Quantum State Reconstruction[1]

Quantum state reconstruction, or state reconstruction for short, aims at identifying an unknown →quantum state on the basis of experimentally accessible data. The Quantum Optics community usually refers to this inverse problem as *quantum (state) tomography* while the expression *quantum state estimation* is often used in the field of →Quantum Information. Reconstruction procedures depend on the physical context defined by the system carrying the unknown state, the experimentally accessible observables, the size of the ensemble of systems prepared in the unknown state, and the precision of the measured data.

A two-level system (such as a spin-1/2, a qubit, or the two polarizations of a photon) prepared in a state with density matrix $\hat{\rho}$ is sufficient to illustrate the idea of state reconstruction. With two positive eigenvalues summing to one, the density matrix is a *positive* operator, and it depends on *three* real parameters. In its Bloch representation, the parameters combine to a real vector $\mathbf{n}$ with length $|\mathbf{n}| \leq 1$,

$$\hat{\rho} = \frac{1}{2}\left(\mathbb{I} + \mathbf{n}\cdot\hat{\boldsymbol{\sigma}}\right) ,$$

where $\mathbb{I}$ denotes the identity operator, and the components of the spin operator $\hat{\boldsymbol{\sigma}}$ are given by the →Pauli matrices $\hat{\sigma}_x, \hat{\sigma}_y$, and $\hat{\sigma}_z$. This parametrization of the density matrix $\hat{\rho}$ is immediatly useful for state reconstruction since the components of the vector $\mathbf{n}$ coincide with the →expectation values of the →Pauli matrices in the state $\hat{\rho}$,

$$n_j = \mathrm{Tr}[\hat{\sigma}_j\hat{\rho}] \equiv \langle\hat{\sigma}_j\rangle_\rho\,, \quad j = x, y, z\,.$$

The three observables $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ are *informationally complete*: any state $\hat{\rho}$ of the two-level system is determined uniquely by the values of the measured expectations $\langle\hat{\sigma}_x\rangle_\rho$, $\langle\hat{\sigma}_y\rangle_\rho$, and $\langle\hat{\sigma}_z\rangle_\rho$. No *pair* of observables allows one to reconstruct the state of a two-level system but many other *triples* (and larger sets) of observables exist which are also informationally complete.This flexibility is highly desirable from an experimental point of view. Specific reconstruction procedures will take into account any additional information: if a system is known to reside in a →pure state, for example, it will be sufficient to measure a smaller number of →expectation values.

The reconstruction of a quantum state in a laboratory is necessarily based on →expectation values which are known only *approximately*: any ensemble used to measure an →expectation value such as $\langle\hat{\sigma}_x\rangle_\rho$ is *finite*, and any measuring apparatus invariably introduces uncertainties. Consequently, the collected data will be compatible with a continuous family of quantum states. The reconstruction is complicated by the fact that unacceptable density matrices with *negative* eigenvalues may arise upon inverting the information contained in experimentally observed mean values. To determine the 'best' candidate among the acceptable states requires additional selection criteria such as the maximum-likelihood method, for example.

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[1]Stefan Weigert (University of York, UK) in *Compendium of Quantum Physics: Concepts, Experiments, History and Philosophy*, edited by F. Weinert, K. Hentschel, D. Greenberger, and B. Falkenburg (Springer, in print)

In 1933, W. Pauli raised he question [1] whether the probability distributions $|\langle q|\psi\rangle|^2 dq$ (to find a particle located near position q) and $|\langle p|\psi\rangle|^2 dp$ (to find the particle with a momentum close to p) determine a single →pure state $|\psi\rangle$. This is an early instance of quantum state reconstruction, with a negative answer: in general, there is a number of →pure states, called Pauli partners, which give rise to the *same* pair of probabilities, also known as Pauli data.

E. Schrödinger suggested in 1935 to think of the →wave function as a *catalogue of expectations*, that is, a tool which succinctly holds the information about the →expectation value of any observable [2]. *In nuce*, this remark contains the concept of quantum state reconstruction. Knowing the →expectation values of all observables effectively means to know the quantum state, and only a technical problem remains to be solved, namely to identify an informationally complete set of observables, or *quorum*.

The tomography of classical objects has inspired a successful method of quantum state reconstruction. Quantum tomography is based on the →*Wigner function*, an intuitively appealing way to represent the state $\hat{\rho}$ of a quantum particle. This real function resembles a classical probability distribution for two real variables q and p although it may take negative values and, therefore, cannot be observed experimentally. It is not difficult, however, to derive *marginals* from the →Wigner function which are legitimate probability distributions. As shown in 1989, suitable families of marginals provide sufficient information to recover the →Wigner function and, *a fortiori*, the unknown state $\hat{\rho}$ [3]. The marginals can be measured through optical homodyning, a well-established technique of quantum optics, as has been demonstrated experimentally in 1993 [4].

Regarding the efficiency of different reconstruction schemes, some quantitative results are known for states residing in a d-dimensional space. Given a finite ensemble of quantum systems in one and the same state, the statistical error is minimal if measurements are performed with respect to $d + 1$ sets of *mutually unbiased* bases, each containing d observables [5]. So far, the required set of observables has been found to exist only if the dimension d equals the power of a prime number. To extract maximal information about an unknown state of which N copies of are provided, it is often advantageous to go beyond the traditional framework of →projective measurements, using →positive operator-valued measurements instead, for example. Within the field of →quantum cloning, the quality of a given reconstruction procedure is measured by its *fidelity* which compares the estimated state to the original one.

Literature

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