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# The problem of quantum integrability

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The notion of integrability in quantum mechanics is investigated in order to prepare rigorous grounds for the study of regular and irregular behaviour of quantum systems. Its common-sense definition turns out to have deficiencies which are illustrated by various explicit examples. Part of the ambiguities are shown to have their origin in the difficulty of transferring the concept of independent constants of motion into quantum mechanics, due to a fundamental theorem on sets of commuting operators by von Neumann. Taking into account the classical limit with coherent states does not resolve the problems. As a result, it is pointed out why the appealing phenomenological distinction between regular and chaotic quantum systems cannot be traced back to the present notion of "quantum integrability" in a mathematically rigorous way.

## 1. Introduction

The intention of this paper is to clarify the relation between the phenomenological separation of quantum systems into regular and chaotic ones, and the present notion of integrability in quantum mechanics. In particular, the possibility to connect this notion in a *mathematically rigorous* way to the observed physical phenomena will be investigated.

First, the theoretical motivation for the introduction of the concept of integrability into quantum mechanics, and some phenomenological evidence for the necessity of such a concept are presented.

Quantum mechanics – taken as the fundamental theory for the description of physical phenomena – should contain at least principally the basic features of classical mechanics. For this reason one would like to understand the phenomenon of chaotic behaviour encountered in classical mechanics on quantum mechanical grounds. Actually, various approaches to this problem are currently investigated constituting the field of "quantum chaos" [1, 2]. Inter alia, three aspects of this problem are to be distinguished.

On the one hand, one asks for "precursors" of classically irregular behaviour in quantum systems. This amounts to determine – mostly by numerical means – the properties of typical quantum-mechanical quantities (energy spectra [3], eigenfunctions [4, 5], etc.) for systems which have a classically chaotic counterpart. Compared to systems with classically integrable counterparts, considerable differences have been found indicating the reasonableness of a classification of quantum systems in analogy with classical mechanics. The second approach which is intimately interwoven with the first one consists of studying the semi-classical regime characterized roughly by assuming $\hbar$ to be a small quantity [6]. To this end WKB-methods have been refined considerably, and in fact they have been extended to classically non-integrable systems [7, 8]. In the semiclassical

limit, the correspondence between classical and and quantum-mechanical behaviour becomes even more pronounced. Thirdly, one may address directly the problem of the occurrence of "deterministic randomness" in the quantum time evolution [9, 10].

Another approach is to carry over the standard definition of integrability to quantum mechanics, thereby defining "quantum integrability" or "integrable quantum systems." This would define the *complement* of the set of those quantum systems which are expected to exhibit manifestations of irregular behaviour.

This approach – to be studied in the following – seems promising because of its generality and its formal equivalence to classical mechanics. Although a successful transfer of the notion of integrability into quantum mechanics would not lead directly to statements about "non-integrable" quantum systems, it is expected to reveal characteristic properties which such systems certainly should *not* possess. In addition, the class of systems which are worthwhile studying in the context of "quantum chaos" would be well-defined and then, possibly, all three aspects mentioned above could be understood coherently in the light of one concept. In spite of these appealing properties not much study has been devoted to the concept of integrability in quantum mechanics. In the following the common-sense definition of quantum integrability is studied in this respect.

The paper is organized as follows. In section 2 the ingredients of various definitions of integrability in quantum mechanics – leaning heavily on the classical notion – are presented. This definition is then shown to possess some deficiencies due to the fact that the structure of quantum mechanics is not taken into account with sufficient rigour. Explicit examples illustrate the difficulty of giving quantum integrability, as it stands, an indisputable, physical meaning. Subsequently, the impact of the classical limit on the problem of integrability is discussed briefly. In section 3, the ambiguities encountered in section 2 are shown to be partially related to a fundamental theorem by von Neumann. Section 4 contains a summary and discusses the problems in defining quantum integrability from a general point of view.

## 2. Integrability in quantum mechanics

### 2.1. The definition of integrability

In classical mechanics the concept of integrability is of importance because it entails a physically reasonable partition of all Hamiltonian systems into two classes.

In integrable systems [11], the solutions of the equations of motion globally foliate the phase space into smooth manifolds which have the topology of $N$-dimensional tori. As a consequence, the Fourier spectrum of trajectories is discrete, and only small regions of phase space are visited even after infinite times.

In non-integrable systems [12] there is no such global phase-space structure. Usually, the time evolution of neighbouring trajectories is totally different in the long run. The associated Fourier spectrum contains generically a continuous component, and reliable forecasts of the time evolution are a delicate matter because of the (algorithmic) complexity of possible motions.

A classical Hamiltonian system with Hamiltonian $H(\boldsymbol{p}, \boldsymbol{q})$ is called integrable if

($\alpha$) there exist $N$ *single-valued constants of motion* $\boldsymbol{I} = (I_1, I_2, \ldots, I_N)$ defined smoothly over all phase space $\Gamma$. Thus, one has

$$\{H, \boldsymbol{I}\} = 0, \tag{1}$$

where $\{\ ,\ \}$ denotes the Poisson bracket.

($\beta$) The constants of motion $\boldsymbol{I}$ are *functionally independent* of each other.

($\gamma$) All $N$ constants $\boldsymbol{I}$ are *in involution*:

$$\{I_n, I_{n'}\} = 0, \quad \forall n, n'. \tag{2}$$

The second condition ensures that on each hypersurface – defined by values $I_0$ – the directions of the phase-space flow $\mathrm{d}I_n$, $n = 1, 2, \ldots, N$, generated by the constants $I$, are linearly independent (except on separatrices).

