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Lüders theorem for coherent-state POVMs

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Lüders' theorem states that two observables commute if measuring one of them does not disturb the measurement outcomes of the other. We study measurements which are described by continuous positive operator-valued measurements (or POVMs) associated with coherent states on Lie groups. In general, operators turn out to be invariant under the *Lüders* map if their P- and Q-symbols coincide. For a spin corresponding to SU(2), the identity is shown to be the only operator with this property. For a particle, a countable family of linearly independent operators is identified which are invariant under the *Lüders* map generated by the coherent states of the Heisenberg–Weyl group, H_3. The *Lüders* map is also shown to implement the anti-normal ordering of creation and annihilation operators of a particle. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1623001]

I. INTRODUCTION

In this article we determine operators B which are invariant under a generalized *Lüders* map

$$B \mapsto \Lambda(B) = \int_{\mathbb{X}} d\mu(\Omega)\, E(\Omega) B E(\Omega)\,, \tag{1}$$

where each $E(\Omega)$ is a projection operator labeled by a point Ω of a manifold $\mathbb{X}$. These operators constitute a continuous positive operator-valued measure, or POVM, with a resolution of unity:

$$\int_{\mathbb{X}} d\mu(\Omega)\, E(\Omega) = I\,. \tag{2}$$

Any operator B, bounded or not, will be called *Lüders* if it is invariant under *Lüders*' map,

$$\Lambda(B) = B\,. \tag{3}$$

The operator B acts on a complex separable Hilbert space $\mathcal{H}$, and the operator $E(\Omega)$ is a member of a (over-) complete family of projectors on coherent states $|\Omega\rangle$ associated with an irreducible, unitary representation of a Lie group G in the space $\mathcal{H}$.

This setting generalizes the traditional approach to minimally disturbing (or *ideal*) *Lüders* measurements. Given a self-adjoint operator with spectral decomposition $A = \Sigma_i^N a_i E_i$, $N \leqslant \infty$, the projectors E_i are complete and orthogonal,

$$\sum_{i=1}^{N} E_i = I\,, \quad E_i E_j = E_i \delta_{ij}\,, \quad i,j = 1,\ldots,N \leqslant \infty\,. \tag{4}$$

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If a nonselective, ideal measurement of A is performed on a quantum system with density operator ρ, its state undergoes a *Lüders* transformation:

$$\rho \mapsto \Lambda(\rho) = \sum_{i=1}^{N} E_i \rho E_i \,, \tag{5}$$

which extends to a linear, completely positive map. If, for some operator B, one has

$$\mathrm{Tr}\,[\rho B] = \mathrm{Tr}\,[\Lambda(\rho) B]\,, \quad \text{for all } \rho\,, \tag{6}$$

then the *Lüders* measurement of A does not disturb the measurement of B. In other words, the expectation value of B with respect to *any* density operator ρ is not affected by measuring A. Introduce the *dual Lüders* map Λ^D, acting on operators defined on $\mathcal{H}$, by

$$\mathrm{Tr}\,[\Lambda(\rho) B] = \mathrm{Tr}\,[\rho \Lambda^D(B)]\,. \tag{7}$$

Since Eq. (6) is supposed to hold for any ρ, one must have

$$\Lambda^D(B) = B\,, \tag{8}$$

which, after dropping the superscript, is the discrete counterpart of Eq. (3). Now we can state *Lüders*' theorem:

$$\Lambda(B) = B \quad \Leftrightarrow \quad [B, E_i] = 0\,, \quad \text{for all } i = 1,2,\ldots\,, \tag{9}$$

i.e., it is necessary and sufficient for $A = \Sigma_i^N a_i E_i$ to commute with a (bounded) operator B if the measurement of A should not disturb any measurement of B.

Originally, this theorem has been shown to hold for orthogonal projections;[1] after generalizations to some discrete POVMs had been obtained,[2] the theorem was expected to hold under very general conditions. However, the existence of a nonintuitive counterexample has been proved nonconstructively in Ref. 3. It is our purpose to extend the validity of *Lüders*' theorem to *continuous* POVMs which are associated with coherent states on Lie groups.

A. Outline and summary

In the following, we will consider POVMs which consist of continuous families of one-dimensional projections onto coherent states, or CS-POVMs, for short. The CS-POVMs for a spin and for a particle provide well-known examples, being associated with the group SU(2) and the Heisenberg–Weyl group H_3, respectively. However, coherent states can be defined for general Lie groups G while retaining many of their properties. We will begin to discuss the *Lüders* map in general terms and specialize to particular groups only later.

When considering *Lüders*' map generated by coherent states of an arbitrary (simple and simple connected) Lie group G, a first general observation is that

- the P- and the Q-symbol of a *Lüders* operator coincide for the CS-POVM associated with a Lie group G.

Subsequently, we will derive a simple form of this constraint by expanding the symbol of the operator in terms of harmonic functions associated with the group G. The resulting condition on the expansion coefficients will be shown to imply that

- for the CS-POVM of a *spin* only multiples of the identity operator are *Lüders*;
- for the CS-POVM of a *particle* a countable family of linearly independent, unbounded *Lüders* operators exists, none of which commutes with the elements of the POVM.

