INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 4169–4181 PII: S0305-4470(02)30013-1

Quantum parametric resonance

Stefan Weigert

Institut de Physique, Université de Neuchâtel, Neuchâtel, Switzerland
and
Department of Mathematics, University of Hull, Hull HU6 7RX, UK

E-mail: s.weigert@hull.ac.uk

Received 19 October 2001, in final form 12 February 2002
Published 26 April 2002
Online at stacks.iop.org/JPhysA/35/4169

Abstract
The quantum mechanical equivalent of parametric resonance is studied. A simple model of a periodically kicked harmonic oscillator is introduced which can be solved exactly. Classically stable and unstable regions in parameter space are shown to correspond to Floquet operators with qualitatively different properties. Their eigenfunctions, which are calculated exactly, exhibit a transition: for parameter values with classically *stable* solutions the eigenstates are normalizable while they cannot be normalized for parameter values with classically *unstable* solutions. Similarly, the spectrum of quasi energies undergoes a specific transition. These observations remain valid qualitatively for arbitrary linear systems exhibiting classically parametric resonance such as the paradigm example of a frequency modulated pendulum described by Mathieu's equation.

PACS numbers: 03.65.Sq, 05.45.−a

1. Introduction

The highly complicated behaviour of classically chaotic systems does not translate in an obvious and straightforward way into properties of their quantum counterparts. Nevertheless, various features such as energy level statistics and the spatial structures of wave functions have been identified [1, 2] as more or less faithful quantum indicators of chaos in the classical limit. The extreme sensitivity of classically chaotic systems to the variation of initial conditions appears, for fundamental reasons, to figure less prominent in quantum systems—although deterministic randomness is compatible with Schrödinger dynamics [3].

Parametric resonance is a useful concept in order to understand the transition from regular to chaotic motion in classical dynamic systems. Each of the phase space filling tori of an integrable system (with two degrees of freedom) is characterized by a frequency ratio, and the value of this parameter determines the fate of a torus under a perturbation according to the KAM theorem [4]. When the strength of the perturbation increases, initially small areas of instability proliferate in the form of *Arnold tongues*. They echo the intricate structure of phase space regions where motion on tori and irregular trajectories coexist. Apart from its conceptual importance, parametric resonance has found applications in systems with many degrees of freedom providing, for example, an important mechanism in the formation of patterns [5, 6].

The purpose of this paper is to study the phenomenon of parametric resonance in a quantum context. Hamiltonians quadratic in position and momentum are known to provide relevant, if not paradigmatic, examples of classical parametric resonance. Since the classical and quantum mechanical behaviour of such systems are intimately related, the program here is to make explicit what the *quantum manifestations* of the *classical instability* look like. In particular, the focus will be on the quasi-energy spectrum of the quantum mechanical evolution operator or *Floquet* operator and on its eigenstates. It will be shown that the properties of the quantum system unambiguously reflect the stability and instability of its classical counterpart.

In section 2 classical parametric resonance is discussed from a general point of view. An analytically solvable model exhibiting parametric resonance is introduced in section 3: a classical harmonic oscillator is subjected to a perturbation which periodically dilates and squeezes volumes in phase space. The parameter ranges for the instability of the classical system are determined. In section 4, Floquet eigenstates and quasi energies of the associated quantum system are calculated in the unstable case. By means of an *effective* Hamiltonian, a comprehensive point of view for the possible scenarios is developed in section 5. Subsequently, the result is shown to persist for linear systems with arbitrary periodic frequency modulation. The generalization includes systems which are described classically by Mathieu's equation. Finally, the results are summarized and links to related models are pointed out.

2. Classical parametric resonance

Parametric *resonance* occurs if an appropriate parameter of a classical dynamical system is varied periodically in time. Stable fixed points of the flow in phase space become unstable for specific values of the period of the parameter variation. As an example, consider a pendulum with mass m and length l under the influence of gravity, the support of which is moved up and down by an amount $\Delta l < l$ with frequency ω_0. The equilibrium position of the undriven system corresponds to the bob resting vertically below the support. For small oscillations, the pendulum is described by Mathieu's equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2(t)x = 0 \tag{1}$$

with

$$\omega^2(t) = l - \Delta l \cos(\omega_0 t) > 0 \tag{2}$$

where $x(t)$ is the vertical coordinate of the bob. This equation provides the paradigm for parametric resonance in classical mechanics: the behaviour of its solutions has been investigated in great detail [7] as a function of the parameters l, Δl and ω_0 (see [8] for an interesting presentation). The equilibrium position is *destabilized* for appropriately chosen values of Δl and ω_0: arbitrarily small deviations from the fixed point will grow exponentially in time. This is characteristic for parametric resonance. For other parameter values, the elliptic fixed point of the undriven system survives the perturbation. Smooth boundaries decompose the parameter space into regions with stable and unstable behaviour. On the separating boundary, a third marginal type of behaviour occurs which interpolates between stable and unstable motion.

Parametric excitation can also have the opposite effect which is less intuitive: an unstable fixed point may become a stable one. Indeed, the *inverted* pendulum turns stable under specific conditions on Δl and $\omega(t)$, a phenomenon known as parametric *stabilization*. Small deviations from the vertical position do *not* grow without bound but the pendulum remains in the neighbourhood of its initial position for all times.

Qualitatively, these phenomena continue to exist for other types of *periodic* driving, $\omega^2(t+T) = \omega^2(t)$; the actual division of parameter space into stable and unstable regions will depend on the function $\omega(t)$. In the following, a particularly simple quantum system, known to exhibit parametric resonance classically, will be studied in detail.

