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Many-path interference and topologically suppressed tunneling

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Abstract. – Quantum interference is studied for a charged particle on a multiply connected configuration space. The holes of the space are assumed to accommodate independent magnetic fluxes. As follows from an instanton calculation, tunneling of the particle is suppressed completely in such a geometry for appropriately chosen field strengths. This set of values defines an algebraic variety in parameter space, not a manifold. The quenching persists even if the geometric symmetry of the system is broken. Aharonov-Bohm–type experiments and mesoscopic tunneling devices are natural candidates to observe many-path interference.

For a quantum particle on the real line, all paths contributing to the propagator from one point P_a to another point P_b are topologically equivalent since they can be deformed smoothly into each other. On a closed loop, however, paths connecting two points come in different types labelled by a winding number which indicates how often a given path wraps around the loop [1]. Hence, the propagator is given by a sum of terms which may have nonzero relative phases when added up in point P_b. The resulting interference of amplitudes can be tuned if the particle interacts with a magnetic field penetrating the loop. This topological mechanism allows one to suppress the *tunneling* of a charged particle [2] if a gauge field is present. Tunneling of a spin can be modified similarly if its classical equilibrium positions are connected by two paths in phase space instead of only one path [3].

The purpose of this work is to study such phenomena if the underlying space is topologically more complicated than a loop. A charged particle in a symmetric double-well potential will be considered from now on. Its minima are separated by a barrier having $N = 2, 3, 4, \ldots$ equivalent saddles and a magnetic field $\mathbf{B}$ is present. The relevant features of this situation are captured by a model system defined as follows. Consider two points $P_\pm$ on the z-axis at a distance $2R$, and connect them [4] by N semicircles ("legs") with radius R consisting of a normally conducting metal. The resulting object is required to be invariant under the elements of the dihedral group D_{Nh}; it is denoted by $\mathcal{C}_N$, and it will be called a *carambola* [5]. A carambola

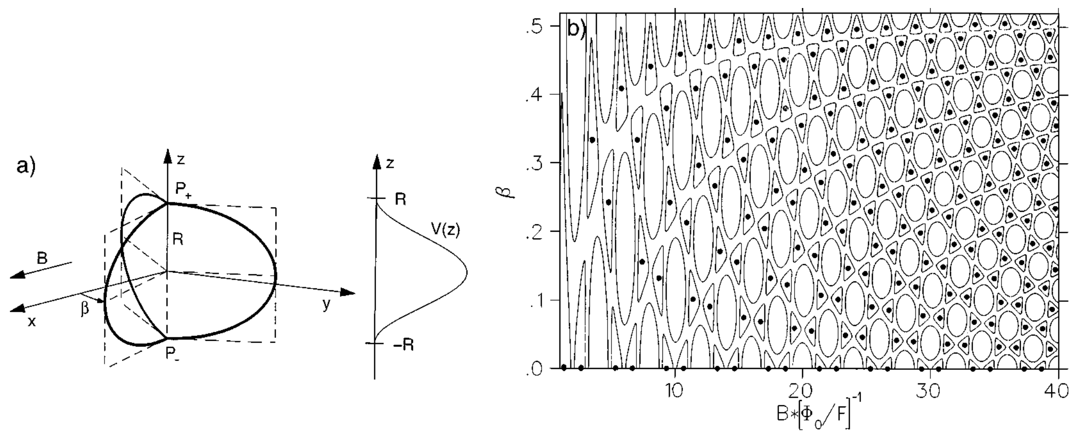

Fig. 1. – a): Carambola $\mathcal{C}_3$ with symmetry D_{nh}. The potential $V(z)$ turns the points $P_\pm$ into stable minima. A homogeneous magnetic field B points along the x-axis, and the angle β determines the amount of flux piercing the three areas. b): Equipotential lines of the squared modulus of the interference term $\varDelta$ as a function of the deflection angle $\beta \in [0, \pi/6]$ defined in a) and the rescaled field strength B (F is the area enclosed by two legs). Full circles correspond to zeroes of $|\varDelta|^2$.

$\mathcal{C}_N$ is topologically equivalent to the retraction of the plane with $N-1$ points removed or of a sphere punched N times [6], corresponding thus to a circle for $N = 2$ and a "figure-eight" for $N = 3$ (cf. fig. 1a)). If $N \geq 3$, the carambola has a *non-Abelian* homotopy group $\pi_1(\mathcal{C}_N)$, given by the free group with $N-1$ generators [7]. In other words, the composition of fundamental paths in this space is not commutative [8].

