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KEEPING TRACK OF CHAOS BY QUANTUM-NONDEMOLITION MEASUREMENTS

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INTRODUCTION

The aim of this contribution is twofold. On the one hand, the ingredients of deterministic randomness ubiquitous in classical mechanics are to be identified clearly and their presence in quantum mechanics is to be shown. On the other hand, the possibility of actually observing deterministic randomness in quantum mechanics is to be pointed out. To this end quantum-nondemolition measurements are used. The intention of the present work is to give a nontechnical survey of the ideas relating deterministic randomness and quantum-nondemolition measurements; for a more detailed study of this topic the reader is referred to the paper [9].

The developments given here are one particular way to look at the problems related to the field "quantum chaos" (see e.g. [3] and this volume). Contrary to many other investigations the question of existence of deterministic randomness in the time-evolution of quantum systems is addressed—no recourse is made to the investigation of level statistics or the spatial structure of wave functions, for example. The theory-independent tool of algorithmic complexity is used in order to allow a direct comparison between the quantum evolution and classical dynamics. It is known that this concept catches the relevant features of classically chaotic motion [7]. For example, the difficulty of making reliable long-time predictions of the behaviour of a physical system is reflected in nonvanishing algorithmic complexity of the associated dynamical model.

The discussion of these question is organized as follows. In the next section algorithmic complexity is introduced in an abstract way meaning that no reference is made to either classical mechanics or quantum mechanics. Subsequently, in Section III this mathematical concept is shown to coincide with deterministic randomness when applied to classical mechanics. A brief illustration of these features with Arnold's Cat Map follows, being one of the standard examples of chaotic motion. Section IV deals with another physical realization of the abstract scheme of algorithmic complexity. It is explained that (time-dependent) quantum systems in principle may evolve deterministically random in time. From the discussion given in Section V it follows that quantum-nondemolition measurements emerge naturally as a possibility to observe the *deterministic* time evolution of a quantum system due to Schrödinger's equation. Performing this particular type of measurements in general allows to suppress the occurrence of the intrinsic quantum-mechanical probabilities. In the last section a summary of the relevant points is given.

ALGORITHMIC COMPLEXITY ...

In this section the notion of algorithmic complexity is introduced in mathematical terms without reference to either classical or quantum mechanics. Consider a compact manifold Γ of points $\gamma \in \Gamma$ on which a linear map $U : \Gamma \to \Gamma$ is defined, U being one-to-one and invertible. The automorphism U, when repeatedly applied to the manifold Γ, later on will be interpreted as the time evolution of the "state space" Γ for a physical system. Other state spaces—having a finite or a countable number of elements—may occur (cf. [9]) but are not taken into account here.

Algorithmic complexity comes into play if one is interested in the amount of numerical work to determine the n^{th} iterate $\gamma_n = U^n\gamma$ of (generic) points $\gamma \in \Gamma$, for arbitrary large numbers $n \in \mathbb{N}$. From a numerical point of view the calculation of γ_n amounts to implement on a computer the shortest possible program $\mathcal{P}$ generating that number. A fixed number N_A of bits is needed to store the algorithm which effects numerically the map U of the manifold Γ. Realistically, the determination of the image points γ_n is possible only up to an (arbitrarily small) uncertainty Δ which will depend on the given accuracy Δ_0 of the initial point $\gamma \equiv \gamma_0$, on the map U and on the number n of iterations. Storage of the starting point in order to obtain a prescribed value of Δ after n steps requires $N_D(\Delta, n)$ bits. Finally, the number of iterations n has to be stored, corresponding to $\log_2 n$ bits. Algorithmic complexity then is defined as the total length of the shortest program in bits divided by n

$$C = \lim_{n\to\infty} \frac{1}{n}(N_A + N_D(\Delta, n) + \log_2 n) \ , \tag{1}$$

in the limit of arbitrarily large values of n. Since the length of the algorithm N_A is a fixed number and $\lim_{n\to\infty}(log_2 n)/n \to 0$, only the second term $N_D(\Delta, n)$ may effectively contribute to a nonzero value of the complexity C.