3. A simple model

Consider a harmonic oscillator with frequency ω and Hamiltonian $H_0 = p^2/2m + m\omega^2x^2/2$. Subject it to a periodic perturbation [9],

$$H(t) = H_0 + H_k(t) \tag{3}$$

where

$$H_k(t) = \frac{\alpha}{2}(xp + px)\delta_T(t) \qquad \alpha \in \mathbb{R} \tag{4}$$

with an infinite comb of delta functions,

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT). \tag{5}$$

In between the times $T_n = nT$, $n \in \mathbb{Z}$, the particle with mass m oscillates in a quadratic potential, while at times T_n it experiences an impulsive kick with amplitude αxp. In fact, the Hamiltonian does not depend on three but only on two parameters (ωT and α) as seen from introducing $\tau = t/T$, and rescaling simultaneously $p \to \sqrt{m\omega}p$, $q \to q/\sqrt{m\omega}$.

From a general point of view, the momentum dependence of the kick is a particular feature of this system. Other driven systems which have been studied as models for chaotic motion in both classical and quantum mechanics, such as the kicked rotator [10], have a position-dependent amplitude only. Various authors pointed out the consequences which result from this difference [11, 12], and explicitly solvable models have been studied [13].

Consider the evolution of the system over one period, from $t = -\varepsilon$ to $t = T - \varepsilon$, say, for $\varepsilon \ll T$. The equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} \alpha\delta_T(t) & 1/m \\ -m\omega^2 & -\alpha\delta_T(t) \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}. \tag{6}$$

The integration over the time interval $-\varepsilon < t < \varepsilon$ is carried out by first introducing a smooth approximation $\delta_T^\varepsilon(t)$ of the delta functions in (6). Then the solution of (6) is given by

$$\begin{pmatrix} x(+\varepsilon) \\ p(+\varepsilon) \end{pmatrix} = \mathcal{T}\exp\left[\int_{-\varepsilon}^{+\varepsilon} \mathrm{d}t\, M_\varepsilon(t)\right]\begin{pmatrix} x(-\varepsilon) \\ p(-\varepsilon) \end{pmatrix} \tag{7}$$

where the symbol $\mathcal{T}$ denotes time ordering, and $M_\varepsilon(t)$ is the smooth 2×2 matrix from (6). A straightforward calculation gives a simple result in the limit $\varepsilon \to 0$,

$$M_k \equiv \lim_{\varepsilon\to 0}\exp\left[\int_{-\varepsilon}^{+\varepsilon} \mathrm{d}t\, M_\varepsilon(t)\right] = \exp\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\alpha} & 0 \\ 0 & \mathrm{e}^{-\alpha} \end{pmatrix}. \tag{8}$$

Consequently, the interaction $H_k(t)$ rescales instantaneously position and momentum by the amounts $\exp[\pm\alpha]$, respectively. In other words, it generates periodically *dilations* [14], which preserve volume in phase space.

Next, equations (6) are integrated over the remaining time interval between the kicks, $\varepsilon < t < T - \varepsilon$. In the limit $\varepsilon \to 0$, one obtains

$$M_0 = \lim_{\varepsilon\to 0} M_0^{\varepsilon} = \lim_{\varepsilon\to 0} \exp\left[\begin{pmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{pmatrix}(T-\varepsilon)\right]$$
$$= \begin{pmatrix} \cos\omega T & (m\omega)^{-1}\sin\omega T \\ -m\omega\sin\omega T & \cos\omega T \end{pmatrix}. \tag{9}$$

Writing $z = (x, p)^T$ for points in phase space, the time evolution of the kicked harmonic oscillator (3) over one full period from $t = 0^-$ to $t = T^-$: $z(T^-) = Mz(0^-)$, is thus due to a kick followed by oscillatory motion, generated by the matrix

$$M = M_0 M_k = \begin{pmatrix} \mathrm{e}^{\alpha}\cos(\omega T) & (\mathrm{e}^{\alpha} m\omega)^{-1}\sin\omega T \\ -\mathrm{e}^{\alpha} m\omega\sin\omega T & \mathrm{e}^{-\alpha}\cos\omega T \end{pmatrix} \tag{10}$$

being symplectic and of unit determinant.

Dilations and oscillatory motion compete with each other. The overall character of the motion over one period depends on the actual values of the parameters. The matrix M does not vary over phase space. Hence, for given values of the parameters ω, α and T, all phase space points z ($\neq 0$) evolve in a similar way. The matrix M in (10) can generate three qualitatively different types of motion, conveniently characterized by their eigenvalues

$$\begin{aligned} \lambda_{\pm} &= \tfrac{1}{2}\mathrm{Tr}\, M \pm \sqrt{\tfrac{1}{4}(\mathrm{Tr}\, M)^2 - 1} \\ &= \cosh\alpha\cos\omega T \pm \sqrt{\cosh^2\alpha\cos^2\omega T - 1}. \end{aligned} \tag{11}$$

The eigenvalues may come as

(i) a complex conjugate pair with modulus one: $\lambda_{\pm} = \exp[\pm \mathrm{i}\Omega]$;
(ii) a real reciprocal pair: $|\lambda_{\pm}| = \exp[\pm\mu]$;
(iii) a real degenerate pair with modulus one: $\lambda_+ = \lambda_- = \pm 1$.

If the eigenvalues of M are purely imaginary (i), the images of a point of phase space z under the composed action of M_0 and M_k are located on an *ellipse*. Appropriate rescaling of the axes transforms it into a circle such that during one period the angular coordinate of a point is seen to increase by an angle $\tilde{\omega}$ defined through $\cos\tilde{\omega} \equiv \cosh\alpha\cos\omega T$. The perturbation changes the original motion only *quantitatively*, i.e. the frequency of the oscillator now is $\tilde{\omega}$ instead of ω.

