies of the space with the same  $J$  label. An arbitrary element  $R$  of the group  $SU(2)$  maps a state in any  $J, \alpha'$  subspace into a linear combination of states *within the same subspace*;

$$R |J_M^J; \alpha'\rangle = \sum_{M'} |J_{M'}^J; \alpha'\rangle \langle J_{M'}^J | R |J_M^J\rangle = \sum_{M'} |J_{M'}^J; \alpha'\rangle \mathfrak{D}_{M'M}^J(R). \quad (5.4a)$$

Referring to Fig. 2, we say that the group  $SU(2)$  “acts vertically.”

### C. Subspace Invariant under Permutation Operations

Operators which commute with all elements of  $P_N$  have the same eigenvalues on all bases within an irreducible representation. These eigenvalues can then be used as labels for the irreducible representations. The operators in  $P_N$  analogous to  $J^2$  in  $SU(2)$  are the class sums<sup>5</sup>; their eigenvalues are the non-negative integers  $\lambda_1 \geq \lambda_2$  which obey  $\lambda_1 + \lambda_2 = N$ .

The number of distinct basis vectors within any irreducible representation is equal to<sup>5</sup>

$$\dim \Gamma^{\lambda_1, \lambda_2} = \binom{\lambda_1 + \lambda_2}{\lambda_2} - \binom{\lambda_1 + \lambda_2}{\lambda_2 - 1}. \quad (5.2b)$$

These bases can be labeled by eigenvalues of operators analogous to  $J_z$ : These operators are the  $\frac{1}{2}N$  mutually commuting adjacent interchanges  $P_{12}, P_{34}, P_{56}, \dots$ . However, it is more convenient in this case to label the bases simply by the number  $i$ ,

$$i = 1, 2, \dots, \dim \Gamma^{\lambda_1, \lambda_2}. \quad (5.3b)$$

Bases are therefore labeled  $|\alpha; \lambda_1, \lambda_2\rangle$ . Here  $\alpha$  is the set of all other quantum numbers necessary to distinguish different copies of the space with the same  $\lambda_1, \lambda_2$  label. An arbitrary element  $P$  of the group  $P_N$  maps a state in any  $\lambda_1, \lambda_2; \alpha$  subspace into a linear combination of states *within the same subspace*,

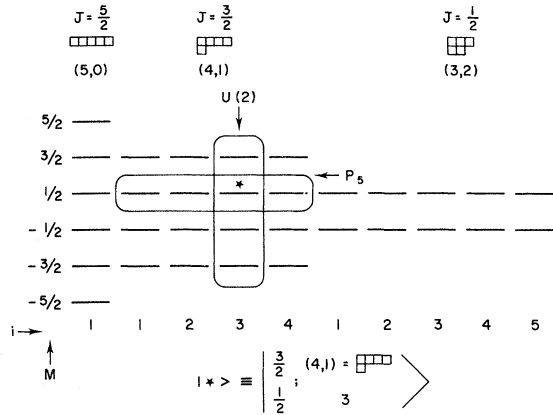


FIG. 2. An arbitrary state in the symmetry-adapted basis is labeled by (i) the  $U(2)$  invariant subspace  $J$  in which it lies, and its position  $M$  within that space, and by (ii) the  $P_N$  invariant subspace  $\tilde{\lambda}$  in which it lies, and its position  $i$  within that space. Moreover, the intersection of any  $P_N$  invariant subspace with any  $U(2)$  invariant subspace is at most one dimensional, so the quantum numbers  $J$ ,  $M$  and  $\tilde{\lambda}$ ,  $i$  are sufficient for a complete labeling of states. The intersection is exactly one dimensional when the partitions describing the  $U(2)$  and  $P_N$  invariant subspace are identical:  $2J = \lambda_1 - \lambda_2$ . The state marked  $\star$  in the figure is labeled  $|_{M=1/2}^{J=3/2}; \tilde{\lambda}=(4,1)_{i=3}^{(4,1)}\rangle$ . Under a  $U(2)$  operation  $R$ , this state is mapped into a linear combination of states within the same vertical box,

$$R |_{M=1/2}^{J=3/2}; \tilde{\lambda}=(4,1)_{i=3}^{(4,1)}\rangle = \sum_M |_{M'}^{J=3/2}; \tilde{\lambda}=(4,1)_{i=3}^{(4,1)}\rangle \mathcal{D}_{M,1/2}^{3/2}(R).$$

Under a  $P_5$  operation  $\rho$ , it is mapped into a linear combination of states within the horizontal box,

$$\rho |_{M=1/2}^{J=3/2}; \tilde{\lambda}=(4,1)_{i=3}^{(4,1)}\rangle = \sum_j |_{M'}^{J=3/2}; \tilde{\lambda}=(4,1)_{i=3}^{(4,1)}\rangle \Gamma_{j,3}^{(4,1)}(\rho).$$

In other words, group operations do not affect the labels classifying the invariant subspace (upper line); they only affect the appropriate *internal* state labels (lower line). Note that states belonging to the same  $M$  but different  $\tilde{\lambda}$  values are not necessarily degenerated.

$$P | \alpha; \tilde{\lambda}; i \rangle = \sum_j | \alpha; \tilde{\lambda}; j \rangle \langle \tilde{\lambda}; j | \tilde{\lambda}; i \rangle P | \tilde{\lambda}; j \rangle = \sum | \alpha; \tilde{\lambda}; j \rangle \Gamma_{ji}^{\tilde{\lambda}}(P). \tag{5.4b}$$

Referring to Fig. 2, we say that the group  $P_N$  “acts horizontally.”

D. Classification of Atomic States

It can easily be shown that

$$\sum_{2J = \lambda_1 - \lambda_2 \geq 0} (\dim \mathcal{D}^J) (\dim \Gamma^{\lambda_1, \lambda_2}) = 2^N. \tag{5.5}$$

This is a manifestation of a very deep and beautiful theorem.<sup>24</sup>

*Theorem.* The intersection of any  $SU(2)$  invariant subspace with any  $P_N$  invariant subspace is at most one dimensional. It is exactly one dimensional only when

$$2J = \lambda_1 - \lambda_2, \quad N = \lambda_1 + \lambda_2. \tag{5.6}$$

This theorem is sufficient to tell us what the

sets  $\alpha$ ,  $\alpha'$  of “all other quantum numbers” consist of. In short, a symmetrized state is specified *uniquely* by

$$|_{M'}^J; \tilde{\lambda}; i \rangle. \tag{5.7}$$

This specification is, in fact, even redundant. The labels  $J$ ,  $\tilde{\lambda}$  are not independent but related by (5.6). This intersection theorem is illustrated in Fig. 2.

Under an arbitrary Hamiltonian (5.1), only states with the same  $\tilde{\lambda}$ ,  $J$ ,  $M$  values are necessarily degenerate. But if the Hamiltonian consists of a sum of single-particle terms, all states with the same  $M$  value are, in general, degenerated. The introduction of particle-particle interactions lifts the high- $M$  degeneracy (such as in the case of pressure broadening in a gas).

It is a simple matter to transform between the symmetrized bases (5.7) and the direct product bases (1.4). For simplicity we label the states (1.4) by  $|\vec{N}\rangle$ , where  $\vec{N}$  is the  $N$ -component vector  $(i_1, i_2, \dots, i_N)$ . We let  $n_1$  and  $n_2$  be the total number of one's and two's in  $\vec{N}$ , respectively. Then

$$n_1 + n_2 = N, \quad n_1 - n_2 = 2M. \tag{5.8}$$

The states  $|\vec{N}\rangle$  and  $|_{M'}^J; \tilde{\lambda}; i \rangle$  are connected by a unitary transformation which is, in fact, also real and therefore orthogonal,

$$|_{M'}^J; \tilde{\lambda}; i \rangle = \sum_{(n_1, n_2)} |\vec{N}\rangle \langle \vec{N} |_{M', i}^{\tilde{\lambda}} \rangle = \sum_{(n_1, n_2)} |\vec{N}\rangle C_{\tilde{\lambda}; i M}^{\vec{N}}. \tag{5.9}$$

Here  $\sum_{(n_1, n_2)}$  indicates a summation over all distinct permutations of one's and two's with fixed  $M$ . The coefficients  $C_{\tilde{\lambda}; i M}^{\vec{N}}$  vanish unless  $2M = n_1 - n_2$ . Thus, the structure of  $C$  is that of a block diagonal matrix. The block-connecting states with  $2M = n_1 - n_2$  is a square matrix of order

$$\binom{N}{n_1} = \binom{N}{n_2}.$$

We illustrate what we have in mind in Fig. 3. This transformation will be used in discussing leakage from symmetrized states (Sec. VII D).

In the same manner as (5.7), Bloch states produced by rotation  $R_{\theta, \varphi}$  of an  $M = -J$  state are denoted

$$|_{\theta, \varphi}^J; \tilde{\lambda}; i \rangle = R_{\theta, \varphi} |_{-J}^J; \tilde{\lambda}; i \rangle. \tag{5.10}$$

VI. RADIATIVE PROPERTIES OF BLOCH STATES

A. Position of Problem

The description of the atomic states given above is now completed by considering a problem central to quantum optics, namely, the interaction of the assembly of two-level systems with the radiation field. This interaction introduces a *spatial* dependence, which we have so far neglected, and which



$$(e^{i\vec{k}\cdot\vec{r}_n\sigma_+^n})|\psi_2^{\vec{k},n}\rangle = |\psi_1^{\vec{k},n}\rangle, \quad (6.12a)$$

$$(e^{-i\vec{k}\cdot\vec{r}_n\sigma_-^n})|\psi_1^{\vec{k},n}\rangle = |\psi_2^{\vec{k},n}\rangle. \quad (6.12b)$$

The new step-up and step-down operators within the parentheses are precisely those which enter in the definition of  $J_{\pm}^{\vec{k}}$  in Eqs. (6.6c)–(6.6d). In terms of these new states and operators all the formulas of Sec. III remain valid provided one defines new operators  $J_x^{\vec{k}}$  and  $J_y^{\vec{k}}$  as required by Eq. (3.2). In particular, Dicke states can be defined as the simultaneous eigenstates of  $J_z$  and  $(J^{\vec{k}})^2 = J_z^2 + \frac{1}{2}(J_x^{\vec{k}}J_x^{\vec{k}} + J_y^{\vec{k}}J_y^{\vec{k}})$ , and Bloch states are obtained from the lowest Dicke states by application of the operator  $\exp(\xi J_+^{\vec{k}} - \xi^* J_-^{\vec{k}})$ .

These states are ideally suited for the treatment of the single-mode problem.<sup>25</sup> Indeed the angular momentum operator  $(J^{\vec{k}})^2$  commutes with the single-mode interaction Hamiltonian of same  $\vec{k}$ . This means that the corresponding Dicke and Bloch states move in subspaces of constant  $(J^{\vec{k}})^2$  eigenvalue. This symmetry is not preserved in the many-mode problem (6.8), as a consequence of the commutation relation (6.7). Leakage into other symmetry types results, as shown in detail for the case of the interaction with a classical field in Sec. VII.

