Quantum Chaos[1]

The term Quantum Chaos designates a body of knowledge which has been established in an attempt to understand the implications of Classical Chaos for quantum theory. Classical Mechanics successfully describes many aspects of the macroscopic world in a phenomenological way. Chaotic behaviour being ubiquitiuos, its presence begs for an explanation in terms of (non-relativistic) quantum mechanics, the fundamental theory to describe matter. Only the *deterministic* part of the quantum time evolution generated by →Schrödinger's equation is of interest here while the probabilistic element introduced by →quantum measurements is ignored.

Classical Hamiltonian systems with $N \geq 2$ degrees of freedom are either integrable or non-integrable. The time evolution of integrable systems is *quasi-periodic*, hence simple: N global constants of motion exist which force trajectories in phase space to evolve on tori of dimension N. The distance between initially close trajectories increases at most *linearly* with time; the Lyapunov exponent, a measure for the rate of divergence of nearby trajectories, is equal to zero. In the vast majority of cases, however, fewer than N constants of motion exist and the system is non-integrable. A typical trajectory again evolves deterministically but now may explore a larger part of phase space. Due to their highly complicated – apparently chaotic – time evolutions, trajectories with similar initial conditions tend to diverge at an *exponential* rate. This property makes long-term predictions of the system's dynamics unreliable if not effectively impossible.

A considerable amount of studies relevant to Quantum Chaos revolve around three questions: (1) Is it possible to (approximately) quantize classically chaotic systems by exploiting their phase-space structure? (2) What are quantum mechanical manifestations–also known as *precursors* or *signatures*–of Classical Chaos? (3) Does a rigorous distinction between regular and chaotic quantum systems exist?

To answer these questions, quantum systems from many branches of physics and chemistry have been studied afresh from a new perspective. They include nuclei, atoms and molecules in the presence of strong electromagnetic fields, and microwaves in cavities, for example. The approaches to explore the properties of these systems range from experimental and numerical to rigorously mathematical.

For a long time, complicated dynamical behaviour has been assumed (tacitly) to require *many* interacting constituents such as the molecules of a gas. Their large number justifies the use of powerful statistical methods. Dynamical chaos, however, results from non-linear interactions between only a few degrees of freedom. This fundamental property of Classical Mechanics has been widely recognized only in the second half of the 20$^{\text{th}}$ century, when it became one of the driving forces to study quantum mechanical counterparts of classical systems with effectively unpredictable time evolution.

Widely studied models include quantum particles restricted to move in two-dimensional regions known as *billiards*, pairs of coupled spins or a single periodically driven spin. Reducing the continuous time evolution of a classically chaotic system

[1]Stefan Weigert (University of York, UK) in *Compendium of Quantum Physics: Concepts, Experiments, History and Philosophy*, edited by F. Weinert, K. Hentschel, D. Greenberger, and B. Falkenburg (Springer, in print)

to an iterated *map* has proved advantageous in many cases. Maps are simple to formulate but capture essential features of the dynamics. A thoroughly studied example is the (classical or quantum) *standard map* describing a kicked rotor. Many other systems such as an electron in a one-dimensional hydrogen atom in the presence of a periodically modulated electric field give rise to the same or structurally similar maps.

(1) If a quantum system has a classically chaotic limit, it is usually hard to extract useful information from its →Schrödinger's equation. Often, extensive numerical calculations are the only means to determine (the spatial structure, say, of) excited states and the corresponding energy levels. A substantial amount of work has thus been devoted to generalize the *torus quantization*, an early method to 'quantize' classical systems which precedes and thus bypasses →Schrödinger's equation. Its original formulation relies on the phase space of the system being foliated entirely by tori. This structure, however, only exists if the system is *integrable*, i.e., it must possesses as many global constants of motion as it has degrees of freedom. The foliation is destroyed if a perturbation is added to the system, and only a skeleton of closed trajectories known as *periodic orbits* continues to exist. Einstein realized in 1917 that the quantization conditions are not generally applicable [1]. The new approach, initiated in the early 1970ies, relies on the fact that, even in a non-integrable system, isolated periodic orbits survive and continue to determine the quantum properties of the system to a large extent. To see this, one uses the →path-integral formulation of quantum mechanics. The resulting *trace formula* provides an alternative and often efficient road to (approximately) quantize a classically chaotic system [2].

(2) The statistics of energy levels exhibit striking differences for different quantum systems. After appropriate normalization, the spacings between the energy eigenvalues of systems with a classically *regular* limit are described well by a Poisson distribution: small spacings dominate. The small spacings are suppressed for systems with a chaotic limit, resulting in a distribution derived by E.P. Wigner in 1951 to statistically describe observed energy spectra of nuclei [3]. The overall shapes of the distributions are universal in the sense that they only depend on symmetry properties such as the presence or absence of time reversal invariance of the system. It turns out that the spectra of *random matrices*, with matrix elements drawn from specific distributions determined by the symmetries, have very similar spectral properties [4]. This confirms the intuitively appealing picture that a Hamiltonian describing a quantum system with a classically chaotic limit corresponds to a matrix with 'random' entries.

The spatial structure of energy →eigenstates of a quantum system may also anticipate whether it has a classically chaotic counterpart or not [5], as do scattering amplitudes. It is the classical periodic orbits which, to a large extent, determine the properties of both bounded and open quantum systems in the →quasi-classical regime defined by $S/\hbar \ll 1$, where S is the value of the classical action associated with a typical periodic orbit.

The Anderson model of conduction in a one-dimensional disordered solid predicts that its energy eigenstates are confined to only small parts of the available space. Mathematically, the quantum standard map is structurally identical to the Anderson Hamiltonian if discrete time is thought to label lattice sites [6]. The resulting *dynamical localization* is used to explain that electron diffusion in a driven hydrogen atom

[7] deviates from classically expected behaviour: the atom is ultimately not ionized since the diffusion is suppressed quantum mechanically.

(3) Ideally, a concept such as Quantum Chaos should rest upon a definition which is inherently quantum mechanical: it should not depend on properties of quantum system which emerge only in the classical limit. The challenge is to put each (non-relativistic) quantum system with only a few degrees of freedom, say, in one of two disjoint classes using quantum mechanical concepts only. So far, no such division entailing sets of systems with *provably* different properties has been agreed upon.

Another fundamental aspect is the question to what degree →Schrödinger's equation, as a *linear* equation, is capable to generate complicated time evolutions. Is it conceivable that the evolution of a quantum state is as difficult to predict as a trajectory of a classically chaotic system, typically resulting from coupled non-linear differential equations? An appropriate Fourier transform of such a trajectory will reveal a *continuous* spectrum of frequencies, an unmistakable sign for the trajectory being highly irregular. If a similar approach is taken within a time-independent quantum system, the resulting spectrum will be determined by the energy eigenvalues of the system which are a *discrete* set if the quantum system has bound states only. This observation explains why externally driven quantum systems and scattering processes are promising candidates when searching for chaotic behaviour in quantum quantum mechanics.

The tendency of quantum mechanics to suppress chaos is supported by a phase space perspective: quantization can be thought of as introducing a 'granular' structure. Its scale relates to the non-commutativity of position and momentum operators measured by the value of →Planck's constant $\hbar$. Thus, the evolution of arbitrarily fine structures in phase space, a hallmark of Classical Chaos, appears forbidden. Nevertheless, the time evolution of a quantum system may be as difficult to predict as a classical irregular trajectory if *commuting* observables such as two (or more) position operators undergo a complicated dynamics in configuration space.

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