This definition of integrability is sufficient for the purpose intended here – mathematically more elaborate definitions can be found e.g., in refs. [11, 13, 14].

It seems natural to apply to the definition of classical integrability Dirac's prescription of replacing Poisson brackets by commutators of corresponding quantum operators

$$\{\,,\,\} \to \frac{\mathrm{i}}{\hbar}[\,,\,] \tag{3}$$

in order to introduce the notion of quantum integrability. This has been done by a number of authors, see, e.g., refs. [15–19]. Their definitions will not be discussed separately, because, essentially, they all agree in calling a quantum system with Hamiltonian operator $\hat{H}$ quantum integrable if

($\alpha'$) there are $N$ *quantum constants of the motion* $\hat{I} = (\hat{I}_1, \hat{I}_2, \ldots, \hat{I}_N)$, i.e.,

$$\left[\hat{H}, \hat{I}\right] = 0. \tag{4}$$

($\beta'$) All $N$ constants $\hat{I}$ *commute* with each other:

$$\left[\hat{I}_n, \hat{I}_{n'}\right] = 0, \quad \forall n, n'. \tag{5}$$

The following discussion will take this "working definition" (abbreviated QI) as a starting point. Important features of the classical definition have been carried over into quantum mechanics: the definition of constants of motion in quantum mechanics is unambiguous as it is in classical mechanics and the condition of involution (which classically is necessary in order to derive the phase-space foliation into tori (see, e.g. ref. [20])) is transferred easily into quantum mechanics, too[#1]. However, the notion of functional independence, crucial for the classical concept of integrability, has been dropped.

Does this definition imply a partition of the set of all quantum systems into two classes with physically different properties? This being so, quantum integrable systems should be endowed with particular properties to be derived from this definition. However, a number of problems arises when attempting to define quantum integrability in this way. The next section points out that the difficulties are due to the very structure of quantum mechanics.

### 2.2. *Loopholes*

In this subsection it is argued that the present notion of quantum integrability is not a well-defined concept. For simplicity, the discussion is restricted to systems which can be quantized unambiguously: the Hamiltonian is supposed not to contain products of non-commuting operators. Two results are established.

**(A)** The class of quantum systems representing counterparts of classically non-integrable systems is expected to contain quantum non-integrable systems only. However, each member of this class can be shown to possess constants of motion sufficient in number to fulfill the requirements of the definition QI, hence, to render the system quantum integrable.

**(B)** The class of quantum systems representing counterparts of classically integrable systems is expected to contain quantum integrable systems only. However, it is possible to show that all commuting quantum constants of motion – neces-

[#1] It should be noted, however, that its meaning has changed: the vanishing commutators express the simultaneous measurability of the involved constants of motion which is a purely quantum-mechanical statement and does not reflect a classical concept in an obvious way. Considering the constants of motion as generators of transformations in phase space and Hilbert space, respectively, it is possible to obtain a somewhat closer connection.

sary to render the system quantum integrable – can be encoded into one single conserved quantity without loss of information. This leads to formal non-integrability of systems which one would expect to be quantum integrable.

It will be sufficient to discuss specific examples in order to realize the general validity of these statements. The investigation of the set of constants of motion in simple oscillator systems is particularly advantageous since explicit constructions can be performed. The laws of quantum mechanics are applied rigorously and, as far as possible, no reference is made to classical mechanics.

**(A)** In order to construct *two* constants of motion in a quantum system with a classically non-integrable counterpart a one-dimensional harmonic oscillator is considered first. It is described by the Hamiltonian operator

$$\hat{h} = \hbar\omega\left(a^{+}a^{-} + \tfrac{1}{2}\right) \equiv \hbar\omega\left(\hat{n} + \tfrac{1}{2}\right), \tag{6}$$

$\omega$ being the frequency of the oscillator. The creation and annihilation operators $a^{\pm}$ fulfill the commutation relation

$$[a^{-}, a^{+}] = 1, \tag{7}$$

and $\hat{n}$ is the number operator. There is a countable set of eigenstates of $\hat{n}$ and, simultaneously, of the Hamiltonian $\hat{h}$,

$$\hat{n}|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \ldots \tag{8}$$

constituting an orthonormal basis of the Hilbert space of the oscillator. The operator projecting on the energy eigenstate $|m\rangle$ is denoted by

$$\hat{P}(m) \equiv |m\rangle\langle m| \,. \tag{9}$$

All projection operators $\hat{P}$ commute with each other:

$$\left[\hat{P}(n), \hat{P}(m)\right] = 0. \tag{10}$$

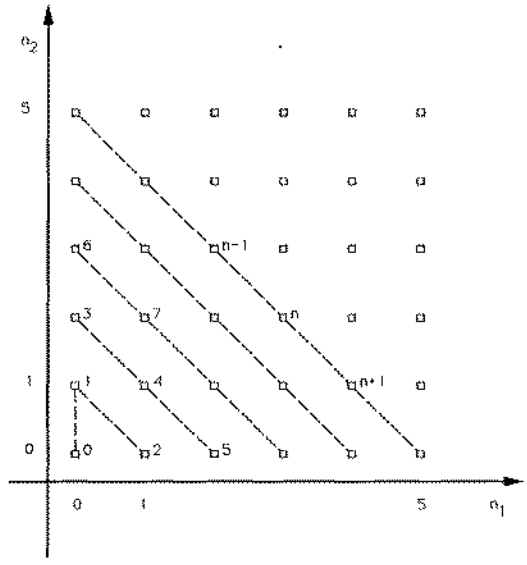


Fig. 1. Arranging the positive integers $n \in \mathbb{N}_0$ in a "two-dimensional" grid – or labeling two copies of $\mathbb{N}_0$ by one integer $n$.