#### C. Interaction with Classical Field

In certain cases, when field statistics and spontaneous emission are not important, one is allowed to treat the field classically, defining

$$\vec{E}(\vec{r}, t) = \text{Re}[\vec{E}(\vec{r}, t)e^{i\omega t}], \quad (6.13)$$

the time dependence of  $\vec{E}(\vec{r}, t)$  being slow compared to  $\omega$ . In this case the interaction Hamiltonian (6.4) becomes

$$\mathcal{H}_{AF} = -\frac{1}{2}\sum_n [\sigma_+^n \vec{p} \cdot \vec{E}^*(\vec{r}_n, t)e^{-i\omega t} + \sigma_-^n \vec{p}^* \cdot \vec{E}(\vec{r}_n, t)e^{i\omega t}], \quad (6.14)$$

where we have assumed that  $\vec{p} \cdot \vec{E} = \vec{p}^* \cdot \vec{E}^* = 0$ . If this selection rule does not apply, (6.14) implies the so-called rotating wave approximation, which consists in neglecting terms in  $\sigma_+^n e^{i\omega t}$  and  $\sigma_-^n e^{-i\omega t}$  on the grounds that these contribute only to small double-frequency terms in the final result. The total Hamiltonian now being  $\mathcal{H}(t) = \mathcal{H}_A + \mathcal{H}_{AF}(t)$ , the equations of motion of the expectation values take the form

$$\langle \dot{\sigma}_-^n \rangle = -i\omega \langle \sigma_-^n \rangle - (i/2\hbar) \vec{p} \cdot \vec{E}^*(\vec{r}_n, t) \langle \sigma_z^n \rangle e^{-i\omega t}, \quad (6.15a)$$

$$\langle \dot{\sigma}_z^n \rangle = (i/\hbar) \vec{p} \cdot \vec{E}^*(\vec{r}_n, t) \langle \sigma_-^n \rangle e^{-i\omega t} + \text{c. c.}, \quad (6.15b)$$

which are just the ordinary Bloch equations.<sup>8</sup>

For a pointlike medium  $\vec{E}(\vec{r}_n, t) \simeq \vec{E}(t)$  for all  $n$ 's, and Eqs. (6.15) can be summed over  $n$ , giving

$$\langle \dot{J}_- \rangle = -i\omega \langle J_- \rangle - (i/\hbar) \vec{p} \cdot \vec{E}^*(t) \langle J_z \rangle e^{-i\omega t}, \quad (6.16a)$$

$$\langle \dot{J}_z \rangle = (i/2\hbar) \vec{p} \cdot \vec{E}^*(t) \langle J_+ \rangle e^{-i\omega t} + \text{c. c.} \quad (6.16b)$$

These equations could have been derived directly from (6.10) by making a self-consistent-field approximation  $\langle J_\mu a \rangle = \langle J_\mu \rangle \langle a \rangle$ , and writing

$$\vec{E}e^{i\omega t} = 2\sum_{\vec{k}} i\hat{e}_{\vec{k}} (\hbar\omega_{\vec{k}}/2\epsilon_0 V)^{1/2} \langle a_{\vec{k}}^\dagger \rangle. \quad (6.17)$$

For an extended medium, assuming that the spatial dependence of  $\vec{E}$  is strictly of the form  $e^{-i\vec{k}\cdot\vec{r}}$ , the summation of (6.15) over  $n$  gives equations similar to (6.16) but where  $J_{\pm}$  is replaced by  $J_{\pm}^{\vec{k}}$ . These equations could also be derived from (6.9) by a self-consistent field approximation.

#### D. Emission Rates

Returning to the pointlike medium, it is interesting to calculate, using first-order perturbation theory, the emission probability of single photons for systems which are initially either in a Dicke state or in a Bloch state. The interaction Hamiltonian is, from (6.6a),  $\mathcal{H}_{AF} = \sum_{\vec{k}} \mathcal{H}_{\vec{k}}$  with

$$\mathcal{H}_{\vec{k}} = \hbar g_{\vec{k}} a_{\vec{k}} J_+ + \hbar g_{\vec{k}}^* a_{\vec{k}}^\dagger J_-. \quad (6.18)$$

The transition probability from an initial atomic state  $i$  to a final atomic state  $f$ , with emission of one photon in mode  $\vec{k}$ , is simply

$$W_{\vec{k}} = (2\pi/\hbar) \delta(\hbar\omega - \hbar\omega_{\vec{k}}) |\langle f, n_{\vec{k}} + 1 | \mathcal{H}_{\vec{k}} | i, n_{\vec{k}} \rangle|^2, \quad (6.19)$$

where the second symbol in the state labeling indicates the photon number in the field. For a single atom initially in the upper state  $|\psi_1\rangle$ , and in the vacuum of photons ( $n_{\vec{k}} = 0$  for all  $\vec{k}$ ), the usual spontaneous emission intensity in the solid angle  $d\Omega$  around  $\vec{k}$  is obtained by summing over  $\vec{k}$  in this solid angle,

$$I_0 d\Omega = \hbar\omega \sum_{\vec{k} \text{ in } d\Omega} W_{\vec{k}} = \frac{\omega^4 |\hat{e}_{\vec{k}} \cdot \vec{p}|^2}{8\pi^2 \epsilon_0 c^3} d\Omega. \quad (6.20)$$

For many atoms initially in the Dicke state  $|M\rangle$  and making transition to the state  $|M-1\rangle$ , one obtains, using  $\langle M-1 | J_- | M \rangle = [(J+M)(J-M+1)]^{1/2}$ , the spontaneous emission intensity

$$I_D = (J+M)(J-M+1)I_0. \quad (6.21a)$$

This is the familiar result of Dicke<sup>7</sup> indicating superradiance for  $M \simeq 0$ . On the other hand, the stimulated intensity in a certain mode  $\vec{k}$ , which is proportional to

$$|\langle M-1 | J_- | M \rangle|^2 - |\langle M+1 | J_+ | M \rangle|^2 = 2M, \quad (6.22a)$$

is the normal stimulated rate, simply proportional to the population inversion.

If the initial state is the Bloch state  $|\theta, \varphi\rangle$ , the spontaneous emission intensity is

$$\begin{aligned} I_B &= \sum_M |\langle M | J_- | \theta, \varphi \rangle|^2 I_0 = \langle \theta, \varphi | J_+ J_- | \theta, \varphi \rangle I_0 \\ &= [J^2 \sin^2 \theta + 2J \sin^4(\frac{1}{2}\theta)] I_0, \end{aligned} \quad (6.21b)$$

TABLE II. Comparison of emission rates and dipole moments of Dicke and Bloch states of the same energy expectation value.

	Dicke state $ M = -J \cos \theta\rangle$	Bloch state $ \theta, \varphi\rangle$
Spontaneous emission	$[J^2 \sin^2 \theta + 2J \sin^2 \frac{1}{2} \theta] I_0$	$[J^2 \sin^2 \theta + 2J \sin^4 \frac{1}{2} \theta] I_0$
Stimulated emission	$-2J \cos \theta$	$-2J \cos \theta$
Dipole moment	0	$J \sin \theta (\vec{p} e^{i\omega t + i\varphi} + \text{c. c.})$
Classical emission	0	$[J^2 \sin^2 \theta] I_0$

the last equality being obtained from Eqs. (A2a) and (B6). This is also superradiant for  $\theta \simeq \frac{1}{2}\pi$ . The intensity can be compared to that obtained from the Dicke state having the same energy expectation value. This fact was already noted by Senitzky.<sup>3</sup> With  $M = -J \cos \theta$  one obtains from (6.21a)  $I_D = [J^2 \sin^2 \theta + 2J \sin^2(\frac{1}{2}\theta)] I_0$ . This is practically identical to (6.21b) for  $J \gg 1$ . Similarly, the stimulated intensity is now proportional to

$$\sum_M (|\langle M | J_- | \theta, \varphi \rangle|^2 - |\langle M | J_+ | \theta, \varphi \rangle|^2) = -2J \cos \theta. \quad (6.22b)$$

This is identical to the stimulated intensity obtained from the Dicke state having the same mean energy.

Though their radiation rates are very similar, there is a considerable difference between Dicke and Bloch states of the same mean energy. This is seen by calculating the expectation value of the total dipole moment  $\vec{D}$ . In the Heisenberg picture, using the unperturbed Hamiltonian  $\mathcal{H} = \mathcal{H}_A + \mathcal{H}_F$  (which amounts to neglecting the field reaction, and is equivalent to first-order perturbation theory), this dipole moment is

$$\begin{aligned} \vec{D} &= e^{i\mathcal{H}t/\hbar} \sum_n \vec{d}_n e^{-i\mathcal{H}t/\hbar} \\ &= \vec{p} e^{i\omega t} J_+ + \vec{p}^* e^{-i\omega t} J_-, \end{aligned} \quad (6.23)$$

where  $\vec{d}_n$  is given by (6.5). For the Dicke state  $|M\rangle$  one has

$$\langle M | \vec{D} | M \rangle = 0, \quad (6.24a)$$

whereas for the Bloch state  $|\theta, \varphi\rangle$ ,

$$\langle \theta, \varphi | \vec{D} | \theta, \varphi \rangle = J \sin \theta (\vec{p} e^{i\omega t + i\varphi} + \vec{p}^* e^{-i\omega t - i\varphi}). \quad (6.24b)$$

This last result is easily obtained using the technique of Appendix B, or the relation  $\langle \theta, \varphi | J_x | \theta, \varphi \rangle = -J \cos \theta$  together with the eigenvalue equations (3.10d) and (3.10e). While the Dicke state has zero expectation value for the dipole moment, the Bloch state is characterized by a macroscopic di-

pole which is able to radiate classically. In the wave zone, the classical radiation intensity of this dipole turns out to be

$$I_C = (J^2 \sin^2 \theta) I_0. \quad (6.21c)$$

This is practically identical to (6.21b), except for  $\theta$  close to  $\pi$ . For the totally inverted state there is no classical emission, whereas first-order perturbation theory gives a finite intensity  $2JI_0$ . These results are summarized in Table II. An inspection of this table suggests that Bloch states are the closest quantum analogs to "classical" atomic states, in the same manner as Glauber states are the quantum analogs of classical fields.

## VII. SEMICLASSICAL LIMIT

### A. Semiclassical Theorem

A known property of field coherent states is that they can be produced by classical currents acting on the vacuum of photons.<sup>2,4</sup> A similar property exists for Bloch states.

*An atomic system of small dimension which is initially in a Bloch state, or in particular in its ground state, and which is acted upon by classical fields, remains in a Bloch state.* By a classical field is meant that the interaction Hamiltonian is of the form (6.14) or, for a point system,

$$\mathcal{H}_{AF} = -\frac{1}{2} \vec{p} \cdot \vec{E}^*(t) J_+ e^{-i\omega t} - \frac{1}{2} \vec{p}^* \cdot \vec{E}(t) J_- e^{-i\omega t}. \quad (7.1a)$$

In the interaction picture this Hamiltonian becomes

$$\mathcal{H}_{AF} = -\frac{1}{2} \vec{p} \cdot \vec{E}^*(t) J_+ - \frac{1}{2} \vec{p}^* \cdot \vec{E}(t) J_-. \quad (7.1b)$$

As the field  $\vec{E}$  is turned on, the state vector  $|t\rangle$ , which was previously fixed in this picture and equal to  $|\theta, \varphi\rangle$ , changes according to

$$i\hbar \frac{\partial |t\rangle}{\partial t} = \mathcal{H}_{AF} |t\rangle. \quad (7.2)$$

Writing  $\vec{p}^* \cdot \vec{E} = i|\vec{p}^* \cdot \vec{E}| e^{i\varphi'}$ , this equation gives, for a small time increment  $\Delta t$ ,

$$\begin{aligned} |(t + \Delta t)\rangle &= [1 - i(\Delta t) |\vec{p}^* \cdot \vec{E}| / \hbar] \\ &\quad \times (J_x \sin \varphi' - J_y \cos \varphi') | \theta, \varphi \rangle. \end{aligned} \quad (7.3)$$

The operator within the bracket on the right-hand side produces an infinitesimal rotation by an angle  $\theta' = \Delta t |\vec{p}^* \cdot \vec{E}| / \hbar$  around an axis  $(\sin \varphi', -\cos \varphi', 0)$ . Using Eq. (A9), one has from (7.3)

$$\begin{aligned} |(t + \Delta t)\rangle &= R_{\theta', \varphi} R_{\theta, \varphi} | -J \rangle = R_{\theta, \varphi} e^{-i\psi J_x} | -J \rangle \\ &= e^{i\psi J} | \theta, \varphi \rangle. \end{aligned} \quad (7.4)$$

The new state is therefore a Bloch state, and the statement is proved. In the spirit of Sec. VI B, this also applies to an extended medium excited by a field  $\vec{E}(t) e^{i\omega t - i\vec{k} \cdot \vec{r}}$ . In this case however the reaction of the medium on the field will not allow, in general, the preparation of a strictly space-in-

dependent field  $\vec{E}(t)$ . This is related to the concepts of maximum cooperation number and cooperation time which have been exposed elsewhere.<sup>26</sup> It leads to the leakage into Bloch states of other symmetry types, to be treated at the end of this section.