Complexity as defined above measures the difficulty to locate the images γ_n accurately on the manifold Γ as a function of the inevitable inaccuracy of the initial position γ_0. According to the properties of the map U this inaccuracy, for example, may increases algebraically or exponentially, thus possibly giving rise to positive complexity. Consequently, this notion allows to distinguish in a computationally relevant way qualitatively different types of dynamical evolution.

Nonzero complexity occurs whenever the term $N_D(\Delta, n)$ increases—for large values of n—at least linearly as a function of n. This corresponds to an *exponential* growth of the initial inaccuracy Δ_0 or, correspondingly, of a small volume of initial conditions with typical size Δ_0, due to the application of the automorphism U. Since the set Γ is compact and is mapped to itself under U, the increase of inaccuracies in one direction must be a accompanied by a decrease in another direction, leading even after a small number of iterations to an intricate image of the original "volume" of initial conditions. Initially "distant" points γ, γ' on the manifold Γ may be mapped by U^n onto "neighbouring" ones and vice versa. In the next section a simple example of such behaviour in an abstract dynamical system will be presented.

...UNDERLYING CHAOTIC MOTION IN CLASSICAL MECHANICS ...

The relevance of the concept of algorithmic complexity in classical mechanics can be seen by making the following identification. Consider the manifold Γ as phase space (or as a hypersurface of fixed energy in this space) of a classical Hamiltonian system with conserved energy H. The evolution of the system over a time interval Δt being conservative and symplectic is described by a map $U(\Delta t) : \Gamma \to \Gamma$. The time translation $U(\Delta t)$ of phase space densities represents a linear automorphism of Γ, as required for the general scheme described in the previous section. Consequently, algorithmic complexity applied to classical mechanics may entail a division of the set of Hamiltonian systems into two classes the elements of which are different with respect to the possibility of predicting their long-time behaviour. Roughly speaking, this distinction

coincides with the difference between integrable and nonintegrable systems, although there are exceptions [6].

The consequences of a time evolution being algorithmically complex are easily explained in physical terms. The expression "deterministic randomness" has been coined in order to describe the seemingly irregular motion of nonintegrable systems. Although the law governing the time evolution of a classically chaotic system is completely deterministic, accurate predictions of the final state even after relatively short times become illusionary due to the fact that small deviations in the initial conditions are blown up by the dynamics. It is essentially this physically relevant sensitive dependence on initial conditions which is enciphered in algorithmic complexity since the actual work to generate quantitative predictions is taken into account.

The Arnold Cat Map [1] serves as a paradigm to illustrate the features mentioned above although it is not conservative due to the explicit time-dependence of its Hamiltonian. The unit square (with opposite boundaries identified) considered as the phase space of a dynamical system is mapped onto itself according to

$$\boldsymbol{z}' = (\boldsymbol{M} \cdot \boldsymbol{z}) \bmod 1\,, \tag{2}$$

where $\boldsymbol{z} = (p, q)$, $p, q \in [0, 1)$ and $\boldsymbol{M}$ is a 2×2 matrix with all entries equal to 1, except the upper left one which is equal to 2. In Figure 1 A the image of a phase space density concentrated about the point γ is shown. The process of stretching and folding which