The construction of two constants of motion proceeds as follows. The label $n = 0, 1, 2, \ldots$ of the basis $\{|n\rangle\}$ of the oscillator, eq. (6), can be replaced by *two* indices $n_1$ and $n_2$, both taking on all non-negative integer values. To this end the discrete set of integers $\mathbb{N}_0$ simply has to be arranged in a "two-dimensional" grid with axes $n_1$ and $n_2$ as depicted in fig. 1. The associated transformation can be given explicitly

$$n = n(n_1, n_2) \equiv n_1 + \tfrac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1). \tag{11}$$

Every pair $(n_1, n_2)$ determines exactly one number $n$ and vice versa, as is obvious from fig. 1. Consequently, one may label the eigenstate of the Hamiltonian by $n_1$ and $n_2$:

$$|n\rangle = |n(n_1, n_2)\rangle = |n_1, n_2\rangle, \quad n, n_1, n_2 \in \mathbb{N}_0. \tag{12}$$

The completeness relation reads

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}|n_1,n_2\rangle\langle n_1,n_2| = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1 \tag{13}$$

and the orthonormality becomes

$$\langle n_1,n_2|n_1',n_2'\rangle = \delta_{n_1n_1'}\delta_{n_2n_2'}. \tag{14}$$

The eigenvalues of the energy can be written as

$$h_n = \tilde{h}_{n_1,n_2} \equiv \hbar\omega\big[n_1+\tfrac{1}{2}(n_1+n_2)(n_1+n_2+1)+\tfrac{1}{2}\big], \tag{15}$$

where $n_1, n_2 \in \mathbb{N}_0$. Considering the labels $n_1$ and $n_2$ as quantum numbers it is natural to ask for corresponding operators $\hat{n}_1, \hat{n}_2$ and their properties. Observing that the sequence of states ($k \in \mathbb{N}_0$)

$$|0\rangle, |1\rangle, |3\rangle, |6\rangle, \ldots, |\tfrac{1}{2}k(k+1)\rangle, \ldots \tag{16}$$

belongs to the value $n_1 = 0$ and that

$$|2\rangle, |4\rangle, |7\rangle, 11\rangle, \ldots, |1+\tfrac{1}{2}(k+1)(k+2)\rangle, \ldots \tag{17}$$

and

$$|5\rangle, |8\rangle, |12\rangle, |17\rangle, \ldots, |2+\tfrac{1}{2}(k+2)(k+3)\rangle, \ldots \tag{18}$$

etc., are related to $n_1 = 1$ and $n_1 = 2$, respectively, one finds

$$\begin{aligned}\hat{n}_1 = 0 &\sum_{k=0}^{\infty}\hat{P}\big(\tfrac{1}{2}k(k+1)\big) \\ &+ 1\sum_{k=0}^{\infty}\hat{P}\big(1+\tfrac{1}{2}(k+1)(k+2)\big) \\ &+ 2\sum_{k=0}^{\infty}\hat{P}\big(2+\tfrac{1}{2}(k+2)(k+3)\big)+\cdots \\ = &\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} l\hat{P}\big(l+\tfrac{1}{2}(k+l)(k+l+1)\big).\end{aligned} \tag{19}$$

The operator $\hat{n}_2$ is defined analogously:

$$\hat{n}_2 = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} k\hat{P}\big(l+\tfrac{1}{2}(k+l)(k+l+1)\big). \tag{20}$$

Clearly, being linear combinations of commuting operators, the quantities $\hat{n}_1$ and $\hat{n}_2$ commute:

$$\big[\hat{n}_1,\hat{n}_2\big] = 0. \tag{21}$$

Finally, one can write the Hamiltonian itself as a function of the Hermitian operators $\hat{n}_1$ and $\hat{n}_2$:

$$\hat{h} = \hbar\omega\big[\hat{n}_1+\tfrac{1}{2}(\hat{n}_1+\hat{n}_2)(\hat{n}_1+\hat{n}_2+1)+\tfrac{1}{2}\big], \tag{22}$$

indicating that (trivially) $\hat{n}_1$ and $\hat{n}_2$ are both constants of the motion,

$$\big[\hat{n}_1,\hat{h}\big] = \big[\hat{n}_2,\hat{h}\big] = 0. \tag{23}$$

Note that one can choose the labels $n_1$ and $n_2$ completely independently of each other in order to specify energy eigenstates. All eigenvalues $n_1$ of the eigenvalue problem $\hat{n}_1|\psi\rangle = \hat{n}_1|\psi\rangle$, for example, are countably infinitely degenerate and the second label, $n_2$, is necessary in order to determine uniquely one particular state.

Consider now a system with two degrees of freedom possessing a classically non-integrable counterpart and Hamiltonian

$$\hat{H} = \sum_{n=0}^{\infty} E_n\hat{\mathscr{P}}_n. \tag{24}$$

This structure is typical – but not necessary – for systems representing candidates for non-integrability. It is possible to apply the above procedure to the sequence of eigenvalues $E_n$ ($\neq 0$), $n = 0,1,2,\ldots$ after performing an appropriate scaling of the energy axis.