In case (ii), the real eigenvalues of M indicate a 'hyperbolic rotation' parametrized conveniently by the positive real number μ with $\cosh\mu \equiv \cosh\alpha\cos\omega T$. The time evolution of the oscillator is changed *qualitatively* by the perturbation $H_k(t)$ from stable to unstable motion. According to the sign of the eigenvalues $\lambda_{\pm}$ the iterates of a phase space point are located on one (ordinary hyperbolic) or two (hyperbolic with reflection) branches of a *hyperbola*. The parameter regions associated with cases (i) and (ii) exhaust the parameter space almost completely.

Stable and unstable regions are separated by boundaries defined by the condition $|\cosh\alpha\cos\omega T| = 1$. For the parameter values of case (iii), iterates of phase space points are either on *one* ($\lambda_+ = \lambda_- = +1$) or on *two* ($\lambda_+ = \lambda_- = -1$) *straight lines*. This situation may be thought to interpolate between the elliptic and hyperbolic cases.

The three cases are related with specific invariant curves in phase space. To see this, associate a quadratic form with the matrix M by

$$Q(z) = z^T \Lambda M z \qquad \Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{12}$$

In fact, $Q(z)$ is invariant under time evolution over one period,

$$Q(Mz) = Q(z) \tag{13}$$

as follows from the symplecticity of M: $M^T \Lambda M = \Lambda$. Using M_0 and M_k as examples for (i) and (ii), respectively, one obtains

$$Q_0(z) \propto \tfrac{1}{2}(p^2 + m^2\omega^2 x^2) \qquad Q_k(z) \propto px \propto (p+x)^2 - (p-x)^2 \tag{14}$$

while the marginal situation (iii) implies

$$Q_m(z) \propto (p \pm x)^2 \tag{15}$$

using $\cos\omega T = \pm 1/\cosh\alpha$ and $\sin\omega T = \tanh\alpha$. These results can be summarized by writing

$$Q(z) \propto \tfrac{1}{2}(P^2 + \Omega^2 X^2) \tag{16}$$

where (X, P) are related to (x, p) by appropriate linear canonical transformations. Then, a variation of the original parameters is reflected in a change of the factor Ω^2 multiplying the quadratic term in X: Ω is either real, purely imaginary or zero. In physical terms, the time evolution of the system over one period is effectively that of a particle in an attractive or repulsive quadratic potential, or in the absence of a potential. Not surprisingly, the quadratic forms $Q(z)$ will play an important role for quantum parametric resonance.

If $\cos\omega T = 1$, or $\omega T = 2\pi k$, $k \in \mathbb{Z}$, an arbitrarily small kick amplitude α renders the system unstable. This particularly simple situation is called *resonant*—the harmonic evolution has *no* net effect: $M_0 = 1$. Similarly, for $\omega T = 2\pi(k + 1/2)$, $k \in \mathbb{Z}$, the time evolution is *resonant with reflection*: each point z in phase space is mapped to $(-z)$ by the harmonic time evolution: $M_0 = -1$.

4. Quantum parametric resonance

The quantum mechanical harmonic oscillator with an impulsive force [9] is described by the Hamiltonian operator

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 + \frac{\alpha}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})\delta_T(t). \tag{17}$$

Position and momentum operators $\hat{x}$ and $\hat{p}$ satisfy the fundamental commutation relation $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$. The long-time behaviour of the system is determined by the Floquet operator $\mathcal{F} = U(t_0 + T, t_0) \equiv U_T$. It maps a state at time t_0 to the state at time $t_0 + T$, i.e. over one period: $\mathcal{F}|\psi(t_0)\rangle = |\psi(t_0 + T)\rangle$. Choose the time $t_0 = T_0^- = 0^-$ just before the kick at $t = 0$. Then, due to the δ-type interaction the Floquet operator is a product of two unitary operators:

$$\mathcal{F} = U_0 U_k. \tag{18}$$

The operator U_0 generates harmonic motion with frequency ω from $t = 0^+$ to $t = T^-$:

$$U_0(T^-, 0^+) = \mathcal{T}\int_{0^+}^{T^-} \mathrm{d}t\ \exp[-\mathrm{i}\hat{H}(t)/\hbar] = \exp\left[-\frac{\mathrm{i}}{\hbar}\left(\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}\right)T\right] \tag{19}$$

where $\mathcal{T}$ denotes time ordering again. The operator U_k describes the effect of the kick at time $t = 0$. As before, the time-ordered product is evaluated explicitly by approximating the δ distribution as a strongly peaked, smooth function δ_T^ε and taking the limit $\varepsilon \to 0$ [13],

$$U_k(0^+, 0^-) = \mathcal{T}\int_{0^-}^{0^+} \mathrm{d}t \exp[-\mathrm{i}\hat{H}(t)/\hbar] = \exp\left[-\frac{\mathrm{i}\alpha}{2\hbar}(\hat{x}\hat{p} + \hat{p}\hat{x})\right]. \tag{20}$$

As shown in [16], the resulting 'squeeze operator' [15] has an implementation in a quantum optical context. Some properties of the operator $(\hat{x}\hat{p} + \hat{p}\hat{x})$ have been studied in [17]. Using the shorthand $\hat{z} \equiv (\hat{x}, \hat{p})^T$, the action of the Floquet operator $\mathcal{F}$ on position and momentum operators works out as in the classical case:

$$\hat{z}(T^-) = \mathcal{F}\hat{z}(0^-)\mathcal{F}^\dagger = M_0 M_k \hat{z}(0^-) = M\hat{z}(0^-) \tag{21}$$

where

$$\hat{z}(T^-) = U_0 \hat{z}(0^+) U_0^\dagger = M_0 z(0^+) \tag{22}$$

$$\hat{z}(0^+) = U_k \hat{z}(0^-) U_k^\dagger = M_k z(0^-). \tag{23}$$

These relations follow from either integrating the linear Heisenberg equations of motion or from expanding and resumming the exponentials involved. On comparing equations (8), (9) to the result (21), the intimate relation between the classical and quantum time evolutions generated by a quadratic Hamiltonian is evident.