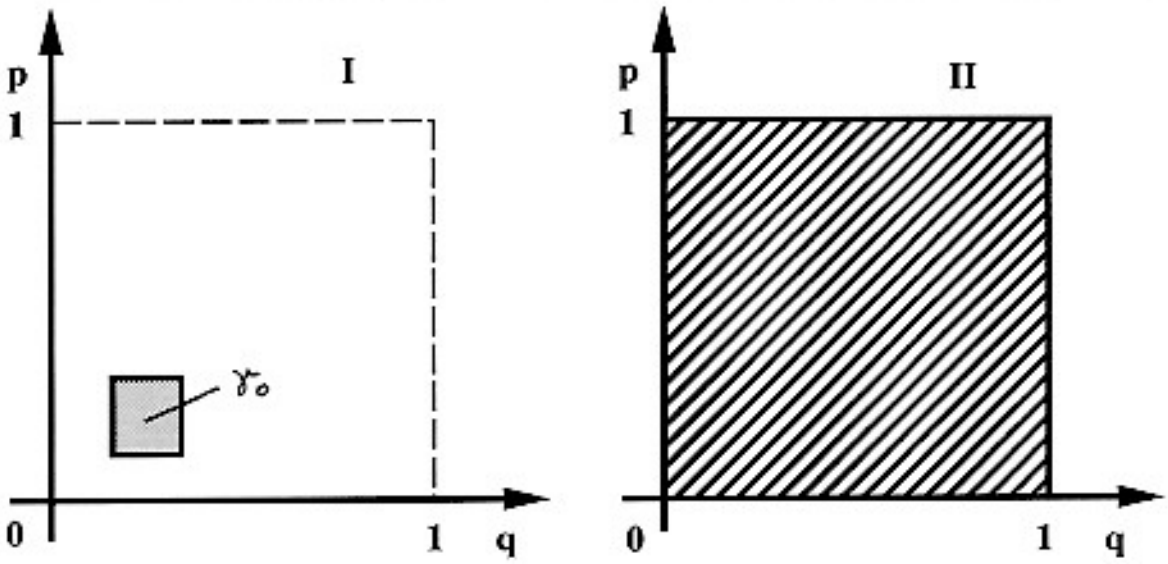

Figure 1: A: Phase space distribution centered about the point γ_0 at $t = 0$ (I) and after n iterations of the map U (II)
B: Distribution of nonzero coefficients centered about the state γ_0 at $t = 0$ (I) and after n iterations of the map U (II)

introduces an ever finer structure of the density over the manifold Γ can be discerned easily indicating the algorithmic complexity of the motion.
In the next section the occurrence of algorithmic complexity in quantum mechanics is investigated.

...ALSO EXISTS IN QUANTUM MECHANICS ...

In order to see algorithmic complexity evolve in a quantum-mechanical time evolution one can proceed in the following way. Let the manifold Γ correspond to the values of continuously varying labels $g_i\,(i = 1, 2, \ldots, \dim\Gamma)$ associated with the eigenvectors $\{|\boldsymbol{g}\rangle\}$ of operators $\hat{g}_i\,(i = 1, 2, \ldots, \dim\Gamma)$, that is

$$\hat{g}_i|\boldsymbol{g}\rangle = g_i|\boldsymbol{g}\rangle, \quad \forall\, i\,. \tag{3}$$

The continuous spectrum of eigenvalues $\{\boldsymbol{g}\}$ necessarily is accompanied by *generalized* eigenvectors $\{|\,\boldsymbol{g}\rangle\}$ as they are known from position operators, for example. It is assumed here that the operators $\hat{\boldsymbol{g}}$ form a complete set of operators for the system under consideration. Consequently, any possible quantum state $|\psi\rangle$ of the system may be expressed as a superposition of the eigenstates $\{|\,\boldsymbol{g}\rangle\}$

$$|\psi\rangle = \int \psi(\boldsymbol{g})|\boldsymbol{g}\rangle\, d\boldsymbol{g}\,, \tag{4}$$

with appropriate coefficients $\psi(\boldsymbol{g}) \equiv \langle \boldsymbol{g} \mid \psi\rangle$ and defining $d\boldsymbol{g} \equiv \prod_i dg_i$. In other words, the manifold Γ is identified with a *complete set of basis vectors* in Hilbert space $\mathcal{H}$—it does *not* correspond to the set of all possible quantum states.