Define an equidistant sequence of eigenvalues $\varepsilon_n$ by introducing the function $g^{-1}(z)$, defined on the energy spectrum of the Hamiltonian $\hat{H}$ such that

$$\varepsilon_n = g^{-1}(E_n), \quad E_n = n+\tfrac{1}{2}, \quad n = 0,1,2,\ldots. \tag{25}$$

The sequence $\varepsilon_n$ is associated with the well-defined operator

$$\hat{\mathscr{R}} = g^{-1}(\hat{H})\,\hat{H} = g^{-1}(\hat{H}) \sum_{n=0}^{\infty} E_n \hat{\mathscr{P}}_n$$

$$= \sum_{n=0}^{\infty} g^{-1}(E_n)\, E_n \hat{\mathscr{P}}_n = \sum_{n=0}^{\infty} \varepsilon_n \hat{\mathscr{P}}_n. \tag{26}$$

The operators

$$\hat{\mathscr{N}}_1 = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} l \hat{\mathscr{P}}\big(l + \tfrac{1}{2}(k+l)(k+l+1)\big),$$

$$\hat{\mathscr{N}}_2 = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} k \hat{\mathscr{P}}\big(l + \tfrac{1}{2}(k+l)(k+l+1)\big), \tag{27}$$

commute with the original Hamiltonian $\hat{H}$ since they are linear combinations of the projection operators $\hat{\mathscr{P}}_n$ on energy subspaces:

$$\big[\hat{\mathscr{N}}_1, \hat{\mathscr{N}}_2\big] = \big[\hat{\mathscr{N}}_1, \hat{H}\big] = \big[\hat{\mathscr{N}}_2, \hat{H}\big] = 0. \tag{28}$$

Making use of eq. (22) one can write

$$\hat{H} = g(\hat{H})\Big[\hat{\mathscr{N}}_1 + \tfrac{1}{2}\big(\hat{\mathscr{N}}_1 + \hat{\mathscr{N}}_2\big)\big(\hat{\mathscr{N}}_1 + \hat{\mathscr{N}}_2 + 1\big) + \tfrac{1}{2}\Big] \tag{29}$$

reproducing correctly the eigenvalues $E_n$ when applied to an eigenstate of the Hamiltonian $\hat{H}$. Since eq. (28) demonstrates that the requirements of the definition of quantum integrability are fulfilled, the present definition of quantum integrability does *not* single out a particular subset of all quantum systems.

By arranging the numbers $n = 0, 1, 2, \ldots$ in an appropriate "$d$-dimensional" grid one can analogously construct $d$ commuting constants of motion. Consequently, without further requirements on the operators to be counted in the definition QI the *number* of constants of motion in quantum systems is not uniquely given. This proves the general validity of statement **(A)**.

**(B)** Similar arguments can be used in order to *decrease* the number of constants of motion. Consider, for definiteness, a two-dimensional harmonic isotropic oscillator with frequencies $\omega_1 = \omega_2 = \omega$:

$$\hat{H} = \hbar\omega(a_1^+ a_1^- + a_2^+ a_2^- + 1) = \hbar\omega\big(\hat{N}_1 + \hat{N}_2 + 1\big). \tag{30}$$

For reasons of correspondence, this simple quantum system unambiguously should fall into the class of quantum integrable systems.

Fig. 1 now shows a natural arrangement of the eigenstates (or their labels) $\{|n_1, n_2\rangle; n_1, n_2 \in \mathbb{N}_0\}$ of this system in a two-dimensional grid. However, one is not forced to use *two* labels $n_1$ and $n_2$: one single index $n$ is sufficient to mark all states uniquely. Clearly, reversing the procedure from **(A)** is the simplest way to achieve this. The operator

$$\hat{\mathscr{R}} = \hat{N}_1 + \tfrac{1}{2}\big(\hat{N}_1 + \hat{N}_2\big)\big(\hat{N}_1 + \hat{N}_2 + 1\big) \tag{31}$$

has positive integer eigenvalues only:

$$\hat{\mathscr{R}}|k\rangle = k|k\rangle, \quad k \in \mathbb{N}_0, \tag{32}$$

and its eigenstates are given as direct products of eigenstates of $\hat{N}_1$ and $\hat{N}_2$:

$$|k\rangle = |n_1\rangle \otimes |n_2\rangle, \quad n_1, n_2 \in \mathbb{N}_0. \tag{33}$$

Note that the spectrum of $\hat{\mathscr{R}}$ is not degenerate: to every eigenvalue $k$ belongs one and only one state $|k\rangle$.

Introduce the "quadratic Gaussian bracket" $[\cdot]_{(2)}$ with

$$[x]_{(2)} = \tfrac{1}{2}n(n+1), \quad n \in \mathbb{N}_0, x \in \mathbb{R}_+, \tag{34}$$

where $n$ is the maximal integer such that

$$x - \tfrac{1}{2}n(n+1) \geq 0. \tag{35}$$

Then, any real number $x$ is decomposed into two parts:

$$x = [x]_{(2)} + \Delta x, \quad \Delta x \in [0, n]. \tag{36}$$

Effectively, this equation is the inversion of eq. (11). The bracket $[\cdot]_{(2)}$ allows to express the "natural" constants of motion, $\hat{N}_1$ and $\hat{N}_2$, as functions of the operator $\hat{\mathscr{K}}$ in a simple form. The Hamiltonian becomes

$$\hat{H} = \tfrac{1}{2}\hbar\omega\left(\sqrt{1 + 8[\hat{\mathscr{K}}]_{(2)}} + 1\right), \tag{37}$$

and the $(n+1)$-fold degeneracy of the $n$th energy level of the original Hamiltonian $\hat{H}$ is reproduced correctly in eq. (37) due to the structure of the bracket $[\hat{\mathscr{K}}]_{(2)}$; all $n+1$ values $\Delta x \in [0, n]$ belong to one eigenvalue of $[\hat{\mathscr{K}}]_{(2)}$. The conserved quantity $\hat{N}_1$ reads explicitly

$$\hat{N}_1 = \hat{\mathscr{K}} - [\hat{\mathscr{K}}]_{(2)}, \tag{38}$$

and for $\hat{N}_2$ one derives a similar expression.

