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WHEN IS THE WIGNER QUASI-PROBABILITY DENSITY NON-NEGATIVE?

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It is shown that a necessary and sufficient condition for the Wigner quasi-probability density to be a true density is that the corresponding Schrödinger state function be the exponential of a quadratic polynomial.

Introduction

The Wigner quasi-probability density W [7], for quantum mechanical momentum and position observables P, Q satisfying a rigorous form of the Heisenberg commutation relation

$$PQ-QP=-i,$$

is well known ([1], [6]) not to be everywhere non-negative in general. The occurrence of "negative probabilities" is related to the incompatibility of the observables P, Q; since they cannot be measured simultaneously there is no experimental procedure for estimating a joint probability distribution and no obligation on quantum theory to postulate one. Nevertheless it has been observed [6] that, for certain quantum mechanical states, the Wigner function is non-negative and thus represents a true joint probability distribution, albeit without direct physical significance. The purpose of this paper is to show that for this to be the case it is necessary and sufficient that the Schrödinger state vector ψ be of the form

$$\psi(x)=e^{-\frac{1}{2}(ax^2+2bx+c)}$$

where a, b are arbitrary complex numbers with $\operatorname{Re} a>0$ and the complex number c is chosen so as to ensure correct normalization. The corresponding Wigner densities are bivariate normal distributions whose covariance matrices Γ satisfy

$$\det\Gamma=\tfrac{1}{4}.$$

2. Properties of Wigner densities

The Wigner density W may be characterised by the equation

$$\int_{R^2} e^{i(xp+yq)} W(p, q)\, dp\, dq = \langle e^{i(xP+yQ)} \rangle$$

where the integral on the left is understood in the sense of a Fourier–Plancherel transform, and the expression on the right is the quantum mechanical expectation of the function $\lambda \to e^{i\lambda}$ of the observable $xP+yQ$. In the Schrödinger (position) representation, in which P, Q are the essentially self-adjoint operators

$$P\psi = -iD\psi, \qquad Q\psi(x) = x\psi(x)$$

acting in the space $L^2(R)$ of square-integrable functions on the real line, the state of the system is characterised by a vector $\psi \in L^2(R)$, normalized by the condition

$$\|\psi\|^2 = \int_R |\psi(x)|^2 dx = 1 .$$

In terms of ψ the Wigner function is given by Wigner's original formula ([7])

$$W_\psi(p, q) = (2\pi)^{-1} \int e^{ixp} \psi(q - \tfrac{1}{2}x) \bar{\psi}(q + \tfrac{1}{2}x)\, dx . \tag{1}$$

For well behaved ψ_1, ψ_2, for instance for infinitely differentiable, rapidly decreasing functions, it can be verified by formal manipulation of this formula that

$$\begin{aligned} \int_{R^2} \overline{W_{\psi_1}(p, q)}\, W_{\psi_2}(p, q)\, dp\, dq &= (2\pi)^{-1} \left| \int \bar{\psi}_1(x) \psi_2(x)\, dx \right|^2 \\ &= (2\pi)^{-1} |\langle \psi_1 | \psi_2 \rangle|^2 . \end{aligned} \tag{2}$$

It can be shown ([4]) that (2) holds without restriction on the states ψ_1, ψ_2.

3. The states $\psi_{a,b}$

Consider now the state determined by the Schrödinger state vector

$$\psi_{a,b}(x) = e^{-\frac{1}{2}(ax^2 + 2bx + c)} \qquad (\mathrm{Re}\, a > 0).$$