Defining the time evolution of the system by the action of a unitary operator U on the basis $|\boldsymbol{g}\rangle$ in Hilbert space $\mathcal{H}$, another realization of the mathematical structure developed in Section II is obtained. Let the operator U act in such a way that any eigenstate $|\boldsymbol{g}\rangle$ is mapped onto *another* eigenstate $|\boldsymbol{g}'\rangle$ of the operators $\hat{\boldsymbol{g}}$, i.e., consider an invertible automorphism of the basis vectors $|\boldsymbol{g}\rangle$ between themselves (cf. Figure 2). In most cases, the time evolution of a quantum system does *not* have this property

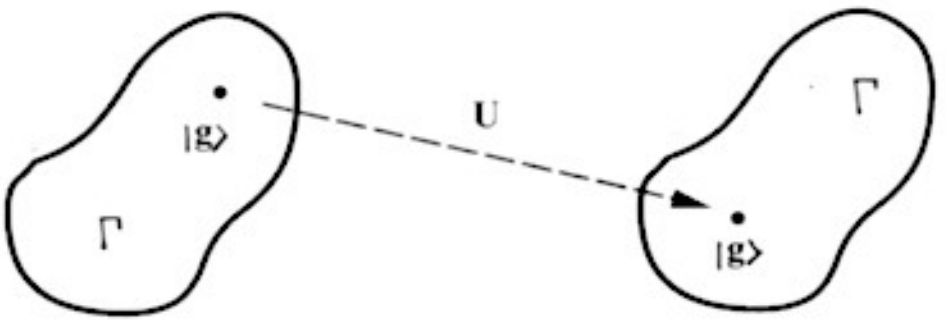

Figure 2: Map U of the manifold Γ parameterized by eigenvalues $\boldsymbol{g}$ of the states $|\boldsymbol{g}\rangle$

because typically an eigenstate $|\,\boldsymbol{f}\rangle$ of a basis $\{|\,\boldsymbol{f}\rangle\}$ in general will be mapped onto a *superposition* of vectors $|\,\boldsymbol{f}\rangle$. However, the above situation also is possible and turns out to be particularly suited to discuss the occurrence of algorithmic complexity in quantum systems. Note that specifying the map U for a set of basis vectors is sufficient to determine the image $U|\psi\rangle$ of any state $|\psi\rangle$ for all times which correspond to the n-fold application of U.

In fact, if the map U is algorithmically complex as defined in the second section the time evolution of the quantum system becomes extremely difficult to follow accurately. Suppose an initial state to be localized about a point γ_0 on the manifold Γ, meaning that only states in the neighbourhood[1] of γ_0 have nonzero coefficients. After a small number of iterations of the map U this distribution of coefficients is no longer localized in a small region of the manifold Γ. One may describe this situation simply by slightly

[1] Note that the notion of neighbourhood here refers to the metric of labels on the manifold Γ and not of Hilbert space $\mathcal{H}$.

changing the statement made in the preceeding section: Although the law governing the time evolution of the quantum system with algorithmic complexity is completely deterministic, accurate prediction of the final probability distribution on the manifold Γ even after relatively short times is illusionary due to the fact that small deviations in the initial distribution are blown up by the dynamics.

It must be emphasized that this drawback in predictability, as presented here, has a *purely dynamical origin*—in exact correspondence to the source of deterministic randomness in classical mechanics. It is important to realize the separation of the deterministic and the probabilistic element of the time evolution which has been effected here. The intrinsic quantum probabilities are an additional independent feature of the theory and do not contribute at this level.

The properties of a particular quantum system system possessing an algorithmically complex time evolution have been worked out elsewhere [8]. The Configurational Quantum Cat Map arises from the description of a charged particle moving on a configurational two-dimensional torus when restricting one's attention to a discrete and periodic set of times only. The system is explicitly time-dependent: free motion on the torus is interrupted periodically by the influence of specific δ-pulsed electro-magnetic fields. They effect the time evolution operator U over one time interval to act on the set of position basis vectors formally as the Arnold's Cat Map does act on the torus-shaped *phase* space.

Figure 1 B illustrates for this system what happens to the nonzero coefficients of a state localized in configuration space under repeated application of the operator U (in this case $q \equiv x_1$ and $p \equiv x_2$ correspond to the spatial coordinates on the configurational torus). Since from a numerical point of view there is no difference to what happens in the classical Arnold's Cat Map, positive algorithmic complexity of the time evolution is immediately obvious entailing deterministic randomness of the quantum motion. A related example has been discussed by Chirikov et al. [5]; a physical realization of the underlying Hamiltonian, however, is not available.