Consider a configuration space C with points $\mathbf{q}$. The amplitude for a transition from position eigenstate $|\mathbf{q}_a\rangle$ at time t_a to state $|\mathbf{q}_b\rangle$ at t_b can be expressed as a path integral,

$$K(\mathbf{q}_b, \mathbf{q}_a; T) = \int_{\mathbf{q}_a}^{\mathbf{q}_b} \mathcal{D}\mathbf{q}\, e^{iS[\mathbf{q}(t)]/\hbar}\,, \tag{1}$$

where the right-hand side is a sum over all paths connecting the points $\mathbf{q}_a$ and $\mathbf{q}_b$ in time $T = t_b - t_a$. Each path contributes a phase factor $\exp[iS[\mathbf{q}(t)]/\hbar]$, where the action $S[\mathbf{q}(t)]$, a functional of the paths $\mathbf{q}(t)$, is defined as the integral over the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ of the system. For a non-relativistic particle moving in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{q})$, it takes the form

$$S[\mathbf{q}(t)] = \int_{t_a}^{t_b} \mathrm{d}t \left(\frac{m}{2}\dot{\mathbf{q}}^2 - V(\mathbf{q}) + \frac{e}{c}\dot{\mathbf{q}} \cdot \mathbf{A}(\mathbf{q}) \right)\,, \tag{2}$$

where the dot denotes differentiation with respect to t.

If the space C is multiply path-connected, its first homotopy group $\pi_1(C)$ is not trivial. Then the propagator (1) decomposes [9] into a sum,

$$K(\mathbf{q}_b, \mathbf{q}_a; T) = \sum_{[\gamma]\in\pi_1(C)} a_{[\gamma]} K^{[\gamma]}(ba; T)\,, \tag{3}$$

over partial propagators $K^{[\gamma]}(ba;T) \equiv K^{[\gamma]}(\mathbf{q}_b, \mathbf{q}_a; T)$ labelled by the elements $[\gamma]$ of the homotopy group which correspond to classes of topologically inequivalent paths in C. The phase factors $a_{[\gamma]}$ constitute a one-dimensional unitary representation of the group $\pi_1(C)$, and they account for the nontrivial interference between the propagators $K^{[\gamma]}(ba; T)$.

The propagator will now be calculated for the carambola $\mathcal{C}_N$ in the presence of a potential $V(\mathbf{q})$ which turns the meeting points into potential minima (cf. fig. 1a)). Upon introducing an imaginary time coordinate $\tau = it$, the *Euclidean* action S_{E} is obtained from integrating the Euclidean Lagrangian L_{E},

$$S_{\mathrm{E}}[\mathbf{q}(\tau)] = \int_{\tau_a}^{\tau_b} \mathrm{d}\tau \left\{ \frac{m}{2}\mathbf{q}'^2 - W(\mathbf{q}) + i\frac{e}{c}\mathbf{q}' \cdot \mathbf{A}(\mathbf{q}) \right\} , \tag{4}$$

where $W(\mathbf{q}) \equiv -V(\mathbf{q})$ is the *inverted* potential [10] and the prime denotes differentiation with respect to τ. The resulting Euclidean propagator,

$$K_{\mathrm{E}}(ba; \tau_b - \tau_a) = \int_{\mathbf{q}(\tau_a)}^{\mathbf{q}(\tau_b)} \mathcal{D}\mathbf{q}\, e^{-S_{\mathrm{E}}[\mathbf{q}(\tau)]/\hbar} , \tag{5}$$

can be used to extract the separation of the two lowest energy levels by looking at large imaginary time $(\tau_b - \tau_a) \to \infty$. For tunneling problems, *kink-solutions* or *instantons* dominate the propagator. They correspond to a classical particle moving not in $V(\mathbf{q})$ but in the inverted potential, $W(\mathbf{q})$, starting at one of its maxima at time $\tau \to -\infty$ and reaching the other at $\tau \to \infty$. The vector potential $\mathbf{A}(\mathbf{q})$ has no influence on the classical motion since the term containing it is a total derivative [2] along each path.

For the carambola configuration $\mathcal{C}_N$, there are N different single instantons connecting the minima, and the Euclidean propagator (5) takes the form

$$K_{\mathrm{E}}(ba; T) \propto \sum_{j=1}^{N} \exp[-S^0_{\mathrm{E},j}/\hbar] , \tag{6}$$

where $S^0_{\mathrm{E},j}$ is the Euclidean action associated with a path γ_j along wire j. The imaginary parts $\sigma_j[\mathbf{q}(\tau)]$ of the actions $S^0_{\mathrm{E},j} = S_{\mathrm{R},j} - i\sigma_j$ depend on the vector potential $\mathbf{A}$. The real parts depend only on the potential $W(\mathbf{q})$ on leg j, hence they may be different for each path. This implies

$$K_{\mathrm{E}}(ba; T) \propto e^{-(S_{\mathrm{R},1} - i\sigma_1)/\hbar}\, \varDelta , \tag{7}$$