Thus, for both the groups SU(2) and H_3, multiples of the identity are found to be the only *bounded Lüders* operator, and they commute with the elements of the corresponding CS-POVM: consequently, *Lüders*' theorem also applies to these CS-POVMs.

Finally, it will be shown that the *Lüders* map implements antinormal ordering for operators which can be written as power series of particle annihilation and creation operators.

II. *LÜDERS* THEOREM FOR POVMS OF COHERENT STATES

A. Coherent states on Lie groups and harmonic functions

Given any finite-dimensional (simple and simply connected) Lie group G, there is a canonical way to introduce coherent states $|\Omega\rangle$ labeled by the points Ω of a well-defined manifold $\mathbb{X}$. To do so, consider a unitary irreducible representation $T(g)$ on a Hilbert space $\mathcal{H}$ of the elements $g \in G$. Following closely the presentation given in Ref. 4, we choose a reference (or fiducial) state $|\psi_0\rangle$ and define the set of coherent states by

$$|\psi_g\rangle = T(g)|\psi_0\rangle , \quad g \in G . \tag{10}$$

Up to a phase, the reference state is left invariant by the elements h of the isotropy subgroup $H \subset \mathrm{G}$,

$$T(h)|\psi_0\rangle = e^{i\phi(h)}|\psi_0\rangle , \quad h \in H \subset G . \tag{11}$$

Therefore, each group element can be written as as product

$$g = \Omega h , \quad \Omega \in \mathbb{X} = G/H , \quad h \in H , \tag{12}$$

where $\mathbb{X}$ is the coset space obtained from dividing G by its subgroup H. As the phase of a state has no physical relevance, the set of coherent states is in a one-to-one correspondence with the points $\Omega(g)$ of the manifold $\mathbb{X}$. This suggests to denote coherent states by $|\Omega\rangle \equiv |\psi_\Omega\rangle$. A fundamental property of the coherent states $|\Omega\rangle$ is their completeness in Hilbert space $\mathcal{H}$,

$$\int_{\mathbb{X}} d\mu(\Omega)\, |\Omega\rangle\langle\Omega| = I , \tag{13}$$

where integration is over the coset space $\mathbb{X}$ with (approximately normalized) invariant measure $d\mu(\Omega)$, and I is the identity in $\mathcal{H}$.

Coherent states $|\Omega\rangle$ can be used to define symbolic representations of operators, i.e., c-number valued functions on the manifold $\mathbb{X}$ which can be understood as the phase space of a classical system associated with the Lie group G.[5] The Q-symbol of an operator B acting in Hilbert space $\mathcal{H}$ is given by its expectation value in coherent states,

$$Q_B(\Omega) = \langle\Omega|B|\Omega\rangle , \quad \Omega \in \mathbb{X} ; \tag{14}$$

due to analyticity properties of $Q_B(\Omega)$, these "diagonal" matrix elements are sufficient to uniquely determine the operator B. The P-symbol of B (Refs. 6 and 7) arises if one expresses B as a linear combination of projection operators $|\Omega\rangle\langle\Omega|$:

$$B = \int_{\mathbb{X}} d\mu(\Omega)\, P_B(\Omega)\, |\Omega\rangle\langle\Omega| . \tag{15}$$

The existence of the symbols $Q_B(\Omega)$ and $P_B(\Omega)$ depends in a subtle way on the properties of the operator B (Ref. 5) but they are unique whenever they exist. Furthermore, one can think of the symbols $Q_A(\Omega)$ and $P_A(\Omega)$ as being dual to each other (cf. Ref. 5), and, at least for particle coherent-states, they are related to normal and anti-normal ordering of creation and annihilation operators.[5,8]

It is useful to introduce the harmonic functions $Y_\nu(\Omega)$ associated with the manifold $\mathbb{X}$ and, hence, with the group G. Consider the Hilbert space $L^2(\mathbb{X},\mu)$ of square integrable functions $u(\Omega)$ on the manifold $\mathbb{X}$, with integration measure $d\mu(\Omega)$. The eigenfunctions $Y_\nu(\Omega)$ of the Laplace–Beltrami operator on $\mathbb{X}$ (Ref. 9) constitute a complete orthonormal set of functions in $L^2(\mathbb{X},\mu)$ since they satisfy

$$\sum_\nu Y_\nu^*(\Omega) Y_\nu(\Omega') = \delta(\Omega - \Omega'), \tag{16}$$

the right-hand side being a delta function with respect to the measure $\mu(\Omega)$, as well as

$$\int_{\mathbb{X}} d\mu(\Omega)\, Y_\nu^*(\Omega) Y_{\nu'}(\Omega) = \delta_{\nu\nu'}. \tag{17}$$

Depending on the manifold $\mathbb{X}$ being compact or not, the right-hand side of (17) must be understood as a Kronecker-delta or a Dirac-delta function (or suitable combinations thereof). There is a simple expression for the (modulus of) the overlap of two coherent states in terms of harmonic functions:

$$|\langle\Omega'|\Omega\rangle|^2 = \sum_\nu \tau_\nu Y_\nu(\Omega') Y_\nu^*(\Omega), \qquad \tau_\nu \in \mathbf{R}, \tag{18}$$

where the numbers or functions τ_ν depend on the actual group.