Therefore, the quantum system is expected to inherit the division of parameter space into stable and unstable regions on the basis of equations (21), i.e. through the properties of the matrix M. How do the different parameter regions manifest themselves in the quantal framework? When moving in parameter space from a classically stable to an unstable region, the eigenstates $|\phi_\mu\rangle$ as well as the quasi energies E_μ of the Floquet operator,

$$\mathcal{F}|\phi_\mu\rangle = \exp(-\mathrm{i}E_\mu T/\hbar)|\phi_\mu\rangle \tag{24}$$

will undergo qualitative changes. The merit of the present model is that one can determine $|\phi_\mu\rangle$ and E_μ explicitly for characteristic cases. This will be done in section 5 by means of an *effective* Hamiltonian $\hat{H}_{\mathrm{eff}}$ which generates the time evolution over one period.

Before presenting the unified treatment, the most interesting case of a Floquet operator associated with a classically *unstable* region of parameters will be studied in detail. Consider, for simplicity, the resonant case, defined by $\omega T = 2\pi k$, $k \in \mathbb{Z}$: the classical motion becomes unstable for any nonzero value of α. The Floquet operator $\mathcal{F}$ reduces to just the kick operator U_k. It is the simplicity of the action of the kick on position and momentum eigenstates which allows one to construct the complete set of (δ-normalized) eigenstates of the time evolution operator.

According to (23), the position operators $\hat{x}_\pm$ just before and after the kick at time $t = 0$ are related by

$$\hat{x}_+ = U_k \hat{x}_- U_k^\dagger = \mathrm{e}^\alpha \hat{x}_- \tag{25}$$

and the eigenvalue equations read

$$\hat{x}_+|x_+\rangle = x_+|x_+\rangle \qquad \hat{x}_-|x_-\rangle = x_-|x_-\rangle. \tag{26}$$

Multiplication of the second equation with U_k from the left and using (25) implies that

$$\hat{x}_-\,(U_k|x_-\rangle) = \mathrm{e}^{-\alpha} x_-\,(U_k|x_-\rangle). \tag{27}$$

Hence, the state $U_k|x_-\rangle$ is an eigenstate of $\hat{x}_-$; in other words, the operator U_k maps a position eigenstate $|x\rangle$ to another position eigenstate with eigenvalue $\mathrm{e}^{-\alpha}x$:

$$U_k|x\rangle = \frac{1}{\sqrt{\rho(x)}}|\mathrm{e}^{-\alpha}x\rangle. \tag{28}$$

The factor $1/\sqrt{\rho(x)}$ accounts for a possible change of the normalization of the states. It guarantees the completeness relation to hold in the form $\int_{-\infty}^{\infty} \mathrm{d}x\,|x\rangle\langle x| = 1$ (cf [18]), and it is fixed by the following condition:

$$\delta(x-x') = \langle x \mid x'\rangle = \langle x|U_k^\dagger U_k|x'\rangle = \frac{1}{\rho(x)}\langle \mathrm{e}^{-\alpha}x \mid \mathrm{e}^{-\alpha}x'\rangle$$
$$= \frac{1}{\rho(x)}\delta(\mathrm{e}^{-\alpha}(x-x')) = \frac{\mathrm{e}^{\alpha}}{\rho(x)}\delta(x-x'). \tag{29}$$

This requires $\rho(x) = \mathrm{e}^{\alpha}$ leading to

$$U_k|x\rangle = \mathrm{e}^{-\alpha/2}|\mathrm{e}^{-\alpha}x\rangle \tag{30}$$

which agrees with the result in [19]. The action of U_k on a momentum eigenfunction $|p\rangle$ reads

$$U_k|p\rangle = \mathrm{e}^{\alpha/2}|\mathrm{e}^{\alpha}p\rangle \tag{31}$$

as follows from a similar argument for $\hat{p}$ or from exploiting $\langle p \mid x\rangle = \langle p|U_k^\dagger U_k|x\rangle$ etc.

The transformation property (30) provides the clue to determine the Floquet eigenstates as linear combinations of position eigenstates. Imagine iterating a position eigenstate $|x_0\rangle$ forwards and backwards by applying powers of U_k and $U_k^{-1} = U_k^\dagger$, respectively. After multiplying with appropriate phase factors, candidates for eigenstates of $\mathcal{F} \equiv U_k$ have the form

$$|x_0, \mu\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu n}\,\mathrm{e}^{-\alpha n/2}\big|\mathrm{e}^{-\alpha n}x_0\big\rangle. \tag{32}$$

As it stands, $|x_0, \mu\rangle$ is indeed mapped to itself under $\mathcal{F}$, except for an additional factor $\exp[-\mathrm{i}\mu]$,

$$\mathcal{F}|x_0, \mu\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu n}\,\mathrm{e}^{-\alpha(n+1)/2}\big|\mathrm{e}^{-\alpha(n+1)}x_0\big\rangle$$
$$= \mathrm{e}^{-\mathrm{i}\mu}\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu n}\,\mathrm{e}^{-\alpha n/2}\big|\mathrm{e}^{-\alpha n}x_0\big\rangle$$
$$= \mathrm{e}^{-\mathrm{i}\mu}|x_0, \mu\rangle. \tag{33}$$