The results concerning the correspondence between Bloch states and classical dipoles, Glauber states and classical fields, and the production of coherent states of one kind by classical excitation of the other kind have been condensed in Table III.

#### B. Evolution of Any Pure State

From (7.2) and (7.3) it is clear that the evolution operator, which is an infinite product of in-

$$\begin{aligned} & \begin{bmatrix} e^{-i\alpha/2} \cos \frac{1}{2} \beta e^{-i\tau/2} & -e^{-i\alpha/2} \sin \frac{1}{2} \beta e^{i\tau/2} \\ e^{i\alpha/2} \sin \frac{1}{2} \beta e^{-i\tau/2} & e^{i\alpha/2} \cos \frac{1}{2} \beta e^{i\tau/2} \end{bmatrix} \\ &= \begin{bmatrix} \cos \frac{1}{2} \xi - in_3 \sin \frac{1}{2} \xi & -i(n_1 - in_2) \sin \frac{1}{2} \xi \\ -i(n_1 + in_2) \sin \frac{1}{2} \xi & \cos \frac{1}{2} \xi + in_3 \sin \frac{1}{2} \xi \end{bmatrix} = (1 + |T|^2)^{-1/2} \begin{bmatrix} e^{-i\Psi/2} & T e^{i\Psi/2} \\ -T^* e^{-i\Psi/2} & e^{i\Psi/2} \end{bmatrix}, \quad (7.5b) \end{aligned}$$

where

$$T = e^{-i\Phi} \tan \frac{1}{2} \Theta. \quad (7.5c)$$

These equalities lead to the appropriate relations, between the various variables.

If we study the evolution from the ground state which is the product state  $|\phi_{i_1 \dots i_N}\rangle$  with  $i_n = 2$  in (1.4), we note that under the rotation operation (7.5a) the state remains a product of single-atom wave functions

$$|\psi^n(t)\rangle = \alpha^n(t) |\psi_1^n\rangle + \beta^n(t) |\psi_2^n\rangle, \quad (7.6)$$

where the coefficients  $\alpha^n(t)$ ,  $\beta^n(t)$  are all equal (for a point system) and determined directly from the transformation matrix (7.5b). The global atomic state is then

$$|(t)\rangle = \sum_{\text{all } \vec{N}} [\alpha(t)]^{n_1} [\beta(t)]^{n_2} |\vec{N}\rangle, \quad (7.7)$$

where  $\vec{N} \equiv (i_1, i_2, \dots, i_N)$  is a short-hand rotation equivalent to (1.4). It is convenient to collect together all states  $|\vec{N}\rangle$  with  $n_1$  one's and  $n_2$  two's as follows:

$$\sum'_{(n_1, n_2)} |\vec{N}\rangle = \binom{N}{n_1}^{1/2} \left\{ \sum'_{(n_1, n_2)} |\vec{N}\rangle / \binom{N}{n_1}^{1/2} \right\}, \quad (7.8)$$

where  $\sum'_{(n_1, n_2)}$  describes a summation over all distinct permutations of  $n_1$  one's and  $n_2$  two's. The state within the curly bracket on the right-hand side of (7.8) is fully symmetric under the permutation group and normalized to unity. Therefore, it is the Dicke state

$$| \begin{smallmatrix} J=N/2 \\ M=(n_1-n_2)/2; 1^N, 0 \end{smallmatrix} \rangle$$

finitesimal rotations, is itself a rotation. This rotation  $R$  can be represented by its three Euler angles  $(\alpha, \beta, \gamma)$ , by a rotation  $\xi$  about an arbitrary axis  $(n_1, n_2, n_3)$ , by the product of a rotation around  $\hat{z}$  by  $\Psi$  with a rotation  $R_{\Theta, \Psi}$ , or in any other convenient manner. These various ways of expressing the evolution operator are easily related using the techniques of Appendixes A and D. For example one simply writes

$$R = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_x} = e^{-i\xi(\hat{n} \cdot \vec{J})} = R_{\Theta, \Psi} e^{-i\Psi J_z}, \quad (7.5a)$$

which leads, using Eqs. (A1) and (A12), to

introduced in Sec. V. Therefore, (7.7) becomes

$$\begin{aligned} |(t)\rangle &= \sum_{M=-J}^J \binom{N}{n_1}^{1/2} \alpha^{n_1} \beta^{n_2} | \begin{smallmatrix} N/2 \\ M; 1^N, 0 \end{smallmatrix} \rangle \\ &= \sum_{M=-J}^J \mathfrak{D}_{M, -J}^J(R) | \begin{smallmatrix} N/2 \\ M; 1^N, 0 \end{smallmatrix} \rangle \\ &= (\beta/|\beta|)^{2J} | \begin{smallmatrix} N/2 \\ 2 \arctan(|\alpha|/|\beta|), \arg(\beta/\alpha); 1^N, 0 \end{smallmatrix} \rangle. \quad (7.9) \end{aligned}$$

The second equality follows by definition<sup>14</sup> of the irreducible representation  $\mathfrak{D}_{M, M'}^J(R)$  for a rotation applied to the  $M' = -J$  state, and the third equality follows from the last form of (7.5a) and the definition of the Bloch state (3.8).

If the initial state is an arbitrary Dicke state  $| \begin{smallmatrix} J \\ M; \hat{\lambda}; i \end{smallmatrix} \rangle$  one has similarly,

$$|(t)\rangle = R | \begin{smallmatrix} J \\ M; \hat{\lambda}; i \end{smallmatrix} \rangle = \sum_{M'} \mathfrak{D}_{M', M}^J(R) | \begin{smallmatrix} J \\ M'; \hat{\lambda}; i \end{smallmatrix} \rangle, \quad (7.10)$$

TABLE III. Classical excitation and coherent states. (The single arrows indicate the direction of production of coherent states starting from classical states. The double arrows indicate states connected by the correspondence principle.)

	Atomic states	Field states
Classical states	Classical current	Classical field
Quantum states	Coherent atomic state	Coherent field state



which, for large  $J$ , is a rather complicated relation in view of Eq. (D6).

### C. Evolution of Statistical Mixture of States

A real physical system will exist, in general, not in a given state, but rather in a statistical mixture of states. Upon application of a classical field it would then be necessary to deal with a summation over states of the type (7.10), which are rather unwieldy to handle except for those arising from the ground state  $M = -J$ , or the most excited state  $M = +J$ .

Fortunately, it can easily be seen, using (5.2b), that

$$\sum_{x=0}^{\lambda_2-1} \dim \Gamma^{N-x,x} \ll \dim \Gamma^{N-\lambda_2,\lambda_2}, \quad (7.11a)$$

as long as

$$\lambda_2 \ll \frac{1}{3}(N+1). \quad (7.11b)$$

The total number of states with fixed- $M$  value lying to the left (Fig. 2) of the states with  $J = |M|$  is given by the left-hand side of (7.11a). The right-hand side gives the total number of states with  $J = |M|$ . Therefore in a statistical mixture in which only those states with  $\lambda_2 \ll \frac{1}{3}N$  are significantly populated, it is possible to neglect all states of the form (7.10), except for those with  $M = -J$ . In short, for fixed negative  $M$  value such that  $|M| \ll \frac{1}{3}N$  it is sufficient to study those states

$$| \frac{|M|}{M}, N/2 + |M|, N/2 - |M| \rangle, \quad (7.12)$$

since there are far fewer states with the same  $M$  value in all other invariant subspaces combined. The evolution of the states (7.12) is given by (7.9).

### D. Evolution of Extended Systems

An extended system, initially in a product state, and acted upon by a classical field, remains in a product state. This is easy to see from the interaction Hamiltonian (6.14). The infinitesimal evolution operator, in the interaction picture, can be written

$$\begin{aligned} U(t, t + \Delta t) &= 1 - \frac{i\Delta t}{\hbar} \mathcal{H}_{AF}^{\text{int}}(t) \\ &= \prod_{n=1}^N \left( 1 - \frac{i\Delta t}{\hbar} \mathcal{H}_n \right) \\ &= \prod_{n=1}^N U_n(t, t + \Delta t), \end{aligned} \quad (7.13a)$$

where

$$\mathcal{H}_n = -\frac{1}{2} \sigma_n^+ \vec{p} \cdot \vec{E}^*(\vec{r}_n, t) - \frac{1}{2} \sigma_n^- \vec{p}^* \cdot \vec{E}(\vec{r}_n, t). \quad (7.13b)$$

$U$  is therefore a product of infinitesimal rotations  $U_n$  applied to each single system. Hence the state remains a product state.

Let us take, for example, the evolution from

the ground state  $|\phi_{22\dots 22}\rangle$ . The coefficients of the single-atom wave functions in (7.6) depend now on the index  $n$ . The total atomic state cannot be written as simply as (7.7). We write

$$| (t) \rangle = \sum_{\text{all } \vec{N}} \{ \alpha^n, \beta^n, \vec{N} \} | \vec{N} \rangle, \quad (7.14)$$

where the notation  $\{ \alpha, \beta, \vec{N} \}$  means the product  $\gamma^1 \gamma^2 \dots \gamma^N$ , where  $\gamma^n$  stands for  $\alpha^n$  or  $\beta^n$  depending on whether the  $n$ th vector component of  $\vec{N}$  is 1 or 2. We collect together all states  $|\vec{N}\rangle$  with  $n_1$  one's and  $n_2$  two's. These correspond to a fixed  $M$  value. Any of these product states can be expanded in terms of the states  $| \frac{J}{M}; \vec{\lambda} \rangle$  with fixed  $M$  value, in the spirit of (5.9). One obtains

$$| (t) \rangle = \sum_M \sum_{\vec{\lambda}, i} | \frac{J}{M}; \vec{\lambda} \rangle \sum'_{(n_1, n_2)} (C_{\vec{\lambda}; iM}^{\vec{N}})^{-1} \{ \alpha^n, \beta^n, \vec{N} \}, \quad (7.15)$$

where  $\sum'_{(n_1, n_2)}$  has the same meaning as in (5.9). In the case of the point system,  $\{ \alpha^n, \beta^n, \vec{N} \}$  depends on  $n_1$  and  $n_2$  but not on which particular permutation of  $n_1$  one's and  $n_2$  two's is selected. As a result, the last sum on the right-hand side of (7.15) is zero in that case except for the totally symmetric state  $\vec{\lambda} = (N, 0)$ ;  $i = 1$ . In the present case all values of  $\vec{\lambda}$  and  $i$  are, in general, obtained, in spite of the fact that we started from a state with  $\vec{\lambda} = (N, 0)$ ;  $i = 1$ . We speak of a *leakage from symmetrized states*. It is due to the fact that the atoms have different space dependences, which allow them to be distinguished, and violate (5.1). A similar difficulty will, of course, occur in a fully quantum treatment. The situation can be saved with a single mode by defining  $\vec{k}$ -dependent Bloch states (Sec. VI B). Similarly, the situation can be saved for classical fields if the spatial dependence is strictly of the form  $e^{-i\vec{k} \cdot \vec{r}}$ . In these cases, however, leakage will also occur in the presence of atomic motion.