B. *Lüders* map for CS-POVMs

It is straightforward to generalize the *Lüders* map (1) to POVMs which can be written in terms of integrals of an operator valued density with respect to a positive measure μ as follows. Let (Ω_0,Σ,μ) be a measure space. Assume that, for the Hilbert space $\mathcal{H}=L^2(\Omega_0,\mu)$, there is a family of positive linear operators $E_\omega \in L(\mathcal{H})$, $\omega \in \Omega_0$, which provide a resolution of unity,

$$\int_{\Omega_0} d\mu(\omega)\, E_\omega = I. \tag{19}$$

Then the operators

$$E(\sigma) = \int_\sigma d\mu(\omega)\, E_\omega, \qquad \sigma \in \Sigma, \tag{20}$$

define a POVM which is of the required form.

It is natural to associate with the POVM in (20) a *Lüders* map $\Lambda(B)$ of an operator B by defining

$$\Lambda(B) = \int_\Omega d\mu(\omega)\, E_\omega^{1/2} B E_\omega^{1/2}, \tag{21}$$

which is a unital, completely positive linear map on $L(\mathcal{H})$. Due to the completeness relation (13), the self-adjoint coherent-state projectors

$$E_\Omega \equiv |\Omega\rangle\langle\Omega| = E_\Omega^{1/2}, \quad \Omega \in \mathbb{X}, \tag{22}$$

are seen to define a POVM in the sense just described.

Any operator B defined on $L^2(\mathbb{X},\mu)$ is *Lüders* with respect to the CS-POVM $E_\Omega, \Omega \in \mathbb{X}$, if it satisfies the relation $B = \Lambda(B)$ with E_ω in (21) replaced by E_Ω,

$$B=\int_{\mathbb{X}} d\mu(\Omega)\,|\Omega\rangle\langle\Omega|B|\Omega\rangle\langle\Omega|=\int_{\mathbb{X}} d\mu(\Omega)\,Q_B(\Omega)|\Omega\rangle\langle\Omega|\,. \tag{23}$$

Upon comparing this equation with (15), we observe that the *Lüders* property has, for any CS-POVM, the following general interpretation: an operator B is *Lüders* if and only if its P- and Q-symbols coincide,

$$P_B(\Omega)=Q_B(\Omega)\,. \tag{24}$$

To the best of our knowledge, this set of operators—which we will call *well-ordered*—has not been introduced before.

The constraint (23) takes a particularly simple form upon expanding the Q-symbol of B in harmonic functions,

$$Q_B(\Omega)=\sum_{\nu} B_\nu Y_\nu(\Omega)\,, \tag{25}$$

which is possible according to (16). The expansion coefficients are given by

$$B_\nu=\int_{\mathbb{X}} d\mu(\Omega)\,Q_B(\Omega)Y_\nu^*(\Omega)\,. \tag{26}$$

Take the expectation value of (23) in the coherent state $|\Omega'\rangle$ and use the relation (18) for the overlap $|\langle\Omega'|\Omega\rangle|^2$. This leads to

$$Q_B(\Omega')=\sum_{\nu} \tau_\nu\left[\int_{\mathbb{X}} d\mu(\Omega)Q_B(\Omega)Y_\nu^*(\Omega)\right]Y_\nu(\Omega')=\sum_{\nu} \tau_\nu B_\nu Y_\nu(\Omega')\,, \tag{27}$$

where (26) has been used. Uniqueness of the expansion (25) implies that the coefficients of a *Lüders* operator must satisfy the condition

$$B_\nu=\tau_\nu B_\nu\,, \quad \text{for all } \nu\,. \tag{28}$$

As mentioned above, the actual form of the quantities τ_ν depend on the group G under consideration. To proceed, we therefore need to specify the system of coherent states we work with, that is, the group G. Explicit conclusions about *Lüders* operators for CS-POVMs will be derived now for the groups SU(2) and H_3.

III. *LÜDERS* OPERATORS FOR THE CS-POVM OF A SPIN

Consider a Hilbert space $\mathcal{H}_s$ of dimension $(2s+1)$, carrying an irreducible representation of the group $G=\mathrm{SU}(2)$. Each space $\mathcal{H}_s$ is associated with a spin of length $s\in\{\frac{1}{2},1,\frac{3}{2},\ldots\}$. To introduce spin-coherent states, it is convenient to select states of highest (lowest) weight $|\pm s\rangle$ as reference states (cf. Refs. 5 and 10). These states are invariant under a change of phase, hence the isotropy group is given by $H=\mathrm{U}(1)$. Therefore, the coset space is the surface of a sphere: $\mathbb{X}=\mathrm{SU}(2)/\mathrm{U}(1)=\mathcal{S}^2$, which corresponds to the phase space of a classical spin.