There is no restriction on the values of $\mu \in [0, 2\pi)$—hence the spectrum of quasi energies $E_\mu = \mu\hbar/T$ is *continuous*. Let the label x_0 be from either one of the two intervals $[1, \mathrm{e}^{\alpha})$ or $[-\mathrm{e}^{\alpha}, -1)$, $\alpha > 0$, say. One can check explicitly that the scalar product of two such states with different labels vanishes,

$$\langle x_0, \mu \mid x_0', \mu'\rangle = \frac{1}{2\pi}\sum_{m,n=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\mu n}\,\mathrm{e}^{\mathrm{i}\mu' m}\,\mathrm{e}^{-\alpha(n+m)/2}\langle x_0|(U_k^\dagger)^n U_k^m|x_0'\rangle$$
$$= \frac{1}{2\pi}\sum_{n,m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(\mu n-\mathrm{i}\mu' m)}\,\mathrm{e}^{-\alpha(n+m)/2}\big\langle \mathrm{e}^{-\alpha n}x_0 \,\big|\, \mathrm{e}^{-\alpha m}x_0'\big\rangle\delta_{nm}$$
$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\mu'-\mu)n}\,\mathrm{e}^{-\alpha n}\delta\left(\mathrm{e}^{-\alpha n}(x_0 - x_0')\right)$$
$$= \delta(\mu-\mu')\delta(x_0 - x_0'). \tag{34}$$

Note that for x_0 and x_0' chosen from the *same* reference interval, the scalar product of position iterates with *different* n and m vanishes and, thus, gives rise to a Kronecker delta δ_{nm}.

The fact that the label x_0 takes values in the interval $[1,\ \mathrm{e}^{\alpha}]$ does not imply a continuous degeneracy of the quasi energies E_μ but only a *countable* one. This is due to the fact that eigenstates of the position operator $\hat{x}$ are not normalizable to one, i.e. they are not elements of a Hilbert space $L^2(\mathbb{R})$—nevertheless, these states can be used to expand its elements. In section 5.3 the countable degeneracy of the quasi energies E_μ will follow from general considerations.

The completeness relation for the states $|x, \mu\rangle$ reads

$$\int_0^{2\pi} \mathrm{d}\mu \int_1^{\mathrm{e}^\alpha} \mathrm{d}x |x, \mu\rangle\langle x, \mu| + \int_0^{2\pi} \mathrm{d}\mu \int_{-1}^{-\mathrm{e}^\alpha} \mathrm{d}x |x, \mu\rangle\langle x, \mu| + \lim_{\epsilon\to 0} \int_{-\epsilon}^{+\epsilon} \mathrm{d}x |x\rangle\langle x| = 1 \qquad (35)$$

where the integration extends over both fundamental intervals, and the state $|x = 0\rangle$ has to be included as an additional (non-normalizable) eigenstate of the Floquet operator. This state is clearly orthogonal to all other eigenstates of $\mathcal{F}$ in equation (32).

The Floquet operator associated with a classically unstable region is thus seen to have a continuous spectrum of quasi energies, and its (generalized) eigenfunctions do not approach zero for $x \to \pm\infty$. In addition, there is one state localized at the origin, $x = 0$, thus 'sitting on top' of the unstable fixed point of the classical map.

5. Effective Hamiltonian

According to equation (18) the Floquet operator $\mathcal{F}$ is a product of two noncommuting operators U_0 and U_k which are quadratic functions of position $\hat{x}$ and momentum $\hat{p}$. Since the Floquet operator is unitary, one can express it in the form $\mathcal{F} = \exp[-\mathrm{i}\hat{H}_{\mathrm{eff}}T/\hbar]$, with an 'effective' Hamiltonian $\hat{H}_{\mathrm{eff}}$. This operator is obtained from entangling the two unitary operators U_0 and U_k into a *single* exponential using Baker–Campbell–Hausdorff technology as presented in [20],

$$\exp\left[-\frac{\mathrm{i}}{\hbar}\left(\frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}\right)T\right]\exp\left[-\frac{\mathrm{i}\alpha}{2\hbar}(\hat{x}\hat{p} + \hat{p}\hat{x})\right] \equiv \exp\left[-\frac{\mathrm{i}}{\hbar}\hat{H}_{\mathrm{eff}}T\right]. \qquad (36)$$

In the present case, it is possible to determine explicitly the effective Hamiltonian since the operators in the exponents of the product in (36) constitute a Lie algebra,

$$[\hat{x}^2, \hat{p}^2] = 2\mathrm{i}\hbar(\hat{x}\hat{p} + \hat{p}\hat{x}) \qquad (37)$$

$$[(\hat{x}\hat{p} + \hat{p}\hat{x}), \hat{p}^2] = 2\mathrm{i}\hbar\hat{p}^2 \qquad (38)$$

$$[(\hat{x}\hat{p} + \hat{p}\hat{x}), \hat{x}^2] = -2\mathrm{i}\hbar\hat{x}^2 \qquad (39)$$

called 'split three-dimensional algebra' in [20], and well known as the algebra of the group $SU(1, 1)$. Physical realizations of this algebra, which is isomorphic to $sl(2, \mathbb{R})$, $so(1, 2)$ as well as $sp(1, \mathbb{R})$ (see [21]), can be given in terms of creation and annihilation operators of a two-dimensional harmonic oscillator or angular momentum operators, for example.