Consequently, all information related to the original constants of motion, $\hat{N}_1$ and $\hat{N}_2$, is contained in the operator $\hat{\mathscr{K}}$. However, taking into account only this operator, the two-dimensional oscillator would not fall into the class of quantum integrable systems. Hence, the ambiguity in the number of constants of motion in a quantum system demonstrates statement **(B)**.

In section 3 it is pointed out that the possibility to combine various commuting operators into a single one is a *general* feature of quantum mechanics, having its origin in a theorem by von Neumann.

In the next section it is investigated whether taking the classical limit of operators into account gives a hint for the selection of relevant quantum constants of motion.

### *2.3. The classical limit as a rescue?*

What can one learn from these formal arguments which indicate the ambiguity of the present notion of quantum integrability? The examples presented in the previous section allow to pinpoint the problem: which constants of motion in a quantum system are "relevant" or "essential," and which are "irrelevant"? In order to exclude manipulations with labels of eigenstates such as presented in the previous section, precise conditions on allowed constants of motion must be given. They, in turn, might entail a classification of quantum systems into *physically* different sets bearing on the observed phenomenological distinction.

One might suspect that studying the behaviour of quantum constants of motion in the transition from quantum mechanics to classical mechanics will reveal their "essential" or "irrelevant" character in a rigorous way. Requesting *relevant* constants of motion to possess an associated (smooth and globally defined) classical counterpart seems to be a natural requirement. In appendix A the effectiveness of this principle of sorting quantum-mechanical operators is investigated for the examples given in the previous section.

The analysis shows that not all additional constants are ruled out by the requirement of a smooth classical limit according to Yaffe's prescription [22]. The attempted definition of "relevant" constants of motion turns out to be insufficient for two reasons. First, since quantum-mechanically inequivalent constants of motion – that is, constants of motion which do not only differ by quantities of the order $\hbar^2$ – can have the same classical limit, the induced distinction between quantum operator is not particularly efficient. Secondly, in spite of the artificial construction, operators used to reduce the number of quantum-mechanical constants of motion are not necessarily excluded. In particular, the operator $\hat{\mathscr{K}}$ defined in eq. (31) has a smooth classical limit.

## 3. von Neumann's comment

A theorem by von Neumann [23] states that for any number of commuting hermitian opera-

tors $\{\hat{Q}_i\}$, $i \in \mathrm{I}$, there exists one hermitian operator $\hat{\mathscr{Q}}$, such that all others can be conceived as functions $f_i$ of this particular operator

$$\hat{Q}_i = f_i(\hat{\mathscr{Q}}), \quad i \in \mathrm{I}, \tag{39}$$

I being a (finite) set of labels. The operators $\{\hat{Q}_i\}$ may have discrete or continuous spectra and are assumed to be continuous: the expectation values of all $\{\hat{Q}_i\}$ are bounded,

$$\|\hat{Q}_i\psi\|^2 \leq C_i\|\psi\|^2, \quad C_i < \infty \text{ and } \forall|\psi\rangle \in \mathscr{H}, \tag{40}$$

where $\|\cdot\|$ denotes the norm in Hilbert space $\mathscr{H}$. The case of operators $\{\hat{Q}_i\}$ with pure point spectra is discussed in von Neumann's book (ref. [24], section II.10) and is of particular interest here.

When discussing the notion of "complete sets of commuting observables" Kemble [25] explains how to visualize the content of this theorem: "To this end it is only necessary to pass a single line through all allowed points of $\alpha'$ space (the space of quantum numbers, St.W.) and to correlate points on this line with distance from its starting point" (ref. [25], p. 287). In principle, this is exactly what has been done in section 2.2 in order to combine the oscillator Hamiltonians $\hat{N}_1$ and $\hat{N}_2$ into one single operator $\hat{\mathscr{H}}$: the labels $n_1$ and $n_2$ (see fig. 1) span "$\alpha'$ space", and the label $k$ measures the "distance" from the starting point, $k = 0$. If the spectra of the operators under consideration have continuous components the "line through all allowed points in $\alpha'$ space" becomes complicated. In the discussion of a simple example von Neumann points out that a finite area has to be mapped onto a line and this is effected by a Peano curve.

One can easily explain von Neumann's theorem in physical terms. Let a particular quantum system be given and consider the measurements of various conserved quantities $\hat{Q}_1, \hat{Q}_2, \ldots$ such as energy, angular momentum, etc. with eigenvalues $\{Q^{\sigma_1}, Q^{\sigma_2}, \ldots\}$. As a result one obtains a set of numbers $Q^{\sigma_1} = E_n$, $Q^{\sigma_2} = l$, etc. If all observables under consideration correspond to commuting operators, it is principally possible to measure them simultaneously. By an appropriate way of encoding, one can report the outcome of all measurements in one single number

$$\mathscr{Q} = \mathscr{Q}(Q^{\sigma_1}, Q^{\sigma_2}, \ldots). \tag{41}$$

The collection of all individual measuring devices in combination with the prescription[#2] how to generate the number $\mathscr{Q}$ from the outcome of the distinct measurements can be conceived as an apparatus measuring the all-embracing operator $\hat{\mathscr{Q}}$. Clearly, as a result of the measurement, the physical system is left in an eigenstate of this operator, and the outcome of the individual measurements can be retrieved from the number $\mathscr{Q}$. It is tempting to actually consider the operator $\hat{\mathscr{Q}}$ as an *observable*.