The resolution of unity I in $\mathcal{H}_s$ using spin-coherent states $|\mathbf{n}\rangle$ reads

$$I=\int_{\mathcal{S}^2} d\mu(\mathbf{n})\,|\mathbf{n}\rangle\langle\mathbf{n}|\,,\quad d\mu(\mathbf{n})=\frac{2s+1}{4\pi}\sin\vartheta d\vartheta\, d\varphi\,, \tag{29}$$

where each unit vector $\mathbf{n}\in\mathbb{R}^3$ denotes a point with spherical coordinates (ϑ,φ), located on the unit sphere $\mathcal{S}^2$. The continuous family of operators

$$E_{\mathbf{n}}=|\mathbf{n}\rangle\langle\mathbf{n}|, \quad \text{with} \quad I=\int_{\mathcal{S}^2} d\mu(\mathbf{n})\, E_{\mathbf{n}}, \tag{30}$$

defines the CS-POVM of SU(2). Being a projector, the positive square root of each operator $E_{\mathbf{n}}$ is equal to itself: $E_{\mathbf{n}}^{1/2}=|\mathbf{n}\rangle\langle\mathbf{n}|$. Therefore, a self-adjoint operator $B\in L(\mathcal{H}_s)$ is *Lüders* with respect to the POVM (30) if

$$B=\int_{\mathcal{S}^2} d\mu(\mathbf{n})|\mathbf{n}\rangle\langle\mathbf{n}|B|\mathbf{n}\rangle\langle\mathbf{n}|\equiv\int_{\mathcal{S}^2} d\mu(\mathbf{n})Q_B(\mathbf{n})|\mathbf{n}\rangle\langle\mathbf{n}|. \tag{31}$$

Following the strategy outlined earlier, we will show now that any operator B satisfying (31) must be a real multiple of unity: $B=\lambda I$, so that B commutes with all elements of the CS-POVM for a spin,

$$[B,E_{\mathbf{n}}]=0, \quad \mathbf{n}\in\mathcal{S}^2. \tag{32}$$

Consider the expectation value of Eq. (31) in the coherent state $|\mathbf{n}'\rangle$,

$$Q_B(\mathbf{n}')=\int_{\mathcal{S}^2} d\mu(\mathbf{n})\, Q_B(\mathbf{n})|\langle\mathbf{n}|\mathbf{n}'\rangle|^2. \tag{33}$$

The function $Q_B(\mathbf{n})$, the Q-symbol of the operator B, is smooth on the sphere $\mathcal{S}^2$, and it can be written as a linear combination of $(2s+1)^2$ spherical harmonics $Y_{lm}(\mathbf{n})$,

$$Q_B(\mathbf{n})=\sqrt{\frac{4\pi}{2s+1}}\sum_{l=0}^{2s}\sum_{m=-l}^{l} B_{lm}Y_{lm}(\mathbf{n}), \tag{34}$$

with expansion coefficients

$$B_{lm}=\sqrt{\frac{4\pi}{2s+1}}\int_{\mathcal{S}^2} d\mu(\mathbf{n})\, Q_B(\mathbf{n})\, Y_{lm}^*(\mathbf{n}). \tag{35}$$

Note that these expressions are connected to the general formulas through identifying $Y_\nu(\Omega)\leftrightarrow\sqrt{4\pi/(2s+1)}Y_{lm}(\mathbf{n})$. Rewrite the scalar product (33) by means of the addition theorem for spherical harmonics,

$$\begin{aligned}|\langle\mathbf{n}|\mathbf{n}'\rangle|^2&=\left(\frac{1+\mathbf{n}\cdot\mathbf{n}'}{2}\right)^{2s}\\ &=\sum_{l=0}^{2s}\frac{2l+1}{2s+1}\left\langle\begin{matrix}s & l\\ s & 0\end{matrix}\middle|\begin{matrix}s\\ s\end{matrix}\right\rangle^2 P_l(\mathbf{n}\cdot\mathbf{n}')\\ &=\frac{4\pi}{2s+1}\sum_{l=0}^{2s}\sum_{m=-l}^{l}\left\langle\begin{matrix}s & l\\ s & 0\end{matrix}\middle|\begin{matrix}s\\ s\end{matrix}\right\rangle^2 Y_{lm}^*(\mathbf{n})Y_{lm}(\mathbf{n}'),\end{aligned} \tag{36}$$

where the functions $P_l(x)$ are the Legendre polynomials. Upon inserting (34) and (36), integration of the right-hand side of Eq. (33) gives (after replacing $\mathbf{n}'$ by $\mathbf{n}$)

$$Q_B(\mathbf{n})=\sqrt{\frac{4\pi}{2s+1}}\sum_{l=0}^{2s}\sum_{m=-l}^{l}\left\langle\begin{matrix}s & l\\ s & 0\end{matrix}\middle|\begin{matrix}s\\ s\end{matrix}\right\rangle^2 B_{lm}Y_{lm}(\mathbf{n}). \tag{37}$$