Due to the commutation relations, the effective Hamiltonian must be a linear combination of the three quadratic generators,

$$-\frac{\mathrm{i}}{\hbar}\hat{H}_{\mathrm{eff}}T = a\hat{p}^2 + b\hat{x}^2 + \frac{c}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}). \qquad (40)$$

The transformation of the operators $\hat{p}$ and $\hat{x}$ over one period T through $\mathcal{F}$ has been determined in equation (21), while in [20] their transformation has been calculated starting from a *given* quadratic expression (40). Combining these results, it is straightforward to determine the effective Hamiltonian: the parameters a, b, c must be expressed in terms of the elements M_{jk} $(j, k = 1, 2)$ of the matrix M. After some algebra one finds

$$a = \frac{\mathrm{i}}{\hbar}\frac{\Delta}{2}M_{12} \qquad (41)$$

$$b = -\frac{\mathrm{i}}{\hbar}\frac{\Delta}{2}M_{21} \qquad (42)$$

$$c = \frac{\mathrm{i}}{\hbar}\frac{\Delta}{2}(M_{11} - M_{22}). \qquad (43)$$

The real number Δ is given by

$$\Delta = \frac{\operatorname{arcsinh} D}{D} \equiv \Delta(D^2) \in \mathbb{R} \tag{44}$$

thus being an even function of the difference between the eigenvalues of the matrix M,

$$D = \tfrac{1}{2}(\lambda_+ - \lambda_-) = \left(\tfrac{1}{4}(M_{11} + M_{22})^2 - 1\right)^{1/2}. \tag{45}$$

Using (10) for the matrix elements M_{jk}, one obtains the expression

$$\hat{H}_{\text{eff}} = \Delta \frac{\sin \omega T}{\omega T} \left(\frac{\hat{p}^2}{2m\, \mathrm{e}^{\alpha}} + \frac{m\omega^2\, \mathrm{e}^{\alpha}}{2} \hat{x}^2 + \omega \sinh \alpha \cot \omega T\, (\hat{x}\hat{p} + \hat{p}\hat{x}) \right). \tag{46}$$

For $\alpha = 0$, the Hamiltonian $\hat{H}_{\text{eff}}$ turns into $\hat{H}_0$ since one finds $\Delta|_{\alpha=0} = \omega T / \sin \omega T$, using (44), (45) and (11). Similarly, taking the limit of $T \to 0$ in (46) can be shown to reproduce correctly U_k via equation (36). It is interesting to note that this result is closely related to the *quantized quadratic form*, i.e. the expression $\hat{Q} = Q(\hat{z})$ obtained from $Q(z)$ in (12) by replacing $z \to \hat{z}$ and appropriately symmetrizing,

$$Q(\hat{z}) \equiv \hat{z}^T \Lambda M \hat{z} = \sigma \hat{H}_{\text{eff}} + \tau \hat{I}, \tag{47}$$

with two constants σ and τ. Retrospectively, it is natural to find that the generator for time displacement over one period is a function of the quantized quadratic form invariant under the corresponding classical motion. Then, obviously, the Floquet operator $\mathcal{F}$ commutes with the corresponding quadratic form.

Therefore, three types of effective quantum Hamiltonians can arise which are in correspondence with the classically stable, unstable and marginal cases. Each of them is unitarily equivalent to one of the following expressions:

$$\hat{H}_{\text{eff}} \propto \tfrac{1}{2}(\hat{P}^2 + \Omega^2 \hat{X}^2) \tag{48}$$

where the operators $\hat{P}$ and $\hat{X}$ are unitarily equivalent to $\hat{x}$ and $\hat{p}$, and there are three possible values for the frequency Ω, nonzero real, purely imaginary or equal to zero. For simplicity, the discussion to follow will take (48) as a starting point. There is no need to use the explicit expressions of $\hat{H}_{\text{eff}}$ in terms of the original variables.

5.1. Stable elliptic case

For parameter values associated with a classically stable region, the effective Hamiltonian operator $\hat{H}_{\text{eff}}$ is unitarily equivalent to that of a harmonic oscillator with frequency $\Omega > 0$. Consequently, the eigenstates of the Floquet operator satisfy

$$\mathcal{F}|n\rangle = \mathrm{e}^{-\mathrm{i}\varepsilon_n}|n\rangle \qquad \varepsilon_n = E_n T/\hbar \tag{49}$$

with the familiar oscillator eigenstates $|n\rangle$,

$$\hat{H}_{\text{eff}}|n\rangle = E_n|n\rangle \qquad E_n = \hbar\Omega\left(n + \tfrac{1}{2}\right) \qquad n \in \mathbb{N}_0 \tag{50}$$

which provide an orthonormal basis in Hilbert space. The spectrum of quasi energies ε_n associated with the operator $\mathcal{F}$ in (49),

$$\varepsilon_n = \Omega T\left(n + \tfrac{1}{2}\right) \bmod 2\pi \qquad n = 0, 1, 2, \ldots \tag{51}$$

is obtained by 'projecting' the oscillator spectrum E_n onto the interval $[0, 2\pi)$. Two possibilities arise:

1. If ΩT is an *irrational* multiple of 2π then all quasi energies ε_n are different, $\varepsilon_n \neq \varepsilon_{n'}$. The union of all ε_n is dense in the interval from 0 to 2π. The set of eigenvalues $\{\varepsilon_n\}$ is countable, and no quasi energy is degenerate.

2. If ΩT is a *rational* multiple of 2π, say r/s, then $\varepsilon_{n+s} = \varepsilon_n$ for all n, and there are only s different quasi energies: $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{s-1} \in [0, 2\pi)$. Each value is countably degenerate, and the Hilbert space $\mathcal{H}$ decomposes into a direct sum,

$$\mathcal{H} = \bigoplus_{\nu=0}^{s-1} \mathcal{H}_\nu \tag{52}$$

where the finite-dimensional degenerate subspace $\mathcal{H}_\nu$ is spanned by the states $\{|\nu + ks\rangle, k \in \mathbb{N}_0\}$.

5.2. Marginal case

Consider now parameter values which imply that the effective Hamiltonian operator is unitarily equivalent to that of a free particle: $\hat{H}_{\text{eff}} \propto \hat{P}^2$, i.e. $\Omega = 0$. Its eigenstates are identical to those of a momentum-type operator,

$$\hat{P}|P\rangle = P|P\rangle \qquad P \in (-\infty, \infty) \tag{53}$$

hence require a δ-normalization. Since these states are also eigenstates of the Floquet operator $\mathcal{F}$, it has a *continuous* set of δ-normalized eigenstates. In the position representation, the expansion coefficients of the states $|P\rangle$ have modulus one throughout, and the states do *not* go to zero for $X \to \pm\infty$.