An explicit, general construction of the operator $\mathscr{Q}$ is not difficult in case the self-adjoint operators $\hat{Q}_1, \hat{Q}_2, \ldots$ are bounded and have a pure point spectrum (cf. ref. [26], p. 304). The commuting observables $\hat{Q}_i$ possess a common set of eigenvectors $\{|\psi_{\sigma_1\sigma_2\ldots}\rangle\}$ constituting a basis of Hilbert space $\mathscr{H}$ with eigenvalues $Q^{\sigma_i} \in \mathbb{R}$

$$\hat{Q}_i|\psi_{\sigma_1\sigma_2\ldots}\rangle = Q^{\sigma_i}|\psi_{\sigma_1\sigma_2\ldots}\rangle, \quad \forall i. \tag{42}$$

All operators $\hat{Q}_i$ can be obtained as functions of the operator

$$\hat{\mathscr{Q}} = \sum_{\sigma} |\psi_\sigma\rangle \mathscr{Q}_\sigma \langle\psi_\sigma| \tag{43}$$

(where $\sigma \sim (\sigma_1\sigma_2\ldots)$), with real numbers $\mathscr{Q}_{\sigma'} \neq \mathscr{Q}_\sigma$, and the self-adjoint operator $\hat{\mathscr{Q}}$ can always be chosen to bounded by the requirement

$$\lim_{\sigma\to\infty} \mathscr{Q}_\sigma = z, \quad |z| < \infty. \tag{44}$$

To this end one has to define functions $f_i$, $i \in \mathrm{I}$,

[#2] For example, by a computer plus an appropriate program.

such that

$$f_i(\mathscr{C}_\sigma) = Q^{\sigma_i}, \quad \text{for } i \text{ fixed and } \forall \sigma_i. \tag{45}$$

The functions $\{f_i\}$ have prescribed values on the discrete set of points $\mathscr{C}_\sigma$ only and one may extend their definition appropriately on the line $\mathbb{R}_1$. The boundedness of the functions $\{f_i\}$ automatically follows from the boundedness of the eigenvalues of the operators $\{\hat{Q}_i\}$.

Many familiar operators such as energy and angular momentum, for example, are not bounded. Nevertheless, it is plausible that this construction still works since the essential point of the prescribed method is the existence of a Hilbert-space basis spanned by simultaneous eigenvectors of the bounded operators $\{\hat{Q}_i\}$. Such a basis still exists for self-adjoint operators (containing the physically relevant class of closed operators) which are not necessarily bounded. One defines the operator $\hat{\mathscr{C}}$ as in eq. (43). The functions $\{f_i\}$ cannot be bounded any longer. The possibility of this construction, furthermore, is supported by the physical picture relating $\hat{\mathscr{C}}$ to actual measurements – the boundedness of the measured observables does not enter here – and by the explicit procedure presented for the oscillator example given in section II, case **(B)**. The operator $\hat{\mathscr{R}}$ of eq. (32) is unbounded but introducing instead of $\hat{\mathscr{R}}$ the operator

$$\hat{\mathscr{R}}' = \sum_{k=0}^{\infty} |k\rangle\chi(k)\langle k| \equiv \sum_{k=0}^{\infty} \frac{|k\rangle\langle k|}{(k+1)^2}, \tag{46}$$

one deals with a bounded operator and $\hat{N}_1$. Then, the operators $\hat{N}_2$ can be expressed as a function of $\hat{\mathscr{R}}'$. Defining, for example,

$$f_1(x) = \frac{1}{\sqrt{x}} - \left[\frac{1}{\sqrt{x}} - 1\right]_{(2)} \tag{47}$$

one can write $\hat{N}_1$ as

$$\hat{N}_1 = f_1(\hat{\mathscr{R}}'), \tag{48}$$

with the square root of the positive operator $\hat{\mathscr{R}}'$

$$\langle\psi|\hat{\mathscr{R}}'|\psi\rangle > 0, \quad \forall|\psi\rangle \in \mathscr{H} \tag{49}$$

being well-defined.

To sum up: the relevant feature of von Neumann's theorem in this context is the impossibility to fix unambiguously the *number of elements* constituting a commuting set of operators.

## 4. Conclusions

In summary, the common-sense definition of quantum integrability does not provide a basis for a rigorous separation of the set of quantum systems into two classes. On the one hand, it has been shown that quantum systems which one would expect to be quantum non-integrable always possess a sufficient number of (counterintuitive) constants of motion to render the system formally quantum integrable. Since, on the other hand, any set of commuting constants of motion can be cast equivalently into one single well-defined invariant, quantum systems which are expected to be quantum integrable, formally should be called non-integrable.

The impact of this result is obvious: the present concept of quantum integrability cannot serve as a starting point for the derivation of properties, specific solely to a subset of all quantum systems. This, however, should be the very idea behind any definition of quantum integrability: to decompose the set of all quantum systems into subsets with distinct physical properties. Classical integrability is a useful concept just for this reason. Having recognized a system to be integrable by finding sufficiently many appropriate constants of motion, one knows – without performing any explicit calculation – that the phase space of the system is endowed with a particular structure. As a consequence, non-trivial predictions, concerning, e.g., the outcome of Poincaré sections or the Fourier spectrum of trajectories can be made. What are unambiguous manifestations of quan-

tum integrability to be compared to – but not necessarily equivalent to – the phase-space foliation of classically integrable systems into tori or, a fortiori, "insensitivity to initial conditions"? In other words, neither unambiguous predictions pertaining to quantum integrable systems can be made at present nor the phenomena observed in systems (obtained from quantizing classically integrable systems) can be explained by the existing notion of quantum integrability[#3]. Consequently, a mathematically more refined concept of quantum integability seems to be necessary in order to arrive at an indisputable classification of quantum systems.