This expansion and Eq. (34) can only hold simultaneously if the coefficients of the harmonics satisfy

$$B_{lm}=\left\langle\begin{matrix}s & l\\ s & 0\end{matrix}\middle|\begin{matrix}s\\ s\end{matrix}\right\rangle^2 B_{lm}, \tag{38}$$

which is (28) for the group SU(2). The m-independent Clebsch–Gordan coefficients correspond to the numbers τ_ν introduced in (18), and they take values

$$\left\langle \begin{matrix} s & l \\ s & 0 \end{matrix} \middle| \begin{matrix} s \\ s \end{matrix} \right\rangle^2 = \frac{(2s)!(2s+1)!}{(2s-l)!(2s+1+l)!}. \tag{39}$$

Since

$$\left\langle \begin{matrix} s & 0 \\ s & 0 \end{matrix} \middle| \begin{matrix} s \\ s \end{matrix} \right\rangle = 1, \quad 0 < \left\langle \begin{matrix} s & l \\ s & 0 \end{matrix} \middle| \begin{matrix} s \\ s \end{matrix} \right\rangle < 1, \quad l = 1,2,\dots,2s, \tag{40}$$

the coefficients B_{lm} with $l \neq 0$ in (38) must vanish; thus, the expansion (34) of a *Lüders* operator satisfying (31) contains only one nonzero term, B_{00}, and B is proportional to $Y_{00}(\mathbf{n})$, i.e., the identity. Hence, it commutes with any operator, including the set $E_{\mathbf{n}}$, so that Eq. (32) follows. At the same time we have shown that the identity is the only operator in $\mathcal{H}_s$ such that its Q- and P-symbols coincide.

IV. *LÜDERS* OPERATORS FOR THE CS-POVM OF A PARTICLE

The kinematics of a quantum particle on the real line $\mathbb{R}$ is described by the creation and annihilation operators a and its adjoint $a^\dagger$ which satisfy $[a,a^\dagger]=I$. The operators a, $a^\dagger$, and the identity I generate the Heisenberg–Weyl algebra h_3; finite transformations, that is, elements of the *group* H_3, are given by the phase-space displacement or shift operators

$$D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a], \quad \alpha \in \mathbb{C}. \tag{41}$$

In fact, they provide an irreducible projective representation of the group H_3 in $L_2(\mathbb{R})$,

$$D(\alpha)D(\alpha') = \exp\left[\frac{i}{2}(\alpha\alpha'^* - \alpha^*\alpha')I\right]D(\alpha+\alpha'). \tag{42}$$

The (overcomplete) family of coherent states $|\alpha\rangle$ in the Hilbert space $L_2(\mathbb{R})$ is obtained by displacing the fiducial state $|0\rangle$, say, with $a|0\rangle = 0$, by arbitrary amounts $\alpha \in \mathbb{C}$:

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{43}$$

The isotropy subgroup of H_3 is again isomorphic to $\mathrm{U}(1) \sim \exp[i\gamma I], \gamma \in [0,2\pi)$, so that the manifold labeling coherent states is given by the complex plane $\mathbb{X} = H_3/\mathrm{U}(1) = \mathbb{C}$, corresponding indeed to the phase space of a classical particle on the real line.

The completeness relation for the particle-coherent states reads

$$I = \int_{\mathbb{C}} d\mu(\alpha)\, |\alpha\rangle\langle\alpha|, \quad d\mu(\alpha) = \frac{1}{\pi} d^2\alpha, \tag{44}$$

and it can be understood as defining a POVM for the continuous family of projection operators

$$E_\alpha = |\alpha\rangle\langle\alpha| = E_\alpha^{1/2}, \quad \alpha \in \mathbb{C}. \tag{45}$$

The operator B on $L_2(\mathbb{R})$ is *Lüders* with respect to the POVM $E_\alpha, \alpha \in \mathbb{C}$, if it is invariant under the *Lüders* map $B \mapsto \Lambda(B)$, i.e.,

$$B = \int_{\mathbb{C}} d\mu(\alpha)\, |\alpha\rangle\langle\alpha|B|\alpha\rangle\langle\alpha| = \int_{\mathbb{C}} d\mu(\alpha)\, Q_B(\alpha)|\alpha\rangle\langle\alpha|, \tag{46}$$

where $\langle \alpha | B | \alpha \rangle = Q_B(\alpha)$ is the Q-symbol of the operator B. As shown earlier, this relation forces the Q-symbol of a *Lüders* operator to coincide with its P-symbol,

$$B = \frac{1}{\pi} \int_{\mathbb{C}} d\mu(\alpha) P(\alpha) |\alpha\rangle\langle\alpha| , \tag{47}$$

if it exists.

We will now search for *bounded Lüders* operators B which commute the members E_α of the CS-POVM (44) for a particle. We begin to look at simple examples of *Lüders* operators, followed by a systematic construction of all well-ordered *Lüders* operators. In addition to the identity, a countable family of *unbounded*, linearly independent *Lüders* operators will emerge, none of which commutes with the elements of the CS-POVM. Finally, an unexpected relation of the *Lüders* map to operator orderings is established for particle coherent states.