## VIII. CALCULATION OF THERMAL AVERAGES

In this last section we show how the disentangling theorem, the contraction procedure, and other group theoretical methods presented in the Appendixes, can be used to obtain statistical results beyond those considerations presented in Secs. IID, IIID, and VIIC. It frequently happens that thermal expectation values must be taken for operators which can be written as exponentials of other operators. Under these circumstances the Baker-Campbell-Hausdorff formulas may be used to compute these thermal averages explicitly.

To begin we consider a single column of Fig. 2. The  $2J+1$  energy levels are assumed to be equally spaced by  $E = \hbar\omega$ . The partition function is

$$Z = \text{Tr} e^{-\beta E J_z}, \quad \beta = 1/kT. \quad (8.1)$$

Let us compute the thermal average of the exponential operator  $e^{i(\alpha_+ J_+ + \alpha_- J_-)}$ ,

$$\langle e^{i(\alpha_+ J_+ + \alpha_- J_-)} \rangle_J = \frac{\text{Tr} e^{-\beta E J_z} e^{i(\alpha_+ J_+ + \alpha_- J_-)}}{\text{Tr} e^{-\beta E J_z}}. \quad (8.2)$$

This is most easily done by considering, in the spirit of Appendix A, the  $J = \frac{1}{2}$ ,  $2 \times 2$  matrix representation of the associated group. Then the  $2 \times 2$  matrix whose trace is to be taken in the numerator of (8.2) is

$$\begin{bmatrix} \cos \gamma e^{-\beta E / 2} & i \alpha_+ (\sin \gamma / \gamma) e^{-\beta E / 2} \\ i \alpha_- (\sin \gamma / \gamma) e^{\beta E / 2} & \cos \gamma e^{\beta E / 2} \end{bmatrix}, \quad (8.3)$$

where  $\gamma^2 = \alpha_+ \alpha_-$ . This results directly from the use of (A1). In order to simplify the calculation for higher- $J$  values, it is useful to transform (8.3) by a unitary transformation. The matrix being non-Hermitian it cannot be diagonalized. However, it can be transformed to upper triangular form. Both the trace and the determinant being preserved by this transformation, the diagonal elements  $\lambda_+$ ,  $\lambda_-$  of this upper triangular matrix are simply given by

$$\lambda_+ + \lambda_- = \cos \gamma (e^{-\beta E / 2} + e^{\beta E / 2}), \quad \lambda_+ \lambda_- = 1. \quad (8.4)$$

We write

$$\lambda_+ = 1/\lambda_- \equiv \lambda = \frac{1}{2} \left\{ \cos \gamma (e^{-\beta E / 2} + e^{\beta E / 2}) + [\cos^2 \gamma (e^{\beta E / 2} + e^{-\beta E / 2})^2 - 4]^{1/2} \right\}. \quad (8.5)$$

The trace in the numerator of (8.2) is written

$$\lambda + \frac{1}{\lambda} = \sum_{m=-1/2}^{1/2} (\lambda)^{2m} = \frac{\lambda^2 - \lambda^{-2}}{\lambda - \lambda^{-1}}. \quad (8.6)$$

The denominator is obtained by setting  $\alpha_+ = \alpha_- = 0$ .

To evaluate the trace for arbitrary  $J$  we merely observe that once (8.3) has been transformed to an upper triangular form all diagonal elements in the representation  $\mathcal{D}^J$  are simply powers of  $\lambda$ . The trace is immediately obtained by extending the summation in (8.6) from  $M = -J$  to  $M = +J$ . The result is

$$\langle e^{i(\alpha_+ J_+ + \alpha_- J_-)} \rangle_J = \frac{[\lambda^{2J+1}(\gamma) - \lambda^{-2J-1}(\gamma)] / [\lambda(\gamma) - \lambda^{-1}(\gamma)]}{[\lambda^{2J+1}(0) - \lambda^{-2J-1}(0)] / [\lambda(0) - \lambda^{-1}(0)]}. \quad (8.7)$$

Now we consider the entire tableau of Fig. 2. The traces are simply obtained by summing over columns

$$\langle e^{i(\alpha_+ J_+ + \alpha_- J_-)} \rangle_N = \frac{\sum_J \dim \Gamma^{\lambda_1, \lambda_2} \text{Tr}_J e^{-\beta E J_z} e^{i(\alpha_+ J_+ + \alpha_- J_-)}}{\sum_J \dim \Gamma^{\lambda_1, \lambda_2} \text{Tr}_J e^{-\beta E J_z}}, \quad (8.8)$$

where  $(\lambda_1, \lambda_2)$  are related to  $N$  and  $J$  by (5.8), and  $\dim \Gamma^{\lambda_1, \lambda_2}$  is given by (5.2b), which can also be written

$$\dim \Gamma^{\lambda_1, \lambda_2} = \binom{N}{\frac{1}{2}N - J} - \binom{N}{\frac{1}{2}N - J - 1}. \quad (8.9)$$

The summation over  $J$  in (8.8) starts from 0 or  $\frac{1}{2}$ ,

and proceeds in integer steps up to  $\frac{1}{2}N$ . Using (8.7) one can write, for the numerator of (8.8),

$$\begin{aligned} & \sum_J \left\{ \binom{N}{\frac{1}{2}N - J} - \binom{N}{\frac{1}{2}N - J - 1} \right\} \sum_{M=-J}^J (\lambda^2)^M \\ &= \sum_{M=-N/2}^{N/2} (\lambda^2)^M \sum_{J=|M|}^{N/2} \left\{ \binom{N}{\frac{1}{2}N - J} - \binom{N}{\frac{1}{2}N - J - 1} \right\}. \end{aligned} \quad (8.10)$$

The last sum in (8.10) is equal to  $\binom{N}{N/2-M}$ . The entire expression reduces then to  $(\lambda + \lambda^{-1})^N$ . Using (8.4) this gives

$$\langle e^{i(\alpha_+ J_+ + \alpha_- J_-)} \rangle_N = \left( \frac{\lambda(\gamma) + \lambda^{-1}(\gamma)}{\lambda(0) + \lambda^{-1}(0)} \right)^N = (\cos \gamma)^N, \quad (8.11)$$

which is temperature independent. This result depends on the fact that the diagonal elements, in the  $2 \times 2$  representation, of the operator whose trace is to be taken are equal. This causes the temperature dependence to factor in the trace (8.4).

Returning to (8.7), we note that this formula can be contracted using the procedure outlined in Table I, and writing  $\alpha_{\pm} = c\gamma_{\pm}$ . After some simple manipulations one obtains

$$\frac{\text{Tr} e^{-\beta \mathcal{H}} e^{i(\gamma_+ a^{\dagger} + \gamma_- a)}}{\text{Tr} e^{-\beta \mathcal{H}}} = \exp \left[ -\frac{1}{2} \gamma_+ \gamma_- \coth \left( \frac{1}{2} \hbar \omega \beta \right) \right]. \quad (8.12)$$

This is a familiar result.<sup>27-29</sup> Equations such as (8.7) and (8.12) are encountered in many physical applications. One example is the calculation of the intensity of a beam scattered by atoms in thermal motion in a lattice.<sup>29,30</sup> Using the techniques which have been exposed here all averages of the type (8.7), (8.11), and (8.12) can be obtained very simply. In order to disentangle exponentials containing harmonic-oscillator operators, it is often more practical to disentangle first similar expressions with spin operators, and to follow this by a contraction. Examples are given in Appendix A.

## IX. CONCLUSION

Starting from the direct-product representation of  $N$  two-level atoms, we have introduced two representations which have been called the Dicke and the Bloch representations. The first one is very suitable to describe problems of cooperative spontaneous emission of a radiation field, insofar as it displays strong atom-atom correlations.<sup>7,25,26</sup> The second one is suitable to describe the resonant interaction of a classical field and of a set of atoms. A qualitative statement on the difference between these two representations has already been given by Senitzky,<sup>3</sup> but this difference has seldom been taken into account in the solution of quantum-electrical problems. In the previous sections we have shown the formal properties of

the two representations, and their relevance to radiation and statistical problems. The properties of the Dicke and Bloch representations are similar, respectively, to those of two well-known representations of the harmonic-oscillator Hilbert space, namely, the Fock (or energy eigenstates) and the Glauber (or annihilation-operator eigenstates) representations. Besides displaying a series of similarities, the two atomic representations contract into the corresponding harmonic-oscillator representations in the limit  $N \rightarrow \infty$ .

In view of their paramount importance in many quantum-electrical problems, we briefly summarize here the main results concerning Bloch states. The atomic coherent states, and the operators involved in their description, obey a number of properties: (i) The states are defined by a unitary transformation operator acting on the ground state; (ii) the states obey simple eigenvalue equations; (iii) these states are nonorthogonal and overcomplete; (iv) the angular momentum operators obey a large number of Baker-Campbell-Hausdorff formulas; (v) within a fixed Bloch subspace the statistical operators have a diagonal representation in the coherent-state representation; (vi) generating functions for normal, anti-normal, and fully symmetrized orderings of powers of the operators  $J_+$ ,  $J_z$ , and  $J_-$  can be constructed; (vii) minimum-uncertainty relations for noncommuting operators can be constructed within the atomic coherent states.

The relationship between the atomic and field coherent states has been effected using a group-contraction procedure. Specifically, the Lie algebra of the group  $U(2)$  is contracted to the "harmonic-oscillator algebra." In this limit the commutation properties of the four operators  $J_z$ ,  $J_+$ ,  $J_-$ , and  $J_0$  go over to the commutation relations of the operators  $a^\dagger a$ ,  $a^\dagger$ ,  $a$ , and  $I$ , respectively. In the limit  $c \rightarrow 0$  the Bloch sphere surface contracts to the phase plane of the harmonic oscillator. The matrix elements of the shift operators  $J_+$ ,  $J_-$  contract to the matrix elements of the creation and an-

nihilation operators  $a^\dagger$ ,  $a$ . All properties of the atomic coherent states (listed above) contract immediately to the corresponding well-known properties of the field coherent states. We have also shown how, in this limit, the spherical harmonics, as well as their orthogonality and completeness relations, contract to the harmonic-oscillator eigenfunctions and their orthogonality and completeness relations. Finally, we have shown how the Baker-Campbell-Hausdorff (BCH) formulas for  $SU(2)$  contract to the BCH relations useful and familiar for the field coherent states.

As this work was being completed we became aware that Radcliffe has defined coherent spin states analogous to the atomic coherent states presented here, and that he derived their overlap and completeness properties.<sup>31</sup> On the other hand, Barut *et al.* have defined coherent states for a group different from ours, but contracted these states to harmonic-oscillator coherent states.<sup>32</sup> It should also be noted that the angular momentum coherent states defined by Arkins *et al.*<sup>33</sup> are based on a Hilbert space different from ours, and therefore are not appropriate to the description of  $N$  two-level atoms.

#### APPENDIX A: DISENTANGLING THEOREM FOR ANGULAR MOMENTUM OPERATORS

In dealing with noncommuting exponential operators it is very useful to be able to change a symmetrized exponential operator into an ordered product of exponential operators. The well-known BCH formula (2.11) is of this type. Similar expressions can be obtained for angular momentum operators. We proceed to the derivation of these expressions by first considering the  $2 \times 2$  matrix representation of the rotation group's algebra,

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which is the faithful representation of smallest dimension.