Clearly, the difficulties exhibited in section 2.2 are due to the fact that the operators occurring in the definition of quantum integrability are *not* required to possess particular properties: a priori, any operator is allowed. Some features, indispensable in the definition of classical integrability, have been dropped in defining its quantum counterpart. For example, single-valuedness and smoothness represent necessary conditions on the allowed phase-space functions but the corresponding properties for operators are not self-evident. In addition, the concept of "functional independence" is not mentioned in the definition of quantum integrability[#4]. If, in the classical definition of quantum integrability, the functional independence of the constants of motion were dropped, one easily could devise sufficiently many constants of motion to render (truly) non-integrable systems integrable.

The attempt to restrict the set of allowed operators by the requirement of a smooth classical limit has not led to conclusive results. In addition, a detour via classical mechanics in order to decide on the relevance of quantum-mechanical constants of motion seems unsatisfactory. It is indispensable to distinguish between essential and irrelevant constants of motion without reference to classical mechanics. Finally, it is a formal procedure only to impose the condition that relevant quantum constants must posess a smooth classical limit – it remains to demonstrate that observable facts actually follow from this property. From a more general point of view the difficulty in defining "essential" constants of motion amounts to giving some additional structure to the set of operators in Hilbert space.

It should be emphasized that in order to conceive "quantum integrability" or, in turn, "quantum chaos" as a distinguished phenomenon on its own, the classification of quantum systems into "integrable" and "non-integrable" ones should *not* refer to the properties of their classical counterparts: an intrinsic quantum-mechanical definition is required.

In conclusion, it has been demonstrated that the present notion of quantum integrability, as it stands, is a formal concept only which does not entail a rigorous separation of quantum systems into distinct classes. The refinement of taking the classical limit of operators into account does not resolve the difficulties. The connection of the problem of quantum integrability with the basic problem of defining "intrinsically" physically relevant observables indicates that the ambiguity of the notion "quantum integrability" in bounded time-independent systems, strictly speaking, touches upon a fundamental problem.

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[#3] It is true that there are semiclassical results pertaining to "regular" or "integrable" quantum systems. Berry and Tabor [21], for example, derive the energy-level statistics of such systems. But such derivations usually are based on the *classical* definition of integrable systems which are then quantized semiclassically. Hence, the quantum version of integrability does not at all come into play.

[#4] Hietarinta in fact requires the constants of motion to represent "independent, well defined, global operators" (ref. [16], p. 1833), however, without specifying the meaning of these notions.

versions of the manuscript are gratefully acknowledged.

## Appendix A

In this appendix an explicit prescription to obtain the classical limit of an arbitrary operator by means of coherent states is discussed. The requirement that any operator occurring in the context of quantum integrability must have a smooth classical limit does not eliminate all examples of counterintuitive constants of motion presented in section 2.2.

In order to deal with a definite concept of the "classical limit" the prescription presented by Yaffe [22] is used. It is shown there that in general one formally recovers the structure of classical mechanics from quantum theory by the following procedure. First, coherent states are defined with respect to the commutation relations of the basic operators, e.g., position and momentum. Then, expectation values of quantum-mechanical operators (and identities between operators like Heisenberg equations) in these states are considered, which usually depend on Planck's constant $\hbar$. Performing subsequently the limit $\hbar \to 0$ one obtains a classical theory associated with the original quantum theory.

The coherent states $\{|z\rangle\}$ of the one-dimensional harmonic oscillator are defined as eigenfunctions of the annihilation operator $a^-$

$$a^-|z\rangle = z|z\rangle, \quad z = \frac{1}{\sqrt{2\hbar\omega}}(\omega q + \mathrm{i}p) \in \mathbb{C}. \tag{50}$$

In the energy basis $\{|n\rangle\}$ they read

$$|z\rangle = \exp\left(-\frac{H(p,q)}{2\hbar\omega}\right) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle, \tag{51}$$

where

$$H(p,q) = \tfrac{1}{2}(p^2 + \omega^2 q^2) \equiv \hbar\omega|z|^2 \tag{52}$$

is the classical Hamiltonian function ($m \equiv 1$). Consequently, one has

$$|\langle n|z\rangle|^2 = \frac{x^n}{n!}\mathrm{e}^{-x}, \tag{53}$$

abbreviating $H(p,q)/\hbar\omega \equiv x$.

The following remarks apply to case (A). The quantity $\hat{n}_1$ of eq. (19), renamed $\hat{M}$, can be written as

$$\hat{n}_1 \equiv \hat{M} = \sum_{n=0}^{\infty} M_n P(n), \tag{54}$$

with eigenvalues $M_n$ given by

$$\{M_n\} = (0,0,1,0,1,2,0,1,2,3,0,1,\ldots), \tag{55}$$

depicted in fig. 2. Note that the shape of the "envelope" of the maximal values $M_n$ for large values of $n$ is roughly given by $\sqrt{2n}$ : the eigenvalues of $\hat{M}$ increase as the square root of $2n$ for growing $n$.