A. Examples of unbounded *Lüders* operators

It is straightforward to apply the map Λ to unbounded operators such as position $Q = (a + a^\dagger)/2$ and momentum $P = (a - a^\dagger)/2i$. Using the equation $a|\alpha\rangle = \alpha|\alpha\rangle$ and its adjoint implies that

$$\begin{aligned} \Lambda(Q) &= \int_{\mathbb{C}} d\mu(\alpha)\, |\alpha\rangle\langle\alpha|Q|\alpha\rangle\langle\alpha| = \int_{\mathbb{C}} d\mu(\alpha) \frac{1}{2}(\alpha + \alpha^*)|\alpha\rangle\langle\alpha| \\ &= \frac{1}{2}\int_{\mathbb{C}} d\mu(\alpha)\, a|\alpha\rangle\langle\alpha| + \frac{1}{2}\int_{\mathbb{C}} d\mu(\alpha)\, |\alpha\rangle\langle\alpha|a^\dagger = Q , \end{aligned} \tag{48}$$

and similarly

$$\Lambda(P) = P . \tag{49}$$

While being invariant under Λ, the operators Q and P are neither positive nor bounded, and they do not commute with the projectors E_α since the expectation value of the commutator in the coherent state $|\beta\rangle$ is, in general, different from zero:

$$\langle\beta|[Q,E_\alpha]|\beta\rangle = \tfrac{1}{2}((\alpha - \alpha^*) - (\beta - \beta^*))|\langle\alpha|\beta\rangle|^2 . \tag{50}$$

Using the relation $D^\dagger(\alpha) a D(\alpha) = a - \alpha$, its adjoint, and the commutation relations of a and $a^\dagger$, one shows that *Lüders*' map acts on the operators Q^2 and P^2 according to

$$\begin{aligned} \Lambda(Q^2) &= Q^2 + 2\langle 0|Q^2|0\rangle I = Q^2 + \tfrac{1}{2} I , \\ \Lambda(P^2) &= P^2 + 2\langle 0|P^2|0\rangle I = P^2 + \tfrac{1}{2} I . \end{aligned} \tag{51}$$

Consequently, appropriate quadratic combinations of position and momentum turn out to be *Lüders*,

$$\Lambda_\Gamma(Q^2 - P^2) = Q^2 - P^2 . \tag{52}$$

However, this indefinite, unbounded operator does not commute with all projections E_α as follows from $\langle 0|[Q^2 - P^2, E_\alpha]|0\rangle = (\alpha^2 - \alpha^{*2})|\langle 0|\alpha\rangle|^2$, for example. In the next section a family of similar *Lüders* operators will be constructed.

B. Construction of *Lüders* operators

Let us turn now to the problem of finding all operators which are *Lüders* with respect to the CS-POVM E_α of a particle, i.e, all well-ordered operators. The argument will resemble the one given in the case of a spin.

Expand the Q-symbol of an operator B as

$$Q_B(\alpha)=\int_{\mathbb{C}} d\mu(\xi)\, B_\xi \exp[\alpha\xi^* - \alpha^*\xi], \tag{53}$$

where the coefficients B_ξ are given by

$$B_\xi=\int_{\mathbb{C}} d\mu(\alpha)\, Q_B(\alpha)\exp[-(\alpha\xi^*-\alpha^*\xi)]. \tag{54}$$

Here, the functions $\exp[\alpha\xi^*-\alpha^*\xi]$ are the complete orthonormal set of harmonic functions in the complex plane, corresponding to $Y_\nu(\Omega)$. Since the Q-symbol of a Hermitian operator is real, $Q_B(\alpha)=\langle\alpha|B|\alpha\rangle^*=Q_B^*(\alpha)$, the coefficients must satisfy the relation

$$\begin{aligned} B_\xi^* &= \int_{\mathbb{C}} d\mu(\alpha)\, Q_B^*(\alpha)\exp[-(\alpha^*\xi-\alpha\xi^*)] \\ &= \int_{\mathbb{C}} d\mu(\alpha)\, Q_B(\alpha)\exp[-(\alpha(-\xi)^*-\alpha^*(-\xi))]=B_{-\xi}. \end{aligned} \tag{55}$$

We will turn (46) into a condition for the expansion coefficients B_ξ of a *Lüders* operator which can be solved explicitly. Take the expectation value of the operator B in (46) in the coherent state $|\beta\rangle$, and use the identity

$$|\langle\alpha|\beta\rangle|^2=\exp[-|\alpha-\beta|^2]=\int_{\mathbb{C}} d\mu(\xi)\, e^{-\xi\xi^*}\exp[\beta\xi^*-\beta^*\xi]\exp[-\alpha\xi^*+\alpha^*\xi], \tag{56}$$

leading to

$$\begin{aligned} Q_B(\beta) &= \int_{\mathbb{C}} d\mu(\xi)\, e^{-\xi\xi^*}\left[\int_{\mathbb{C}} d\mu(\alpha)\, Q_B(\alpha)\exp[-(\alpha\xi^*-\alpha^*\xi)]\right]\exp[\beta\xi^*-\beta^*\xi], \\ &= \int_{\mathbb{C}} d\mu(\xi)\, e^{-\xi\xi^*}B_\xi\exp[\beta\xi^*-\beta^*\xi], \end{aligned} \tag{57}$$