By Maclaurin series expansion one finds

$$e^{w_+ J_+ + w_- J_- + w_z J_z} = \begin{pmatrix} \cosh K + \frac{1}{2} w_z (\sinh K)/K & w_+ (\sinh K)/K \\ w_- (\sinh K)/K & \cosh K - \frac{1}{2} w_z (\sinh K)/K \end{pmatrix}, \quad K = (w_+ w_- + \frac{1}{4} w_z^2)^{1/2}. \quad (\text{A1})$$

and similarly,

$$e^{x_+ J_+ + e^{(i \ln x_z) J_z} e^{x_- J_-} = \begin{pmatrix} (x_z)^{1/2} + x_+ x_- / (x_z)^{1/2} & x_+ / (x_z)^{1/2} \\ x_- / (x_z)^{1/2} & 1 / (x_z)^{1/2} \end{pmatrix}, \quad (\text{A2})$$

$$e^{y_- J_-} e^{(i \ln y_z) J_z} e^{y_+ J_+} = \begin{pmatrix} (y_z)^{1/2} & y_+ (y_z)^{1/2} \\ y_- (y_z)^{1/2} & 1 / (y_z)^{1/2} + y_+ y_- (y_z)^{1/2} \end{pmatrix}. \quad (\text{A3})$$

Equating these three matrices element by element gives expressions for each set of coefficients in terms of the others. This procedure gives four equations for three variables, but since the  $J$  matrices are traceless, the determinant of each group operation (A1)–(A3) is unity; therefore only three of these equations are independent. The applicability of the resulting operator equation

$$e^{w_+ J_+ + w_- J_- + w_z J_z} = e^{x_+ J_+} e^{(\ln x_z) J_z} e^{x_- J_-} \\ = e^{y_- J_-} e^{(\ln y_z) J_z} e^{y_+ J_+} \quad (\text{A4})$$

is not restricted to the  $2 \times 2$  matrix representation. The algebra of infinitesimal rotation operators maps onto the rotation group, which is represented by the exponential operators. Any relation between exponential operators, i. e., between group operations, which is valid for one particular faithful representation of the group remains valid for all others. Therefore the equalities (A4) are general.

In general the parameters  $w$ ,  $x$ ,  $y$  appearing in (A1)–(A3) are arbitrary complex numbers. The  $2 \times 2$  matrices given explicitly are then complex  $2 \times 2$  matrices with determinant +1: i. e., members of the special linear group  $Sl(2, c)$ . Reality restrictions on  $w$ ,  $x$ ,  $y$  lead to different subgroups of  $Sl(2, c)$ . The condition that the parameters be real restricts consideration to the subgroup  $Sl(2, r)$  of  $Sl(2, c)$ . The condition  $w_z$  real,  $w_+^* = -w_-$ , and similar conditions for  $x$ ,  $y$  is equivalent to the restriction to the subgroup  $SU(2)$  of  $Sl(2, c)$ . The BCH formula (A4) is valid for the group  $Sl(2, c)$ . It is also valid for its real forms (subgroups) provided the given set of parameters (say  $w$ ) is selected in such a way that the solution of Eqs. (A1)–(A4) for the other sets of parameters (say  $x$  and  $y$ ) satisfies the same restrictive conditions (for instance, reality) obeyed by the given set (say  $w$ ).

Using the disentangling relations, expressions for the rotation operator (3.7) are obtained:

$$R_{\theta, \varphi} = e^{\xi J_+ - \xi^* J_-} e^{\tau J_z} e^{\ln(1+|\tau|^2) J_z} e^{-\tau^* J_-} \\ = e^{-\tau^* J_-} e^{-\ln(1+|\tau|^2) J_z} e^{\tau J_+}, \quad (\text{A5})$$

where  $\xi$  and  $\tau$  are given in (3.7b) and (3.11b). If we let  $\xi = c\alpha$ ,  $\xi^* = c\alpha^*$ ,  $a^\dagger = cJ_+$ ,  $a = cJ_-$ , and  $J_z = a^\dagger a - 1/2c^2$ , following the contraction procedure of Sec. IV, Eq. (A5) gives, in the limit  $c \rightarrow 0$ ,

$$T_\alpha = e^{\alpha a^\dagger - \alpha^* a} = e^{\alpha a^\dagger} e^{-|\alpha|^2/2} e^{-\alpha^* a} \\ = e^{-\alpha^* a} e^{|\alpha|^2/2} e^{\alpha a^\dagger} \quad (\text{A6})$$

which is the BCH formula (2.11).

A more general expression can be obtained by contraction of Eq. (A4). If we let  $w_+ = c\alpha$ ,  $w_- = -c\beta^*$ , and  $w_z = 0$ , we obtain

$$e^{\alpha a^\dagger - \beta^* a} = e^{\alpha a^\dagger} e^{-\alpha\beta^*/2} e^{-\beta^* a} = e^{-\beta^* a} e^{\alpha\beta^*/2} e^{\alpha a^\dagger}. \quad (\text{A7})$$

Using this relation, together with (A6), one obtains, after some manipulations,

$$T_\alpha T_\beta = e^{(\alpha\beta^* - \alpha^*\beta)/2} T_{\alpha+\beta}, \quad (\text{A8})$$

which describes the composition of translations  $T_\alpha$ . The use of this equation allows derivation of Eq. (2.14a) very simply. Of course, Eq. (A8) can also be obtained by contraction of a similar equation for the composition of rotations:

$$R_{\theta, \varphi} R_{\Phi, \phi} = R_{\Theta, \Phi} e^{-i\Psi J_z}, \quad (\text{A9})$$

where  $\Theta$ ,  $\Phi$ , and  $\Psi$  are to be determined. We note that the  $R_{\theta, \varphi}$ 's do not form a group, since we have restricted ourselves to rotations around an axis in the  $(x, y)$  plane. It is therefore necessary to allow for a rotation around the  $z$  axis on the right-hand side of (A9). This rotation simply amounts to changing the phase factor of the single-atom eigenstates  $|\psi_i\rangle$ . The angles  $\Theta$ ,  $\Phi$ , and  $\Psi$  could be obtained by manipulating (A4) and (A5). A simpler procedure is to use the  $2 \times 2$  matrix representation as in (A1)–(A3). By application of (A1) one has

$$R_{\theta, \varphi} = \frac{1}{(1+|\tau|^2)^{1/2}} \begin{pmatrix} 1 & \tau \\ -\tau^* & 1 \end{pmatrix}, \quad (\text{A10})$$

$$e^{-i\Psi J_z} = \begin{pmatrix} e^{-i\Psi/2} & 0 \\ 0 & e^{i\Psi/2} \end{pmatrix}. \quad (\text{A11})$$

With  $\tau' = e^{-i\psi'} \tan \frac{1}{2} \theta'$  and  $T = e^{-i\Phi} \tan \frac{1}{2} \Theta$ , Eq. (A9) becomes

$$[(1+|\tau|^2)(1+|\tau'|^2)]^{1/2} \begin{pmatrix} 1 - \tau^* \tau' & \tau + \tau' \\ -\tau^* - \tau'^* & 1 - \tau \tau'^* \end{pmatrix} \\ = \frac{1}{(1+|T|^2)^{1/2}} \begin{pmatrix} e^{-i\Psi/2} & T e^{i\Psi/2} \\ -T^* e^{-i\Psi/2} & e^{i\Psi/2} \end{pmatrix}, \quad (\text{A12})$$

which determines  $\Psi$  and  $T$  in term of  $\tau$  and  $\tau'$ . The contraction of (A9) proceeds straightforwardly, and gives (A8). Equation (A9) can also be used to derive (3.14a) in a simple manner.

As a final case of interest let us show how to reentangle the expression

$$e^{\lambda a^\dagger} e^{(\gamma_+ a^\dagger + \gamma_- a)}. \quad (\text{A13})$$

The quantity is obtained from the contraction of

$$e^{\lambda J_z} e^{(\alpha_+ J_+ + \alpha_- J_-)} \\ = \exp \left( 2 \cosh \theta \sinh \frac{1}{2} \lambda J_z + \alpha_+ \frac{\sinh \theta}{\theta} e^{\lambda/2} J_+ \right. \\ \left. + \alpha_- \frac{\sinh \theta}{\theta} e^{-\lambda/2} J_- \right) \frac{\frac{1}{2} \Omega}{\sinh \frac{1}{2} \Omega}, \quad (\text{A14a})$$

where  $\theta^2 = \alpha_+ \alpha_-$ ,

$$\cosh \frac{1}{2} \Omega = \cosh \theta \cosh \frac{1}{2} \lambda. \quad (\text{A14b})$$

Making a contraction with  $\alpha_\pm = c\gamma_\pm$  and following the rules of Table I one obtains

$$e^{\lambda a^\dagger} e^{(\gamma_+ a^\dagger + \gamma_- a)}$$

$$= e^{-\lambda\gamma\tau^{-1/4}} [1 + (2/\lambda) \coth \frac{1}{2}\lambda - \coth^2 \frac{1}{2}\lambda] \\ \times \exp[\lambda a^\dagger a + (\frac{1}{2}\lambda/\sinh \frac{1}{2}\lambda) (\gamma_+ e^{\lambda/2} a^\dagger + \gamma_- e^{-\lambda/2} a)], \quad (\text{A15})$$

which is the desired result. A similar expression has been derived in a more involved way by Weiss and Maradudin<sup>30</sup> in order to calculate some thermal averages which have been obtained differently in Sec. VIII.

#### APPENDIX B: GENERATING FUNCTIONS FOR EXPECTATION VALUES WITHIN BLOCH STATES

Using the disentangling theorem of Appendix A, together with the definition of Bloch states as rotation (3.8), or equivalently (3.12), it is easy to construct generating functions for normally ordered, antinormally ordered, and symmetrized expectation values of products of powers of the operators  $J_+$ ,  $J_z$ ,  $J_-$  within Bloch states.

We define the following expectation values:

$$X_N(\alpha, \beta, \gamma) = \langle \theta, \varphi | e^{\alpha J_+} e^{\beta J_z} e^{\gamma J_-} | \theta, \varphi \rangle, \quad (\text{B1a})$$

$$X_A(\alpha, \beta, \gamma) = \langle \theta, \varphi | e^{\gamma J_-} e^{\beta J_z} e^{\alpha J_+} | \theta, \varphi \rangle, \quad (\text{B1b})$$

$$X_S(\alpha, \beta, \gamma) = \langle \theta, \varphi | e^{\alpha J_+ + \beta J_z + \gamma J_-} | \theta, \varphi \rangle, \quad (\text{B1c})$$

and will show that these functions can easily be calculated. These functions are generating functions since one has

$$\left[ \left( \frac{\partial}{\partial \alpha} \right)^a \left( \frac{\partial}{\partial \beta} \right)^b \left( \frac{\partial}{\partial \gamma} \right)^c X_n \right]_{\alpha=\beta=\gamma=0} \\ = \langle \theta, \varphi | J_+^a J_z^b J_-^c | \theta, \varphi \rangle, \quad (\text{B2a})$$

which immediately gives

$$X_A = \left( \frac{e^{-\beta/2} + e^{\beta/2}(\tau + \alpha)(\tau^* + \gamma)}{1 + |\tau|^2} \right)^{2J} = [e^{-\beta/2} \cos^2(\frac{1}{2}\theta) + e^{\beta/2} (\sin \frac{1}{2}\theta e^{-i\varphi} + \alpha \cos \frac{1}{2}\theta) (\sin \frac{1}{2}\theta e^{i\varphi} + \gamma \cos \frac{1}{2}\theta)]^{2J}. \quad (\text{B5})$$