...AND CAN BE OBSERVED BY QUANTUM-NONDEMOLITION MEASUREMENTS:

It has been shown above that in particular cases the separation of the deterministic and the probabilistic element of quantum dynamics is possible. In this section it is argued that under such circumstances the phenomenon of deterministic randomness, if present in the dynamics under investigation, may be observed unambiguously by making use of quantum-nondemolition measurements. These measurements, however, are in no way related to the *formation* of algorithmic complexity—some maps U simply allow to introduce these measurements and hence allow the direct observation of the deterministically random Schrödinger dynamics.

A simple example of a quantum-nondemolition observable [2, 4] is given by the position operator $\hat{x}(t)$ in the Heisenberg picture of a one-dimensional harmonic oscillator with Hamiltonian $\widehat{H} = (\hat{p}^2 + \omega^2 \hat{x}^2)/2$. Measuring the operator $\hat{x}$ at time t_0 forces the oscillator into an eigenstate[2] $|x_0\rangle$. The operator $\hat{x}(t_0)$ commutes with the time evolved position operator $\hat{x}(t_0 + \tau)$ for certain values of τ

$$[\hat{x}(t_0), \hat{x}(t_0 + \tau)] = 0, \quad \text{if } \tau \in \mathcal{T} = \{\tau \mid \tau = n\pi\omega,\, n \in \mathbb{Z}\}\,, \tag{5}$$

whereas for times t in between one has

$$[\hat{x}(t_0), \hat{x}(t_0 + t)] \neq 0, \quad t \notin \mathcal{T}\,. \tag{6}$$

The spatial spreading of the eigenstate $|x_0\rangle$ and its subsequent "relocalization" at the point x_0 is illustrated in Figure 3. Since commuting operators possess a common set of eigenstates (cf. Eq. 5), the time evolution from t_0 up to $t_0 + \tau$ turns out to be a map U

[2] For simplicity it is assumed that ideal measurements exist.

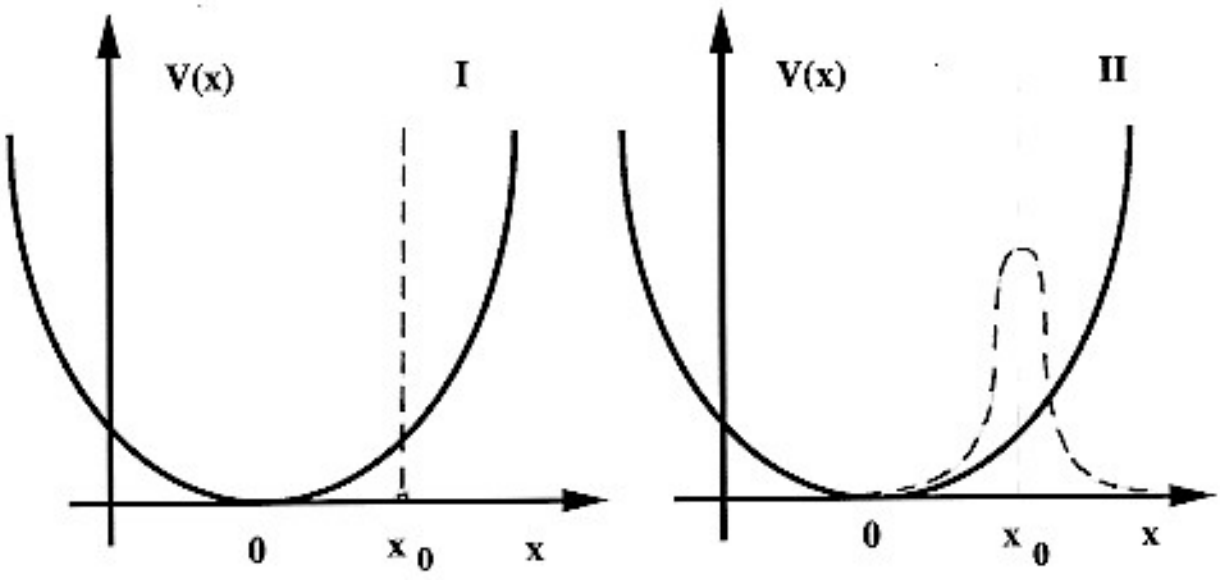

Figure 3: Schematic representation of the state of the harmonic oscillator at times $t_0 + \tau$ where $\tau \in \mathcal{T}$ (I) and at times $t_0 + \tau$ where $\tau \notin \mathcal{T}$ (II)