where (54) has been used. Due to the uniqueness of the expansion (53), the expansion coefficients of any *Lüders* operators must satisfy

$$B_\xi=e^{-\xi\xi^*}B_\xi, \tag{58}$$

which is the equivalent of (38) for continuous variables. Consequently, the coefficients B_ξ are necessarily zero for all values of ξ except $\xi=0$, and there are no solutions in terms of ordinary functions. If allowing for generalized functions, B_ξ is necessarily a distribution of finite order,[11] that is, a linear combination of a δ-distribution and finite derivatives of it,

$$B_\xi=\sum_{n+m=0}^{N} b_{nm}\partial_\xi^n\partial_{\xi^*}^m\delta(\xi), \quad b_{nm}\in\mathbb{C}, \quad n,m=0,1,2,\ldots, \quad N=0,1,2,\ldots. \tag{59}$$

The function B_ξ must satisfy (55) leading to

$$b_{nm}=(-)^{m+n}b_{mn}^*, \quad n,m=0,1,2,\ldots, \tag{60}$$

and the $\delta(\xi)$-function is real,

$$\delta(\xi)=\int_{\mathbb{C}} d\mu(\alpha)\exp[\alpha\xi^* - \alpha^*\xi]=\delta(-\xi)=\delta^*(\xi)\,. \tag{61}$$

Only some of the distributions (59) will satisfy (58) since one must have

$$Q_B(\alpha)=\int_{\mathbb{C}} d\mu(\xi)[D_N\delta(\xi)]e^{-\xi\xi^*}e^{\alpha\xi^*-\alpha^*\xi}=\int_{\mathbb{C}} d\mu(\xi)[D_N\delta(\xi)]e^{\alpha\xi^*-\alpha^*\xi}, \tag{62}$$

where

$$D_N=\sum_{n+m=0}^{N} b_{nm}\partial_\xi^n\partial_{\xi^*}^m\,. \tag{63}$$

Partial integrations in (62) lead to the requirement

$$[D_N^\dagger e^{-\xi\xi^*}e^{\alpha\xi^*-\alpha^*\xi}]_{\xi=\xi^*=0}=[D_N^\dagger e^{\alpha\xi^*-\alpha^*\xi}]_{\xi=\xi^*=0}\,, \tag{64}$$

where the adjoint $D_N^\dagger$ of D_N is obtained from replacing b_{nm} by $(-)^{n+m}b_{nm}$ in (63). It is shown in the Appendix that this condition is satisfied if and only if

$$b_{nm}=0\,,\quad 1\leq m,n\leq N\,, \tag{65}$$

i.e., only terms b_{nm} with at least one index (that is, m or n or both) equal to zero will contribute to the symbol of a well-ordered operator. Therefore, only coefficients of the form

$$B_\xi=\sum_{n=0}^{N} (b_{n0}\partial_\xi^n+(-)^n b_{n0}^*\partial_{\xi^*}^n)\,\delta(\xi) \tag{66}$$

occur which, upon partial integration in (53), give rise to Q-symbols of *Lüders* operators,

$$Q_B(\alpha)=\sum_{n=0}^{N} (b_{n0}\alpha^{*n}+b_{n0}^*\alpha^n)\,. \tag{67}$$

The operators corresponding to these symbols are given by

$$B=b_0I+\sum_{n=1}^{N} (b_n^qB_n^q+b_n^pB_n^p)\,, \tag{68}$$

i.e., a linear combination of the identity and $2N$ Hermitian operators

$$B_n^q=\frac{1}{2}(a^n+a^{\dagger\,n})\quad\text{and}\quad B_n^p=\frac{1}{2i}(a^n-a^{\dagger\,n})\,,\quad n=1,2,\ldots,N\,, \tag{69}$$

which satisfy (46), and $(2N+1)$ real coefficients

$$b_0=2b_{00}\,,\quad b_n^q=b_{n0}+b_{n0}^*\,,\quad b_n^p=\frac{1}{i}(b_{n0}-b_{n0}^*)\,,\quad n=1,2,\ldots,N\,. \tag{70}$$

If $N=2$, for example, it follows that not only the operators Q,P, and Q^2-P^2 are *Lüders* but also

$$B_2^p = \frac{1}{2i}(a^2 - a^{\dagger\, 2}) \propto QP + PQ\,. \tag{71}$$

Every *bounded Lüders* operator is necessarily a multiple of the identity.