Using the well-known integral

$$\int_R e^{-\frac{1}{2}\tau x^2 + zx} dx = \sqrt{\frac{2\pi}{\tau}}\, e^{z^2/2\tau} \qquad \left(\mathrm{Re}\, \tau,\ \mathrm{Re} \sqrt{\frac{2\pi}{\tau}} > 0 \right), \tag{3}$$

we have

$$\int_R |\psi_{a,b}(x)|^2 dx = e^{-\mathrm{Re}\, c} \sqrt{\frac{\pi}{\mathrm{Re}\, a}}\, e^{(\mathrm{Re}\, b)^2/(\mathrm{Re}\, a)}$$

hence correct normalization is ensured by taking c real, with

$$e^{-c}=\left(\sqrt{\frac{\operatorname{Re} a}{\pi}}\, e^{-(\operatorname{Re} b)^2/(\operatorname{Re} a)}\right)^{\frac{1}{2}},$$

The corresponding Wigner density $W_{a,b}$ can be found using (1) and (3); it is

$$W_{a,b}(p,q)=\frac{1}{\pi}\, e^{-(\operatorname{Re} a)^{-1}(p^2+2(\operatorname{Im} a)pq+|a|^2q^2+2\operatorname{Re} a \operatorname{Im} b p+2(\operatorname{Re} a \operatorname{Re} b+\operatorname{Im} a \operatorname{Im} b)q+|b|^2)}. \tag{4}$$

This can be seen to be a bivariate Gaussian density with covariance matrix

$$\Gamma=\tfrac{1}{2}\operatorname{Re} a\begin{bmatrix} 1 & \operatorname{Im} a \\ \operatorname{Im} a & |a|^2 \end{bmatrix}^{-1}$$

satisfying

$$\det \Gamma=\tfrac{1}{4};$$

conversely, it is easily seen that every such density can be expressed in the form (4) by suitable choice of a and b.

4. The main theorem

THEOREM. *A necessary and sufficient condition for the Wigner density W corresponding to the Schrödinger state vector ψ to be a true probability density is that ψ be the exponential of a quadratic polynomial.*

Proof: The sufficiency was established in § 3. To establish the necessity, let $\psi \in L^2(R)$ be a unit vector for which the corresponding Wigner function W is a true density. From (2), we have for all complex numbers z

$$\int \bar{W}(p,q)\, W_{1,z}(p,q)\, dp\, dq=(2\pi)^{-1}|\langle\psi|\psi_{1,z}\rangle|^2. \tag{5}$$

Since W is a true probability density, it is non-negative. But since $W_{1,z}$ is Gaussian, it is everywhere strictly positive. It follows that the left-hand side of (5) must be strictly positive, and hence that the right-hand side is, in particular, non-zero for all $z \in C$. From this it follows that the entire analytic function

$$F(z)=\int \bar{\psi}(x)\, e^{-\frac{1}{2}x^2-zx}dx=e^{\frac{1}{2}c}\langle\psi|\psi_{1,z}\rangle \tag{6}$$

has no zeros. Moreover, by Schwarz's inequality and (3) we have

$$|F(z)|^2\leqslant\|\psi\|^2\sqrt{\pi}\, e^{(\operatorname{Re} z)^2}.$$

It follows that F is an entire function of order not exceeding two, with no zeros. The well-known theorem of Hadamard [5] implies that F is of the form

$$F(z)=e^{\alpha z^2+\beta z+\gamma}.$$

Substituting this form into (6) and setting $z=iy$ shows that

$$F(iy)=e^{-\alpha y^2+i\beta y+\gamma}$$

is the Fourier transform of the integrable function

$$x\to\bar{\psi}(x)\,e^{-\frac{1}{2}x^2}. \tag{7}$$

Since the only functions whose Fourier transforms are exponentials of quadratic polynomials are themselves of this type, it follows that the function (7) and hence in turn ψ itself is such an exponential. This completes the proof of the theorem.

5. Conclusion

Modulo a linear canonical transformation, the states $\psi_{a,b}$ with true Wigner densities are the so-called *coherent states* ([3]) which have proved of great interest and diverse application in several areas of quantum mechanics.

This paper has not considered the question of which mixed (statistical) states give rise to true Wigner densities. Using the multiplicative properties of quasi-characteristic functions ([2]) an extensive class of such states can be constructed. But an effective characterisation of the totality of such states has not yet been found.

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