of the eigenstates $\{|x\rangle\}$ onto themselves. However, in the case of a harmonic oscillator, this map coincides with the identity map $U \equiv 1$ but this need not to be case in general. Consequently, having performed a position measurement at t_0 with the result x_0, one is able to predict *with certainty* the outcome of a position measurement at time $t_0+\tau$. This statement subsequently may be checked by a measurement at $t_0+\tau$, in principle *without disturbing* the quantum state. Consequently, the *same* system may be used again for other measurements at a second , third, ... time $t+\tau'$, $t+\tau''$, ... ($\tau', \tau'', \ldots \in \mathcal{T}$). This results from the fact that the observation of a system prepared in an eigenstate of the measured quantity does not lead to a change in the state of the system. For this reason operators fulfilling Eq. 5 are called quantum-nondemolition.

The generalization to a complete set of commuting operators $\hat{g}$ is straightforward. A complete set of commuting observables consisting of quantum-nondemolition observables is defined by the conditions

$$[\hat{g}_i(t), \hat{g}_j(t+\tau)] = 0 \quad \forall \tau \in \mathcal{T} \text{ and } i, j = 1, 2, \ldots, \dim \Gamma, \tag{7}$$

implying again that there is a map U of the basis vectors $\{|g\rangle\}$ at time t onto the same set $\{|g\rangle\}$ at times $t+\tau$ with $\tau \in \mathcal{T}$, possibly labelled in a different way. In this situation it is possible to predict exactly from the measurement of the quantum state at an initial time t the result of measurement of the same complete set of commuting observables at any later time $t + \tau$ ($\tau \in \mathcal{T}$) *and* to actually observe this evolution by repeated measurements on one single system. Clearly, this situation formally parallels classical mechanics: no probabilistic statements do enter and as a consequence, deterministic randomness, if present, may be observed unambiguously.

SUMMARY AND CONCLUSIONS

Algorithmic complexity underlying chaotic motion in classical mechanics also exists in quantum mechanics and can be observed via quantum-nondemolition measurements.

Starting from the theory-independent notion of algorithmic complexity it is possible to understand under which circumstances "true quantum chaos" [5] may occur. Quantum systems in which a continuous manifold is mapped repeatedly onto itself by the time evolution are candidates to exhibit deterministic randomness. The manifold Γ, however, does not necessarily coincide with the Hilbert space of the system but

may consists of a continuously labelled complete set of (generalized) basisvectors of this space. Due to this property one is lead in a natural way to the consideration of quantum-nondemolition measurements. It turns out that they indeed present an the appropriate framework for the discussion of the existence and observation of deterministic randomness in quantum mechanics, although they are not related to the *formation* of this phenomenon.

The deterministic rearrangement of the set of eigenstates associated with a complete set of commuting operators by the type of time-evolution considered here allows one to make exact predictions about the outcomes of measurements at later times, and to actually check experimentally these statements unambiguously. As a consequence, the occurrence of deterministic randomness of the quantum motion, if present, can be detected unambiguously in such a situation.

In addition, a somewhat surprising feature emerges: only one single quantum system is needed in order to follow the actual time evolution—no *ensemble* of equally prepared systems is necessary.

The conditions for quantum systems to exhibit deterministic randomness are quite specific. Nevertheless, from a fundamental point of view to know about the existence of strongly irregular motion in the innocently looking, linear and deterministic Schrödinger equation may help to better understand the properties of the generic quantum-mechanical dynamics.

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