The classical limit of the operator $\hat{M}$ follows from its expectation value taken in a coherent

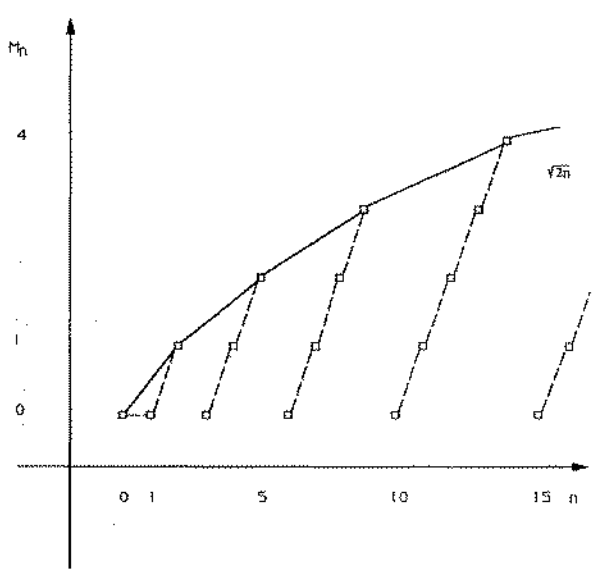


Fig. 2. Eigenvalues of the operator $\hat{M}$.

state $|z\rangle$,

$$\langle z|\hat{M}|z\rangle = e^{-x}\sum_{n=0}^{\infty} M_n \frac{x^n}{n!}. \tag{56}$$

A lengthy calculation shows that for large values of $x \equiv H(p,q)/\hbar\omega$ corresponding to small $\hbar$ one obtains

$$\langle z|\hat{M}|z\rangle = \alpha(\hbar)\sqrt{\frac{2H(p,q)}{\hbar\omega}}\,, \quad \alpha \in [0,1], \tag{57}$$

where $\alpha(\hbar)$ is a function which for $\hbar \to 0$ takes on the values 1 and 0 infinitely often; hence eq. (57) expresses the fact that $\lim_{\hbar\to 0}\sqrt{\hbar\omega}\,\langle z|\hat{M}|z\rangle$ does not exist.

This result can be understood without going through the explicit calculation. First, the sum in eq. (54) is transformed into an integral which is evaluated by the method of stationary phase. For each value of $\hbar$ only a small region of the integrand (depending on the actual value of $\hbar$) contributes to $\langle z|\hat{M}|z\rangle$. Due to the fact that the eigenvalues $M_n$ drop to 0 again and again the dominant term of $\langle z|\hat{M}|z\rangle$ is suppressed again and again. Therefore, the expression $\sqrt{\hbar\omega}\times\langle z|\hat{M}|z\rangle$ never settles down to any fixed value in the limit $\hbar \to 0$[#5]. It is interesting to note that nevertheless in eq. (22) the operators $\hat{n}_1 \equiv \hat{M}$ and $\hat{n}_2$ appear in such a combination that the classical limit of the Hamiltonian $\hat{H}$ is approached correctly.

This result looks promising: the operators $\hat{n}_1$ and $\hat{n}_2$ can indeed be excluded because they do *not* possess an appropriate classical counterpart.

Nevertheless, the requirement of a smooth classical limit of constants of motion involved still does not define "relevant" constants of motion uniquely. In the one-dimensional harmonic oscillator, eq. (6), for example, the operators

$$\hat{h}^{\pm} = \hbar\omega\sum_{n=0}^{\infty}(2n+1\pm\tfrac{1}{2})\hat{P}(2n+\tfrac{1}{2}\pm\tfrac{1}{2}), \tag{58}$$

also represent constants of motion, and they have smooth classical limits. Surprisingly, they become functions of the classical Hamiltonian

$$2\lim_{\hbar\to 0}\langle z|\hat{h}^{\pm}|z\rangle = H(p,q). \tag{59}$$

Intuitively, the three operators $\hat{h}^{\pm}$ and the Hamiltonian $\hat{h}$ are not equivalent quantum-mechanically – the criterion specified above, however, does in no way distinguish between the Hamiltonian $\hat{H}$ and the other two operators.

In case **(B)** two "natural" constants of motion have been represented by a single one. Investigating the classical limit of the operator $\hat{\mathscr{H}}$, eq. (31), along the same lines one finds its classical limit to be a simple function of the $2d$ oscillator Hamiltonian

$$\lim_{\hbar\to 0}(\hbar\omega)^2\langle z|\hat{\mathscr{H}}|z\rangle = \tfrac{1}{2}H(\boldsymbol{p},\boldsymbol{q})^2. \tag{60}$$

The state $|z\rangle$ denotes the direct product of coherent oscillator states associated with the systems 1 and 2, respectively,

$$|z\rangle = \frac{1}{\sqrt{2\hbar\omega}}|\omega\boldsymbol{q}+\mathrm{i}\boldsymbol{p}\rangle = |z_1\rangle\otimes|z_2\rangle. \tag{61}$$

Eq. (60) is consistent with the classical limit of the Hamiltonian $\hat{H}$ in eq. (37) which is required to be the classical Hamiltonian, $H(\boldsymbol{p},\boldsymbol{q})$. Consequently, the operator $\hat{\mathscr{H}}$ cannot be rejected according to the given criterion.

[#5] Scaling the axes in fig. 2 appropriately one observes that in the limit $\hbar \to 0$ (or $\Delta n \to 0$) the *area* under the envelope gets filled densely indicating that the expression $\sqrt{\hbar\omega}\,\langle z|\hat{M}|z\rangle$ does not have a limiting value.

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