where $S_{\mathrm{R},1}$ is taken to be the *largest* real part of the partial Euclidean actions. The quantity $\varDelta$ measures the interference of the N contributions to the propagator and can be expressed as

$$\varDelta = \sum_{n=1}^{N} d_n e^{i\sigma_{n1}/\hbar} , \tag{8}$$

where $\sigma_{n1} \equiv \sigma_n - \sigma_1$ and $0 \le d_n \equiv \exp[(S_{\mathrm{R},1} - S_{\mathrm{R},n})/\hbar] \le 1$. Since [10] the ground-state splitting δE is proportional to $\varDelta$, a quenching will occur whenever $\varDelta$ vanishes:

$$1 + \sum_{n=2}^{N} d_n e^{i\sigma_{n1}/\hbar} = 0 . \tag{9}$$

The differences σ_{jk} are calculated from Stokes' theorem

$$\sigma_{jk} = \frac{e}{c} \oint_{jk} \mathrm{d}\mathbf{q} \cdot \mathbf{A}(\mathbf{q}) = \frac{\varPhi_{jk}}{\varPhi_0}\hbar , \tag{10}$$

where $\varPhi_{jk}$ is the flux through the area enclosed by the path $\gamma_j \circ (-\gamma_k)$ traversing leg j in the positive and leg k in the negative sense; $\varPhi_0 = hc/2e$ is the magnetic flux quantum.

In order to better understand condition (9), let us first look at a situation with all paths being equivalent, *i.e.* with $d_j = 1$ for all legs j. The quenching condition (9) contains $(N-1)$ parameters σ_{n1} corresponding to independent fluxes through $(N-1)$ surfaces defined by the wires. Thinking of (9) as a linear combination of unit vectors in the complex plane which have to add up to zero, its solutions correspond to polygons with N equal sides. The case $N=2$ has already been investigated [2]: a quenching of the tunnel splitting δE occurs whenever $\cos(\varPhi_{21}/\varPhi_0) = 0$. For $N=3$, the quenching condition requires the three vectors in (9) to form a triangle. The vector 1 is fixed, and one can combine the remaining ones in two ways to add up to zero. This happens whenever

$$\begin{aligned} \varPhi_{12} &= \frac{2\pi}{3}(3n+\delta)\varPhi_0\,, \qquad && m,n \in \mathbf{Z}\,, \\ \varPhi_{13} &= \frac{2\pi}{3}(3m+\delta')\varPhi_0\,, \qquad && (\delta \neq \delta') \in \{1,2\}\,. \end{aligned} \tag{11}$$

These conditions give rise to a regular grid of zeroes in a plane parameterized by the two fluxes. As an illustration, fig. 1b) shows the equipotential lines of the quantity $\varDelta$ for a carambola $\mathcal{C}_3$ with full symmetry D_{3h}, expressed in terms of parameters associated with the realistic geometry shown in fig. 1a). The full circles correspond to a vanishing tunnel splitting, *i.e.* the points given in (11). For $N=4$, a *continuous* parameter is needed to label the zeroes of (9): the first two sides of the rhombus define an *angle* taking values between $\pm\pi$; only a finite number of possibilities to close the polygon remains. As a result, the tunnel splitting $\varDelta$ now vanishes on *lines* in the three-dimensional parameter space. The set of all solutions is not a manifold but an algebraic variety [7]: it consists of three circles S^1 with any two of them touching each other at one point. For arbitrary N, eq. (9) defines a variety in an $(N-1)$-dimensional parameter space composed of $(N-3)$-dimensional manifolds glued together in a well-defined way. In other words, the lowest two energy levels of a particle tunneling on the carambola $\mathcal{C}_N$ generically coincide on a variety of codimension 2.

Interference on a loop ($\sim \mathcal{C}_2$) and on $\mathcal{C}_N$ with $N \geq 3$ differ fundamentally from each other for the following reason. The real parts of the two instanton contributions for a particle on a loop are required to be *identical* for completely destructive interference. This restriction does *not* apply if there are three (or more) paths: closed polygons which imply a quenching can also result if N appropriate vectors with *different* lengths are added. Physically, it is possible to satisfy eq. (9) by invoking *inequivalent* wires: the difference may be due to either different potentials on the wires or to wires of different lengths (which do not leave the planes defined by the semicircles).

Going beyond the single-instanton approximation [11] does not change the result (9). At first sight, this is surprising: a multi-instanton calculation could be expected to be sensitive to the full *non-Abelian* homotopy group $\pi_1(\mathcal{C}_N)$. However, for the propagator $K_{\mathrm{E}}(ba;T)$ in (6), only its "abelianization" $\pi_1/[\pi_1,\pi_1]$ is relevant [8], as always in scalar quantum mechanics [9]. The relevant topological information about the carambola is therefore already accounted for by N single instantons.