From this expression it immediately follows that

$$X_N = \left( \frac{e^{\beta/2} |\tau|^2 + e^{-\beta/2} (\alpha\tau^* + 1)(\gamma\tau + 1)}{1 + |\tau|^2} \right)^{2J} = [e^{\beta/2} \sin^2(\frac{1}{2}\theta) + e^{-\beta/2} (\alpha e^{i\varphi} \sin \frac{1}{2}\theta + \cos \frac{1}{2}\theta) (\gamma e^{-i\varphi} \sin \frac{1}{2}\theta + \cos \frac{1}{2}\theta)]^{2J}, \quad (\text{B6})$$

$$X_S = \left( \frac{(1 + |\tau|^2) \cosh K - (1 - |\tau|^2) \frac{1}{2} \beta (\sinh K)/K + (\alpha\tau^* + \gamma\tau) (\sinh K)/K}{1 + |\tau|^2} \right)^{2J} \\ = \left[ \cosh K - \frac{1}{2} \beta \frac{\sinh K}{K} \cos \theta + (\alpha e^{i\varphi} + \gamma e^{-i\varphi}) \frac{\sinh K}{K} \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta \right]^{2J}, \quad (\text{B7a})$$

where

$$K = (\alpha\gamma + \frac{1}{4}\beta^2)^{1/2}. \quad (\text{B7b})$$

#### APPENDIX C: CONTRACTION OF SPHERICAL HARMONICS TO HERMITE POLYNOMIALS

This appendix indicates additional consequences of the contraction of the angular momentum algebra

$$\left[ \left( \frac{\partial}{\partial \alpha} \right)^a \left( \frac{\partial}{\partial \beta} \right)^b \left( \frac{\partial}{\partial \gamma} \right)^c X_A \right]_{\alpha=\beta=\gamma=0} \\ = \langle \theta, \varphi | J_-^c J_z^b J_+^a | \theta, \varphi \rangle, \quad (\text{B2b})$$

$$\left[ \left( \frac{\partial}{\partial \alpha} \right)^a \left( \frac{\partial}{\partial \beta} \right)^b \left( \frac{\partial}{\partial \gamma} \right)^c X_S \right]_{\alpha=\beta=\gamma=0} \\ = \langle \theta, \varphi | S \{ J_+^a J_z^b J_-^c \} | \theta, \varphi \rangle, \quad (\text{B2c})$$

where  $S \{ \}$  means the fully symmetrized sum of products, which is equal to the sum of all distinct permutations of the factors within the brackets divided by the number  $(a+b+c)!/a!b!c!$  of these permutations.

It is sufficient to compute one of the generating functions, as the other two are then given by using the disentangling theorem of Appendix A. It is simplest to do this for  $X_A$ . Using (3.12) one has

$$X_A = (1/[1 + |\tau|^2]^{2J}) \langle -J | e^{(\tau^* + \gamma)J_-} e^{\beta J_z} e^{(\tau + \alpha)J_+} | -J \rangle. \quad (\text{B3})$$

This expression is then put in normally ordered form (A2), in which case only the term  $e^{(\alpha\tau^*)J_z}$  contributes to the expectation value in the ground state  $\langle -J | e^{(\alpha\tau^*)J_z} | -J \rangle = 1/x_z^J$ . One obtains

$$\frac{1}{(x_z)^{1/2}} = \frac{1}{(y_z)^{1/2}} + y_+ y_- (y_z)^{1/2} \\ = e^{-\beta/2} + (\tau + \alpha)(\tau^* + \gamma) e^{\beta/2}, \quad (\text{B4})$$

(3.3) to the harmonic-oscillator algebra (2.3). The mathematical properties derived here are not directly connected to the contraction of Sec. III to Sec. II, but are of independent interest. One should note that the symbols  $\theta$  and  $\varphi$  used here are

the angular coordinates of a Schrödinger representation, and are not related to Bloch-state labels, as elsewhere in this article. The notation  $|l, m\rangle$  is used for Dicke states, in order to conform with the usual spherical harmonic notation.

We first note that the eigenvalue equations

$$J^2|l, m\rangle = l(l+1)|l, m\rangle, \quad (\text{C1a})$$

$$J_x|l, m\rangle = m|l, m\rangle \quad (\text{C1b})$$

contract to

$$[-2n_{op} + (a^\dagger a + aa^\dagger)]|\infty, n\rangle = |\infty, n\rangle, \quad (\text{C2a})$$

$$a^\dagger a|\infty, n\rangle = n|\infty, n\rangle, \quad (\text{C2b})$$

respectively, where  $n_{op}$  on the left-hand side of (C2a) is the number operator. These contractions are easily performed using the rules of Table I. Equation (C2a) can be written in a slightly different form,

$$\frac{1}{2}(a^\dagger a + aa^\dagger)|\infty, n\rangle = (n_{op} + \frac{1}{2})|\infty, n\rangle, \quad (\text{C2c})$$

where the left-hand side is now a Hamiltonian for a model physical system, while the right-hand side describes the eigenvalue spectrum for states which diagonalize  $n_{op}$ .

In the Schrödinger representation Eqs. (C1a) and (C1b) lead to wave functions  $\langle q|l, m\rangle = Y_{l,m}(\theta, \varphi)$  which are the spherical harmonics, whereas (C2b) leads to wave functions  $\langle q|\infty, n\rangle = \psi_n((\omega m/\hbar)^{1/2}q)$  which are related to the Hermite polynomials  $H_n(x)$ , where  $x = (\omega m/\hbar)^{1/2}q$ . It is therefore clear that the spherical harmonics  $Y_{l,m}(\theta, \varphi)$  can be contracted to the Hermite polynomials  $H_n(x)$ , but it remains to show how the coordinates  $\theta, \varphi$  should be contracted to  $x$ . To this effect we consider how the operators  $J_x, J_y, J_z$ , written in the coordinate representation, should be contracted according to the rules of Table I. One has

$$\begin{aligned} \lim_{c \rightarrow 0}(cJ_x) &= \lim_{c \rightarrow 0} \left[ ic \left( \sin\varphi \frac{\partial}{\partial\theta} + \cotan\theta \cos\varphi \frac{\partial}{\partial\varphi} \right) \right] \\ &= \frac{a^\dagger + a}{2} = \frac{x}{\sqrt{2}}, \end{aligned} \quad (\text{C3a})$$

$$\begin{aligned} \lim_{c \rightarrow 0}(cJ_y) &= \lim_{c \rightarrow 0} \left[ ic \left( -\cos\varphi \frac{\partial}{\partial\theta} + \cotan\theta \sin\varphi \frac{\partial}{\partial\varphi} \right) \right] \\ &= \frac{a^\dagger - a}{2i} = -\frac{p}{(2\hbar\omega m)^{1/2}} = \frac{i}{\sqrt{2}} \frac{\partial}{\partial x}, \end{aligned} \quad (\text{C3b})$$

$$\lim_{c \rightarrow 0}(2c^2J_z) = \lim_{c \rightarrow 0} \left( -2ic^2 \frac{\partial}{\partial\varphi} \right) = -1. \quad (\text{C3c})$$

Introducing the last equation in the previous two, one sees that  $\cotan\theta$  must approach zero as fast as  $c$  for the contraction to give a finite result. With  $\theta = \frac{1}{2}\pi - cX$  one then obtains

$$\lim_{c \rightarrow 0} \left( -i \sin\varphi \frac{\partial}{\partial x} + \frac{1}{2}X \cos\varphi \right) = \frac{x}{\sqrt{2}}, \quad (\text{C4a})$$

$$\lim_{c \rightarrow 0} \left( i \cos\varphi \frac{\partial}{\partial x} + \frac{1}{2}X \sin\varphi \right) = \frac{i}{\sqrt{2}} \frac{\partial}{\partial x}. \quad (\text{C4b})$$

Though  $\varphi$  could be kept arbitrary, the simplest contraction procedure is clearly to take  $\varphi = 0$ , in which case  $X = \sqrt{2}x$ . The correct limiting technique is then

$$\lim_{c \rightarrow 0}\varphi = 0, \quad \lim_{c \rightarrow 0}\theta = \frac{1}{2}\pi - c\sqrt{2}x, \quad (\text{C5a})$$

together with

$$\lim_{c \rightarrow 0}(2lc^2) = 1, \quad \lim_{c \rightarrow 0}m = -\infty, \quad l+m = n \text{ (fixed)}. \quad (\text{C5b})$$

The angular momentum eigenfunctions are,<sup>9</sup> with  $u = \cos\theta$ ,

$$Y_{l,m}(\theta, \varphi) = P_{l,m}(u) e^{im\varphi}, \quad (\text{C6a})$$

where

$$\begin{aligned} P_{l,m}(u) &= (-1)^{l+m} \frac{1}{2^l l!} \left( \frac{2l+1}{2} \right)^{1/2} \frac{(l-m)!}{(l+m)!} \\ &\quad \times (1-u^2)^{m/2} \frac{d^{l+m}}{du^{l+m}} (1-u^2) \end{aligned} \quad (\text{C6b})$$

are the associated Legendre polynomials, normalized such that

$$\int_{-1}^1 P_{l,m}^2 du = 1. \quad (\text{C6c})$$

The harmonic-oscillator eigenfunctions are<sup>9</sup>

$$\begin{aligned} \psi_n &= \frac{(-1)^n}{(2^n n! \sqrt{\pi})^{1/2}} e^{-x^2/2} \frac{d^n}{dx^n} e^{-x^2} \\ &= \frac{1}{(2^n n! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_n(x), \end{aligned} \quad (\text{C7a})$$

which are normalized such that

$$\int_{-\infty}^{+\infty} \psi_n^2(x) dx = 1. \quad (\text{C7b})$$

With  $\varphi = 0$ , we just have to contract  $P_{l,m}(u)$ , with  $u = xc/\sqrt{2} = l^{1/2}x$ , as  $c \rightarrow 0$ . Introducing this in (C6c), we see that the normalization will only be preserved if  $\psi_n$  is the limit of  $l^{-1/4}P_{l,m}$ . Indeed, we find

$$\begin{aligned} \lim_{c \rightarrow 0} l^{-1/4} P_{l,m} &= \lim_{c \rightarrow 0} (-1)^n \left( \frac{(2l)! l^{1/2}}{2^{2l} l! l!} \right)^{1/2} \left( \frac{1}{2^n n! (2lc^2)^n} \right)^{1/2} \\ &\quad \times [1 - 2c^2 x^2]^{-(1/2c^2)/2} \left( \frac{d}{dx} \right)^n [1 - 2c^2 x^2]^{1/2c^2} \\ &= (-1)^n \pi^{-1/4} \left( \frac{1}{2^n n!} \right)^{1/2} e^{-x^2/2} \left( \frac{d}{dx} \right)^n e^{-x^2} = \psi_n(x), \end{aligned} \quad (\text{C8})$$

where Stirling's approximation has been used to contract the first quantity within large parentheses.

The orthogonality of the  $\psi_n(x)$ 's results from an orthogonality relation satisfied by the associated

Legendre polynomials

$$\int_{-1}^1 P_{l,m}(u) P_{l,m'}(u) du = \delta_{mm'}, \quad (C9a)$$

which immediately gives

$$\int_{-\infty}^{+\infty} \lim_{l \rightarrow \infty} l^{-1/4} P_{l,m} \lim_{l \rightarrow \infty} l^{-1/4} P_{l,m'} dx = \delta_{mm'} \quad (C9b)$$

or

$$\int_{-\infty}^{+\infty} \psi_n(x) \psi_{n'}(x) dx = \delta_{nn'}.$$

As another example of the derivation of a property of the  $\psi_n$ 's from a property of the associated Legendre polynomials, we show how the completeness relation can be obtained from the addition theorem

$$\left(\frac{2}{2l+1}\right)^{1/2} \sum_{m=-l}^l P_{l,m}(\cos\theta) P_{l,m}(\cos\theta') = P_{l,0}(\cos(\theta - \theta')). \quad (C10a)$$

One has, in the limit,

$$\sum_{n=0}^{\infty} \lim_{l \rightarrow \infty} l^{-1/4} P_{l,m}(\cos\theta) \lim_{l \rightarrow \infty} l^{-1/4} P_{l,m}(\cos\theta') = \lim_{l \rightarrow \infty} P_{l,0}(\cos[(x' - x)/\sqrt{l}]) \quad (C10b)$$

or

$$\sum_{n=0}^{\infty} \psi_n(x) \psi_n(x') = \delta(x' - x). \quad (C10c)$$

In conclusion let us mention that all other properties of associated Legendre polynomials contract to corresponding properties of oscillator eigenfunctions. In this way, one can construct ad-

ditional generating functions, recursion relations, addition theorems, etc., for the oscillator eigenfunctions from those of the associated Legendre polynomials.