The quasi energies $\varepsilon_{0,k\pm} = P^2 T/(2\hbar)$ take all values in the interval $[0, 2\pi)$. Each quasi energy is countably degenerate: given a state $|P_0\rangle$, all states $|P_{0,k_\pm}\rangle$ with

$$P_{0,k}^{\pm} = \pm\sqrt{P_0^2 + \frac{2\hbar}{T} 2\pi k} \qquad k = 1, 2, \ldots \tag{54}$$

are eigenstates with the same eigenvalue: $\exp\left[-\mathrm{i}(P_{0,k}^{\pm})^2 T/(2\hbar)\right] = \exp\left[-\mathrm{i}P_0^2 T \big/ (2\hbar)\right]$.

5.3. Unstable hyperbolic case

The quantum system associated with a classically unstable region is unitarily equivalent to a particle in an inverted quadratic potential, that is, the Hamiltonian in (48) with $\Omega^2 < 0$. The potential takes arbitrarily large negative values for $X \to \pm\infty$, hence the solutions oscillate ever faster for increasing X. No normalizable eigenfunctions exist but there are two solutions of Schrödinger's equation for every value of the energy E. The spectrum of quasi energies E_μ exhibits thus the same features as in the marginal case: the numbers E_μ take any value between 0 and 2π, and each value is countably degenerate. This is due to the exponential function which 'wraps' the real variable $E \in (-\infty, \infty)$ around a circle. For the resonant case, the solutions have been given explicitly in section 4 as a sum of delta functions in the position representation with nonuniform amplitudes.

6. Arbitrary frequency modulation

The results obtained so far have been derived for a model with a particularly simple time evolution. Classically, however, the phenomenon of parametric resonance is known to occur for much more general periodic frequency modulations. In fact, the behaviour of the corresponding quantum systems can be shown to exhibit qualitatively the same behaviour. Consider the classical Hamiltonian

$$H(t) = \frac{p^2}{2m} + \frac{m\omega^2(t)}{2} x^2 \qquad \omega^2(t+T) = \omega^2(t) \tag{55}$$

which generates the equations of motion (1). The system, a pendulum with a harmonically modulated suspension point, exhibits parametric resonance as shown by Mathieu for the choice of $\omega^2(t)$ in (2). Write the Floquet operator of the system as a product of N unitary operators for N consecutive time intervals of length T/N each,

$$\mathcal{F} = \mathcal{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{0^-}^{T^-}\mathrm{d}t\,\hat{H}(t)\right] = \prod_{n=0}^{N-1} U(t_{n+1}, t_n) \qquad t_0 = 0^- \quad t_N = T^-. \tag{56}$$

In the limit $N \to \infty$ the length of the time intervals goes to zero, and one has approximately

$$\begin{aligned} U(t_{n+1}, t_n) &= \mathcal{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_n}^{t_{n+1}}\mathrm{d}t\left(\frac{\hat{p}^2}{2m} + \frac{m\omega^2(t)}{2}\hat{x}^2\right)\right] \\ &\simeq \exp\left[-\frac{\mathrm{i}}{\hbar}\left(\frac{\hat{p}^2}{2m} + \frac{m\omega_n^2}{2}\hat{x}^2\right)\frac{T}{N}\right] \end{aligned} \tag{57}$$

where the number ω_n^2 takes a value between $\omega^2(t_{n+1})$ and $\omega^2(t_n)$. Being quadratic in position and momentum, any two adjacent exponentials can be entangled by means of a Baker–Campbell–Hausdorff relation. The result is another exponential bilinear in $\hat{x}$ and $\hat{p}$ as follows from the algebraic properties (39). Repeating this process for ever larger values of N, the Floquet operator tends to

$$\mathcal{F} = \exp\left[-\frac{\mathrm{i}}{\hbar}\left(u\hat{p}^2 + v\hat{x}^2 + \frac{w}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})\right)\right] = \exp\left[-\frac{\mathrm{i}}{\hbar}\hat{H}^{\omega}_{\mathrm{eff}}T\right]. \tag{58}$$

According to the values of the parameters u, v and w, the effective Hamiltonian $\hat{H}^{\omega}_{\mathrm{eff}}$ necessarily will be (unitarily equivalent to) one of the three possible types discussed in section 5. As long as the frequency modulation is periodic, no qualitatively different behaviour can occur within this class of systems. Clearly, the separation of the parameter space into stable and unstable regions will depend in a subtle way on the actual function $\omega^2(t)$.

7. Discussion and outlook

The main result of the present paper is a global view on the phenomenon of parametric resonance in classical and quantum systems. Classically, in a periodically driven linear system three qualitatively different types of motion are possible, according to the chosen parameter values. This division of parameter space is mirrored quantum mechanically by qualitatively different spectra of the Floquet operator. Its eigenfunctions associated with a classically stable region are normalizable, and they become singular when moving over into a parameter region of classically unstable motion. This has been made explicit by constructing the eigenfunctions in terms of (improper) position eigenstates.

The class of time-dependent harmonic oscillators is well known to provide insight into various aspects of classical and quantum mechanics. Therefore, some of the results can be found implicitly in work dealing with other properties of such systems. Closest to the present approach is, maybe, work by Perelomov [22]: in a discussion of coherent states, a group-theoretical approach to driven linear systems is presented based on work in [23, 24]. The Heisenberg equations of motion for a system showing parametric resonance have been solved in [25]. Further, the paper [16] deals with the generation of parametric resonance within the realm of quantum optics. In this context, [26] is also interesting, where exponential divergence of the energy expectation value has been derived for an oscillator with periodically modulated frequency. In [27], the relationship of an autonomous oscillator with fixed frequency and no driving to a driven oscillator with time-dependent frequency (and even damping) is established.