One experimental realization of a space with homotopy $\pi_1(\mathcal{C}_3)$ relies on the fabrication of nanostructures. Squeezing the three-dimensional carambola configuration of fig. 1a) into a plane while preserving the lengths of the individual wires yields a two-dimensional arrangement of wires (the "Yin-Yang") shown in fig. 2a). The quenching condition (9) holds since the real parts of the complex actions associated with the paths γ_0 and $\gamma_\pm$ are equal; the imaginary parts of the actions are, as before, determined by the amount of flux piercing the two regions into which the area with boundary $\gamma_\pm$ is divided. Applying a magnetic field that is inhomogeneous on a length scale given by the dimension of the mesoscopic structure, one can vary the fluxes

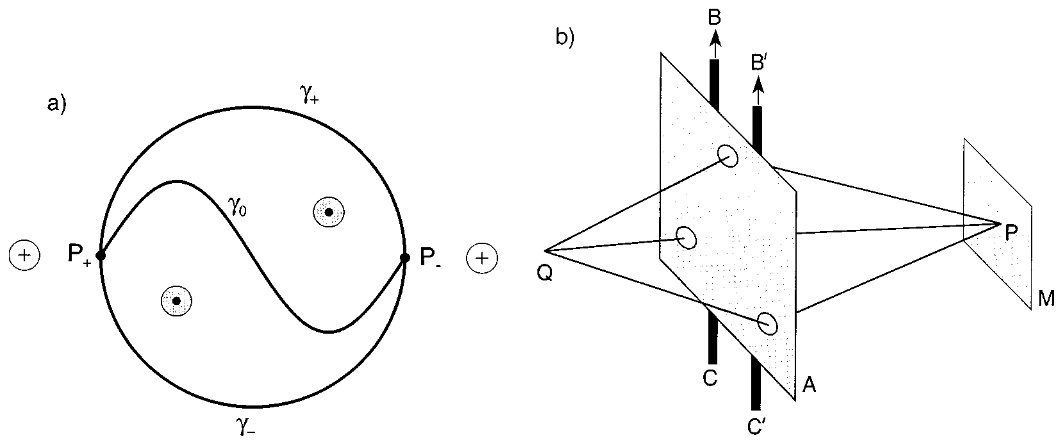

Fig. 2. – a): Planar Yin-Yang configuration with the topology of a carambola; the wire γ_0 is supposed to have the same lenght as the semicircles $\gamma_\pm$. Positive charges near the points $P_\pm$ turn them into equivalent potential minima for an electron. An inhomogeneous magnetic field B (perpendicular to the plane) provides different fluxes passing through the areas enclosed by γ_0, γ_+ and γ_0, γ_-, respectively. b): Aharonov-Bohm–type arrangement for three-path interference: electron source Q, shield A with C_3 symmetric location of three holes, coils C and C' with fields $\mathbf{B}$ and $\mathbf{B}'$, respectively, and screen M.

independently, in principle. Methods to actually measure the tunnel splitting have been discussed briefly in [2].

Further, a variant of an Aharonov-Bohm–type setup allows one to check experimentally properties of a system with multiply connected configuration space in the presence of gauge fields [12]. The *topological* structure of the carambola $\mathcal{C}_3$ is indeed realized by the arrangement shown in fig. 2b). A source Q emits electrons which subsequently pass through three holes located (with C_3 symmetry) on a shield A. The particles only travel through field-free regions since the magnetic fields $\mathbf{B}$ and $\mathbf{B}'$ are constrained to two tiny coils C and C', placed immediately behind the shield. By appropriately varying the fluxes in the coils, one can smoothly tune constructive into destructive interference at the point P on the screen M. The complete two-dimensional interference pattern on the screen M has threefold symmetry, and the maximum at P moves continuously away from the symmetry axis if the magnetic fields are turned on [11]. Complete destructive interference shows up for parameter values satisfying eq. (9) with $d_n = 1$ if the Euclidean actions σ_n are replaced by actions calculated from eq. (2) with $V(\mathbf{q}) \equiv 0$. The dimensions of the holes in the shield, the distances travelled by the electrons and their energy can be chosen similarly to those for standard Aharonov-Bohm experiments [13].

Finally, the structure of the carambola $\mathcal{C}_N$ emerges naturally for a spin in an environment allowing for tunneling [14]. A phase-space calculation takes into account N different instantons connecting the two equivalent minima for the spin. The magnetic monopole at the center, however, does not allow for independent variation of the $(N-1)$ fluxes through the individual areas.

In summary, the influence of topology on interference has been discussed for systems with N paths connecting two points. If three or more paths exist, they do not have to be equivalent to achieve topologically induced destructive interference. Physical systems have been proposed which realize the topology required to observe modifications of both free-particle propagation and tunneling.

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