#### APPENDIX D: RELATIONS BETWEEN BLOCH STATES, SPHERICAL HARMONICS, AND IRREDUCIBLE REPRESENTATIONS OF FULL ROTATION GROUP

In this appendix additional uses of the disentangling theorem are discussed. In particular it is shown how to derive in a simple manner the well-known  $(2J+1) \times (2J+1)$  irreducible representations of the full rotation group. Properties of these representations are then related to properties of the Jacobi polynomials, of the spherical harmonics, and of the associated Legendre polynomials. This allows to derive orthogonality relations in a very simple manner. The relation between spherical harmonics and the Bloch amplitudes of Eq. (3.13) is also shown. These properties are useful to compute integrals on the sphere of Bloch-state projectors with spherical-harmonics weight functions. These integrals are then used to construct the diagonal expansion of operators into Bloch states, as in Eqs. (3.20c) and (3.20d). As a case of special interest the statistical operator of a pure Bloch state is finally derived.

##### 1. Derivation of Irreducible Representations of Full Rotation Group

Let us consider the rotation of an object defined by the three Euler angles  $(-\alpha, -\beta, -\gamma)$ . The result is identical to a coordinate rotation  $(\alpha, \beta, \gamma)$  and will therefore agree with the usual group theoretical expressions.<sup>14</sup> In the  $2 \times 2$  irreducible representation one has, in the spirit of Appendix A,

$$R(-\alpha, -\beta, -\gamma) = e^{iJ_z \alpha} e^{iJ_y \beta} e^{iJ_x \gamma} = \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix} \begin{bmatrix} \cos \frac{1}{2}\beta & \sin \frac{1}{2}\beta \\ -\sin \frac{1}{2}\beta & \cos \frac{1}{2}\beta \end{bmatrix} \begin{bmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{bmatrix} \\ = \begin{bmatrix} (x_+)^{1/2} + x_+ x_- / (x_+)^{1/2} & x_+ / (x_+)^{1/2} \\ x_- / (x_+)^{1/2} & 1 / (x_+)^{1/2} \end{bmatrix}. \quad (D1)$$

By definition one has

$$R(-\alpha, -\beta, -\gamma) |M\rangle = \sum_{M'} \mathfrak{D}_{M'M}^J(\alpha\beta\gamma) |M'\rangle. \quad (D2)$$

The occurrence of the minus signs in the left-hand side has been explained above. Using (D1) and (D2) one obtains

$$\mathfrak{D}_{M'M}^J(\alpha, \beta, \gamma) = \langle M' | R(-\alpha, -\beta, -\gamma) |M\rangle \\ = \langle M' | e^{x_+ J_+} e^{(\ln x_+) J_x} e^{x_- J_-} |M\rangle, \quad (D3)$$

the last equality resulting from (A2). From (D1) one has

$$x_+ = \tan \frac{1}{2}\beta e^{i\gamma}, \quad x_- = -\tan \frac{1}{2}\beta e^{i\gamma},$$

$$x_z = e^{i(\alpha+\gamma)} / \cos^2(\frac{1}{2}\beta).$$

It remains then to calculate the last matrix element in (D3). This is done straightforwardly by expansion of the exponential operators containing  $J_+$  and  $J_-$ :

$$\mathfrak{D}_{M'M}^J(\alpha, \beta, \gamma) = \sum_{\mu\nu} \frac{1}{\mu!} \frac{1}{\nu!} (x_+)^{\mu} (x_-)^{\nu} (x_z)^{M-\nu} \\ \times \langle M' | J_+^{\mu} J_-^{\nu} |M\rangle. \quad (D4)$$

Finally, using the step-up and step-down properties of  $J_+$  and  $J_-$  one obtains

$$\mathfrak{D}_{M',M}^J(\alpha, \beta, \gamma) = \sum_{\mu} \frac{(-1)^{M-M'+\mu}}{\mu! (\mu+M-M')!} \frac{(J-M'+\mu)!}{(J+M'-\mu)!} \left( \frac{(J+M')! (J+M)!}{(J-M')! (J-M)!} \right)^{1/2} (\sin \frac{1}{2} \beta)^{2\mu+M-M'} (\cos \frac{1}{2} \beta)^{-M-M'} e^{i\alpha M'} e^{i\gamma M}. \tag{D5}$$

This expression is identical to the more usual form<sup>14</sup>

$$\mathfrak{D}_{M',M}^J(\alpha, \beta, \gamma) = \sum_{\kappa} (-1)^{\kappa} \frac{[(J+M)! (J-M)! (J+M')! (J-M')!]^{1/2}}{(J-M'-\kappa)! (J+M-\kappa)! \kappa! (\kappa+M'-M)!} (\sin \frac{1}{2} \beta)^{2\kappa+M'-M} (\cos \frac{1}{2} \beta)^{2J+M-M-2\kappa} e^{i\alpha M'} e^{i\gamma M}. \tag{D6}$$

The identity is not easy to show, but this fact should not mask the great simplicity with which (D5) has been derived. To prove the identity, starting from (D6), one replaces  $(\cos \frac{1}{2} \beta)^{2J+2M-2\kappa}$  by  $(1 - \sin^2(\frac{1}{2}\beta))^{J+M-\kappa}$ . This expression is expanded in powers of  $(\sin \frac{1}{2} \beta)^{2\nu}$ , and a double sum results. The summation indices are changed to  $\mu = M' - M + \nu + \kappa$ , which runs from  $M' - M + \kappa$  to  $J + M'$ , and to  $\eta = M' - M + \kappa$ . The summation over  $\eta$  is carried independently, using the relation

$$\sum_{k=0}^p \binom{n}{k} \binom{m}{p-k} = \binom{n+m}{p}.$$

Equation (D5) results.

The construction of the matrix elements (D5)

does not give any new information, as (D6) has been known for a long time. But the procedure which has been followed throws light on mechanisms for computing representation matrix elements for other groups in an explicit form. It turns out that an exactly similar technique can be used for a large number of Lie groups. To expand on this would fall beyond the limits of the present work.<sup>34</sup>

2. Relations to Jacobi Polynomials, Associated Legendre Polynomials, and Spherical Harmonics

The matrix elements  $\mathfrak{D}_{M',M}^J(\alpha, \beta, \gamma)$  are directly related to the Jacobi polynomials. To see this we first write (D5) or (D6) in a different way,

$$\mathfrak{D}_{M',M}^J(0, \beta, 0) = \sum_{\nu=-J}^J \frac{(-1)^{\nu-M'}}{(\nu-M')! (\nu-M)!} \frac{(J+\nu)!}{(J-\nu)!} \left( \frac{(J-M')! (J+M)!}{(J+M')! (J-M)!} \right)^{1/2} (\sin \frac{1}{2} \beta)^{2\nu-M-M'} (\cos \frac{1}{2} \beta)^{M+M'}. \tag{D7}$$

These matrix elements are given by an elegant generating function<sup>35,36</sup>

$$\mathfrak{D}_{M',M}^J(0, \beta, 0) = P_{M',M}^J(\cos \beta) \tag{D8}$$

with

$$P_{m,n}^l(z) = \frac{(-1)^{l-m}}{2^l (l-n)!} \left( \frac{(l-n)! (l+m)!}{(l+n)! (l-m)!} \right)^{1/2} (1+z)^{-(m+n)/2} (1-z)^{-(m-n)/2} \left( \frac{d}{dz} \right)^{l-m} [(1-z)^{l-n} (1+z)^{l+n}]. \tag{D9}$$

This is just the generating function for the Jacobi polynomials.

When  $J=l$  is an integer the labels  $m, n$  can assume all integral values  $-l \leq m, n \leq l$ . Setting  $n=0$  in the generating function above yields the generating function for the associated Legendre polynomials and the spherical harmonics:

$$\begin{aligned} \mathfrak{D}_{m,0}^l(0, \beta, 0) &= P_{m,0}^l(\cos \beta) \\ &= \frac{(-1)^{l-m}}{2^l l!} \left( \frac{(l+m)!}{(l-m)!} \right)^{1/2} (1-z^2)^{-m/2} \\ &\quad \times \left( \frac{d}{dz} \right)^{l-m} (1-z^2)^l \end{aligned} \tag{D10}$$

and

$$\left( \frac{2l+1}{4\pi} \right)^{1/2} \mathfrak{D}_{m,0}^l(\varphi, \theta, -) = (-1)^m Y_m^l(\theta, \varphi). \tag{D11}$$

In short, the spherical harmonics are essentially

matrix elements belonging to the  $n=0$  column of the  $\mathfrak{D}_{m,n}^l$  representation of the full rotation group. Moreover, the matrix element which occurs at the intersection of the  $m=0$  row and the  $n=0$  column is a Legendre polynomial  $P_{0,0}^l(\cos \beta)$ . We see that a number of special functions are associated with  $\mathfrak{D}_{m,n}^l$ . The orthogonality properties of all these functions are simply obtained from the well-known expression<sup>14,37</sup>

$$\begin{aligned} \int_0^{2\pi} d\alpha \int_0^{\pi} \sin \beta d\beta \int_0^{2\pi} d\gamma \mathfrak{D}_{m',n'}^{l,*}(\alpha, \beta, \gamma) \mathfrak{D}_{m,n}^l(\alpha, \beta, \gamma) \\ = \frac{\int_0^{2\pi} d\alpha \int_0^{\pi} d\cos \beta \int_0^{2\pi} d\gamma}{\dim \mathfrak{D}^l} \delta^{l'l'} \delta_{m'n'} \delta_{n'n} \end{aligned} \tag{D12}$$

Thus the functions

$$\left( \frac{2l+1}{4\pi \times 2\pi} \right)^{1/2} \mathfrak{D}_{mn}^l(\alpha, \beta, \gamma) \tag{D13a}$$



form an orthonormal set on the parameter space  $(\alpha, \beta, \gamma)$  with respect to the measure

$$d\alpha |d \cos \beta| d\gamma. \quad (\text{D14a})$$

By setting  $n' = n = 0$  in (D12), and carrying out the integral over  $\gamma$ , we find that the functions

$$\left(\frac{2l+1}{4\pi}\right)^{1/2} \mathfrak{D}_{m,0}^l(\varphi, \theta, -) = (-1)^m Y_m^l(\theta, \varphi) \quad (\text{D13b})$$

are orthonormal on  $(\theta, \varphi)$  with respect to the measure

$$d\varphi |d \cos \theta| = d\varphi \sin \theta d\theta. \quad (\text{D14b})$$

Finally setting  $n = n' = 0$  and  $m = m' = 0$  in (D12), and carrying out the integrals over both  $\alpha$  and  $\gamma$ , we find that the Legendre polynomials  $P_l(\cos \theta)$ ,

$$\left(\frac{2l+1}{2}\right)^{1/2} \mathfrak{D}_{00}^l(-, \theta, -) = \left(\frac{2l+1}{2}\right)^{1/2} P_l(\cos \theta), \quad (\text{D13c})$$

form an orthonormal set on the interval  $(-1, +1)$  with respect to the measure

$$|d \cos \theta| = \sin \theta d\theta. \quad (\text{D14c})$$

The completeness of the functions (D13a)–(D13c) may be proved analogously starting from the completeness relation<sup>37</sup> for the matrix elements  $\mathfrak{D}_{mn}^l$ :

$$\sum_i \sum_m \sum_n \frac{2l+1}{4\pi \times 2\pi} \mathfrak{D}_{mn}^l(\alpha', \beta', \gamma') \mathfrak{D}_{mn}^l(\alpha, \beta, \gamma) = \delta(\alpha' - \alpha) \delta(\cos \beta' - \cos \beta) \delta(\gamma' - \gamma). \quad (\text{D15})$$

Equation (D15) is the dual to Eq. (D12).