Also, parametric resonance is known to play an important role in the collective excitation of atoms when they are subjected to a strong laser field [28].

An important body of related work deals with parametric *stabilization*. For example, the stability regions of an ion in a Paul trap are studied in [29]. Stable fixed points can be created in a Rydberg system by shining an appropriately tuned, circularly polarized wave on an atom; the quantum mechanical properties of this system have been worked out in [30]. In fact, parametric stabilization could be studied along the lines of the present paper by perturbing an initially unstable system, that is, an *inverted* harmonic oscillator (replace ω^2 by $-\omega^2$ in H_0 of equation (3)).

The impact of an instantaneous mass change of a quantum harmonic oscillator also gives rise to squeezing, and the impact on the variance of position, for example, has been studied in [31]. Non-periodic frequency modulations have been studied by various authors, starting with the now well-known Caldirola–Kanai oscillator with exponentially increasing mass [32, 33] which, effectively, gives rise to damped motion of the oscillator. Among other works, exact solutions have been found for a polynomial time-dependence, $\omega^2(t) = \omega^2 t^b$, $b > 0$ in [34].

In the studies mentioned, no global picture of quantum parametric resonance has been established. Further, no simple derivation of the eigenfunctions of the Floquet operator in the position basis can be found.

Finally, it is worthwhile to point out that the periodically driven oscillator systems studied here suggest a natural generalization: modify the amplitude of the kick according to

$$\hat{H}(t) = \hat{H}_0 + \frac{\alpha}{2}(V(\hat{x})\hat{p} + \hat{p}V(\hat{x}))\delta_T(t) \qquad (59)$$

where $V(x)$ is some smooth function. Such a kick can be shown to generate *nonlinear* maps of configuration space onto itself [9]. It will be shown elsewhere that features such as quasi-periodicity and a devil's staircase, which are known as characteristic features of classically chaotic dynamics, also may occur in the time evolution of quantum systems.

Acknowledgments

This work has been supported by the Schweizerischer National-Fonds. Support by Ph Morath in the early stages of preparing this publication as well as the permission to use parts of [9] are acknowledged.

References

[1] Reichl L 1992 *The Transition to Chaos* (Reading, MA: Addison-Wesley)
[2] Haake F 1992 *Quantum Signatures of Chaos* (Berlin: Springer)
[3] Weigert St 1990 *Z. Phys.* B **80** 3–4
[4] Lichtenberg A J and Lieberman M A 1992 *Regular and Chaotic Dynamics* (New York: Springer)
[5] Manneville P 1990 *Dissipative Structures and Weak Turbulence* (Boston, MA: Academic)
[6] Cross M C and Hohenberg P C 1993 *Rev. Mod. Phys.* **65** 851
[7] Abramowitz M and Stegun I 1984 *Pocketbook of Mathematical Functions* (Thun: Deutsch)
[8] Yorke E D 1978 *Am. J. Phys.* **46** 285
[9] Morath Ph 1996 Spektren und Eigenfunktionen magnetisch getriebener eindimensionaler Quantensysteme, *Diplomarbeit* Universität Basel
[10] Casati G, Chirikov B, Ford J and Izrailev F M 1979 Stochastic behaviour in classical and quantum Hamiltonian systems *Lecture Notes in Physics* vol 93, ed G Casati and J Ford (Berlin: Springer)
[11] Casati G and Chirikov B (eds) 1995 *Quantum Chaos: Between Order and Disorder* (Cambridge: Cambridge University Press)
[12] Peres A 1995 *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer)
[13] Weigert St 1993 *Phys. Rev.* A **48** 1780

[14] Moshinsky M and Smirnov Y F 1996 *The Harmonic Oscillator in Modern Physics* (Amsterdam: Harwood)
[15] Stoler D 1970 *Phys. Rev.* D **1** 3217
[16] Facchi P, Nakazato N, Pascazio S, Peřina J and Řeháček J 2000 Stability and instability in parametric resonance and quantum Zeno effect *Preprint* quant-ph/0004039
[17] Bollini C G and Oxman L E 1993 *Phys. Rev.* A **47** 2339
[18] Bohm A 1993 *Quantum Mechanics—Foundations and Applications* (New York: Springer)
[19] Casati G and Molinari J 1989 *Prog. Theor. Phys. Suppl.* **98** 287
[20] Wilcox R M 1967 *J. Math. Phys.* **8** 4
[21] Cornwell J F 1984 *Group Theory in Physics* vol 2 (London: Academic)
[22] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[23] Popov V S and Perelomov A M 1969 *Sov. Phys.–JETP* **29** 738
[24] Popov V S and Perelomov A M 1970 *Sov. Phys.–JETP* **30** 910
[25] Ji J-Y, Kim J-K and Kim S P 1995 *Phys. Rev.* A **51** 4268
[26] Ferrari L 1998 *Phys. Rev.* A **57** 2347
[27] Degli Esposti Boschi C and Ferrari L 1999 *Phys. Rev.* A **61** 010101(R)
[28] Bialynicki-Birula Z and Bialynicki-Birula I 1986 *Phys. Rev.* A **33** 1671
[29] Kelin W, Mang F and Juhao W 1995 *Phys. Rev.* A **52** 1419
[30] Bialynicki-Birula I, Kalinski M and Eberly J H 1994 *Phys. Rev. Lett.* **73** 1777
[31] Janszky J, Adam P and Földesi I 1993 *Phys. Lett.* A **174** 368
[32] Caldirola P 1941 *Nuovo Cimento* **18** 393
[33] Kanai E 1948 *Prog. Theor. Phys.* **3** 440
[34] Kim S P 1994 *J. Phys. A: Math. Gen.* **27** 3927