### 3. Relations between Spherical Harmonics and Bloch Amplitudes

The spherical harmonics (D13b) and the Bloch amplitudes (3.13), respectively,

$$A_{mm'}^{J,L} = \frac{(-1)^M}{2\pi} \left(\frac{2L+1}{4\pi}\right)^{1/2} \int \mathfrak{D}_{m,-J}^{J*}(\varphi, \theta, \gamma) \mathfrak{D}_{M,0}^L(\varphi, \theta, \gamma) \mathfrak{D}_{m',-J}^J(\varphi, \theta, \gamma) d\mu(\varphi, \theta, \gamma). \quad (\text{D19})$$

Here  $d\mu(\varphi, \theta, \gamma)$  is given by (D14a). The integral in (D19) is well known,<sup>38–40</sup> and can be expressed in terms of Clebsch–Gordan coefficients:

$$\int \mathfrak{D}_{a'a}^{j_1*}(R) \mathfrak{D}_{b'b}^{j_2}(R) \mathfrak{D}_{c'c}^{j_3}(R) d\mu(R) = \frac{8\pi^2}{2j_1+1} \begin{matrix} j_2 & j_3 & j_1 \\ b' & c' & a' \end{matrix} \begin{matrix} j_1 & j_2 & j_3 \\ a & b & c \end{matrix}. \quad (\text{D20})$$

Using this fact one finds

$$A_{mm'}^{J,L,M} = (-1)^M \left(\frac{4\pi(2L+1)}{(2J+1)^2}\right)^{1/2} \begin{matrix} L & J & J \\ M & m' & m \end{matrix} \begin{matrix} J & L & J \\ -J & 0 & -J \end{matrix}. \quad (\text{D21})$$

The last Clebsch–Gordan coefficient is given explicitly by<sup>41</sup>

$$\begin{matrix} J & L & J \\ -J & 0 & -J \end{matrix} = \left(\frac{(2J+1)! (2J)!}{(2J+1+L)! (2J-L)!}\right)^{1/2}. \quad (\text{D22})$$

In particular it results that  $A_{mm'}^{J,L,M} = 0$  for  $L > 2J$ .

$$Y_M^L(\theta, \varphi) = (-1)^M \left(\frac{2L+1}{4\pi}\right)^{1/2} \mathfrak{D}_{M,0}^L(\varphi, \theta, -)$$

and

$$\langle J, M | \theta, \varphi \rangle \equiv \mathfrak{D}_{M,-J}^J(-\varphi, \theta, \varphi),$$

have analogous properties because they are functions drawn from different columns of the  $\mathfrak{D}_{m',m}^J$  [SU(2)] representation of SU(2). Since the columns ( $m=0$  for spherical harmonics and  $m=-j$  for the Bloch amplitudes) are separated by half the range of the discrete variable  $m$ , it should come as no surprise that the oscillator eigenfunctions are contracted limits of the spherical harmonics for the argument  $(\frac{1}{2}\pi - c\sqrt{2}x)$  centered around the midpoint in the finite range of the continuous dual variable  $\theta$  [Eq. (C5a)]. This was proved there in a different way.

### 4. Integrals over Bloch-State Projectors

The observations made so far can be used to compute explicitly integrals of the form

$$I = \int |J, \theta, \varphi\rangle Y_M^L(\theta\varphi) \langle J, \theta, \varphi| d\Omega \quad (\text{D16})$$

with  $d\Omega = d\varphi \sin \theta d\theta$ . First, the Bloch states are expanded in terms of matrix elements

$$\begin{aligned} |\theta, \varphi\rangle &= \sum_m |J, m\rangle \mathfrak{D}_{m,-J}^J(-\varphi, \theta, \varphi) \\ &= e^{iJ(\varphi-\psi)} \sum_m |J, m\rangle \mathfrak{D}_{m,-J}^J(-\varphi, \theta, \gamma), \end{aligned} \quad (\text{D17})$$

where  $\gamma$  is a dummy variable, introduced for convenience. Substituting (D17) and (D13b) into (D16) one obtains

$$I = \sum_{m,m'} |J, m\rangle A_{mm'}^{J,L,M} \langle J, m'|, \quad (\text{D18})$$

where

This is precisely what is required to have  $(2J+1)^2$  independent coefficients in the expansion (3.20d).

### 5. Diagonal Expansion of Operators into Bloch States

We now proceed to solve for the coefficients  $G_{lm}$  in (3.20d) in terms of the matrix elements

$\langle M|G|M' \rangle$  of (3.20a). Using (3.20a), (3.20c), (3.20d), and (D18), one obtains

$$\langle M|G|M' \rangle = \sum_{l,m} G_{lm} A_{MM'}^{J_l l m} . \quad (\text{D23})$$

This relation must be inverted, using the value of  $A_{MM'}^{J_l l m}$  from (D21). To this effect we use the relation<sup>41</sup>

$$\begin{aligned} & \left\langle \begin{matrix} j_1 & j_2 & | & J \\ j_2 & m_2 & | & M \end{matrix} \right\rangle \\ &= (-1)^{j_1 - J + m_2} \left( \frac{2J+1}{2j_1+1} \right)^{1/2} \left\langle \begin{matrix} J & j_2 & | & j_1 \\ M & -m_2 & | & m_1 \end{matrix} \right\rangle \end{aligned} \quad (\text{D24})$$

to transform the first Clebsch-Gordan coefficient in (D21). One obtains

$$\begin{aligned} A_{MM'}^{J_l l m} &= (-1)^{l-m} (-1)^{J-M'} \left( \frac{4\pi}{2J+1} \right)^{1/2} \\ & \times \left\langle \begin{matrix} J & J & | & l \\ M & -M' & | & m \end{matrix} \right\rangle \left\langle \begin{matrix} J & | & J \\ -J & 0 & -J \end{matrix} \right\rangle . \end{aligned} \quad (\text{D25})$$

Introducing this in (D23), multiplying both sides by  $\langle l' m' | \begin{matrix} J & J \\ M & -M' \end{matrix} \rangle (-1)^{J-M'}$ , and summing over  $M$  and  $M'$ , one obtains

$$\begin{aligned} & \sum_{M, M'} (-1)^{J-M'} \left\langle \begin{matrix} l' & | & J & J \\ m' & | & M & -M' \end{matrix} \right\rangle \langle M|G|M' \rangle \\ &= (-1)^{l'-m'} G_{l'm'} \left( \frac{4\pi}{2J+1} \right)^{1/2} \left\langle \begin{matrix} J & | & l' & J \\ -J & | & 0 & -J \end{matrix} \right\rangle . \end{aligned} \quad (\text{D26})$$

We made use of the relation

$$\sum_{m_1 m_2} \left\langle \begin{matrix} L & | & j & j \\ M & | & m_1 & m_2 \end{matrix} \right\rangle \left\langle \begin{matrix} j & j & | & L' \\ m_1 & m_2 & | & M' \end{matrix} \right\rangle = \delta^{LL'} \delta_{MM'} .$$

Equation (D26) gives the coefficients  $G_{l'm'}$  in terms of a sum over the matrix elements  $\langle M|G|M' \rangle$ .

These ideas can be used to generate the diagonal representation of any well-behaved operator and, in particular, of the density operator, as explained in Sec. III D. To give a simple example, let us consider the density operator of a pure Bloch state  $|\theta_0, \varphi_0\rangle$ ,

$$\rho = |\theta_0, \varphi_0\rangle \langle \theta_0, \varphi_0| . \quad (\text{D27})$$

From (3.23) it is immediate that the weight function can be written

$$P(\theta, \varphi) = \delta(\theta - \theta_0; \varphi - \varphi_0) . \quad (\text{D28})$$

However this function is not of the form (3.28), since the summation in (3.28) is limited to the first  $(2J+1)^2$  values of  $\lambda = (l, m)$ . The coefficients  $X_\lambda$  are obtained immediately from (3.27),  $X_\lambda = Y_l^m(\theta_0, \varphi_0)$ . The summation (3.28) gives then

$$\begin{aligned} P(\theta, \varphi) &= \sum_{l=0}^{2J} \sum_{m=-l}^l Y_l^m(\theta_0, \varphi_0) Y_l^{-m}(\theta, \varphi) \\ &= \frac{1}{4\pi} \sum_{l=0}^{2J} (2l+1) P_l(\cos\Theta) \end{aligned} \quad (\text{D29})$$

with  $\cos\Theta = \cos\theta \cos\theta_0 + \sin\theta \sin\theta_0 \cos(\varphi - \varphi_0)$ . The two expressions (D28) and (D29) correspond to the same density matrix in view of the fact the integral (D16) vanishes for  $L > 2J$ .

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PHYSICAL REVIEW A

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## Moments of the Sum of Photocounts in Gaussian Light

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The statistics of the sum of photocounts in partially coherent Gaussian light is investigated. Both the case of detection at  $L$  discrete points and the case of an extended surface of detection are considered. The combinatorial-analysis approach is applied to derive the moments and cumulants of the distribution. They all can be expressed in terms of the second-order correlation function. The results are generally valid independent of the ratio of the observation time to the coherence time of light. A comparison between the theoretical moments and the moments computed from experimental distributions as a function of the detector surface is carried out with the help of a pseudothermal source.

### I. INTRODUCTION

A method has been described recently<sup>1</sup> for determining the degree of spatial coherence of Gaussian light on the basis of photocount experiments. The light from a pseudothermal source is simultaneously detected at two points of the observation plane with a single photodetector. The probability distribution of photoelectrons,  $p(n)$ , corresponds then to the probability distribution of the sum of the integrated intensities of light received at each point. The moments of  $n$  play a key role in the above method. In order to derive them the combinatorial-analysis approach is used. This approach turns out to be extremely fruitful also in analyzing multifold photoelectric counting statistics of both Gaussian<sup>2</sup> and a mixed chaotic and coherent light.<sup>3</sup> The purpose of this paper is to extend the results obtained for two points to a larger number of points, and in the limit to an extended photodetector. The results will be generally valid, regardless of the ratio of the counting time to the coherence time of light.

In contrast to correlation and photon coincidence counting experiments which involve a number of

distinct photodetectors and sometimes a signal multiplying device,<sup>4</sup> only one photomultiplier is needed. In statistical terms, we are not interested in the joint probability distribution  $p(n_1, n_2, \dots, n_L)$  that  $n_1$  counts will be produced by the first detector,  $n_2$  counts by the second one, etc., but in the probability  $p(n = n_1 + n_2 + \dots + n_L)$  of recording the sum  $n$  of the random variables  $n_1, n_2, \dots, n_L$ .

Some of the results presented here were already derived<sup>5</sup> by considering the factorial moment-generating function of  $p(n)$ . In the combinatorial-analysis approach the whole involved step of calculating the generating function is superfluous. The moments of  $p(n)$  are directly calculated from the knowledge of the mutual coherence function across the detector surface.

In Sec. II we define the probability distribution of the sum of the random variables  $n_1, n_2, \dots, n_L$  and summarize the various relations between moments and cumulants. Section III contains the derivation of the moments of  $p(n)$  in the case of detection at  $L$  distinct points and for counting intervals much smaller than the coherence time of light. Section IV generalizes the results of Sec. III for the case of an arbitrary counting time in-