C. *Lüders* map and operator ordering

It is easy to understand why the operators $B_n, n=1,2,\ldots,N$, in (70) are *Lüders*. Consider any Hermitian operator B given as a finite polynomial in a and $a^\dagger$. Using their commutation relation, one can bring the annihilation operators either to the right or to the left,

$$B(a,a^\dagger) = \sum_{m,n} \beta_{nm}^{\mathcal{N}} a^{\dagger\, m} a^n = \sum_{m,n} \beta_{nm}^{\mathcal{A}} a^m a^{\dagger\, n}, \tag{72}$$

corresponding to normal and antinormal ordering of B, respectively.[12] It is straightforward to calculate the *Lüders* transform of B if it is written in normal order:

$$\Lambda(B(a,a^\dagger)) = \sum_{m,n} \beta_{nm}^{\mathcal{N}} \Lambda(a^{\dagger\, m} a^n) = \sum_{m,n} \beta_{nm}^{\mathcal{N}} a^n a^{\dagger\, m}, \tag{73}$$

since

$$\Lambda(a^{\dagger\, m} a^n) = \int_{\mathbb{C}} d\mu(\alpha)\, |\alpha\rangle\langle\alpha| a^{\dagger\, m} a^n |\alpha\rangle\langle\alpha| = \int_{\mathbb{C}} d\mu(\alpha)\; \alpha^n |\alpha\rangle\langle\alpha| \alpha^{*\, m}$$
$$= a^n \left(\int_{\mathbb{C}} d\mu(\alpha)\, |\alpha\rangle\langle\alpha| \right) a^{\dagger\, m} = a^n a^{\dagger\, m}. \tag{74}$$

Thus, the effect of Λ is to push each creation operator $a^\dagger$ to the right as if it would commute with the annihilation operator a. In other words, the map Λ provides an explicit form of the operator $\mathcal{A}$ which generates antinormal order of an operator.[8] This operator and its twin $\mathcal{N}$, which brings a given operator into normal order, are useful tools to evaluate expectation values or Baker–Campbell–Hausdorff relations, for example.[8]

To conclude: if an operator B is to be invariant under Λ, the normally and antinormally ordered forms of an operator B must coincide,

$$\sum_{m,n} \beta_{nm}^{\mathcal{N}} a^n a^{\dagger\, m} = \sum_{m,n} \beta_{nm}^{\mathcal{A}} a^m a^{\dagger\, n}, \tag{75}$$

that is, $\beta_{nm}^{\mathcal{N}} = \beta_{nm}^{\mathcal{A}}$. This is obviously true for the linear combinations of powers of a and $a^\dagger$ given in (70), defining the family of well-ordered operators.

V. DISCUSSION

We have shown that there is only one *Lüders* operator, the identity (and its multiples), for the CS-POVM of SU(2) while a countable family of linearly independent, unbounded, and well-ordered operators exists in the case of H_3. Due to the linearity of map Λ, all their linear combinations are well-ordered as well. It is plausible that our study exhausts all possibilities which may arise for CS-POVMs of general (simple and simply connected) Lie groups: we expect only the identity as a *Lüders* operator for *compact* Lie groups such as SU(N), and a countable family for a CS-POVM associated with noncompact groups such as $\mathrm{SU}(N-n,n), 1 \leq n < N$. If we restrict our attention to bounded operators, we conjecture *Lüders*' theorem to hold with respect to the CS-POVM of any Lie group G.

APPENDIX: CONSTRUCTION OF WELL-ORDERED OPERATORS

We will show here that any operator compatible with (46) must have a Q-symbol with expansion coefficients of the following form:

$$B_\xi=\sum_{n=0}^{N}\,(b_{n0}\partial_\xi^n+(-)^n b_{n0}^*\partial_{\xi^*}^n)\,\delta(\xi)\,,\quad N<\infty; \tag{A1}$$

this means, in particular, that most of the coefficients b_{nm} are equal to zero:

$$b_{nm}=0\,,\quad \text{for } 1\leqslant m,n\leqslant N\,. \tag{A2}$$

In a first step, evaluate the right-hand-side of (64):

$$\left[\sum_{n+m=0}^{N}\,(-)^{n+m}b_{nm}\partial_\xi^n\partial_{\xi^*}^m e^{\alpha\xi^*-\alpha^*\xi}\right]_{\xi=0}=\sum_{n+m=0}^{N}\,(-)^m b_{nm}\alpha^m\alpha^{*\,n}\,. \tag{A3}$$

To evaluate the left-hand side, use the relation

$$\partial_\xi(e^{-\xi\xi^*}f(\xi))=e^{-\xi\xi^*}(-\xi^*+\partial_\xi)f(\xi) \tag{A4}$$

and its complex conjugate for any smooth function f. This leads to

$$\partial_\xi^n\partial_{\xi^*}^m e^{-\xi\xi^*}=e^{-\xi\xi^*}(-\xi^*+\partial_\xi)^n(-\xi+\partial_{\xi^*})^m=e^{-\xi\xi^*}\sum_{\nu=0}^{n}\sum_{\mu=0}^{m}\binom{n}{\nu}\binom{m}{\mu}(-\xi^*)^{n-\nu}\partial_\xi^\nu(-\xi)^\mu\partial_{\xi^*}^{m-\mu}\,. \tag{A5}$$

According to Eq. (64), these operators must be applied to the function $e^{\alpha\xi^*-\alpha^*\xi}$. Each derivative ∂_{ξ^*} produces a factor α, while the action of the derivatives ∂_ξ is more complicated:

$$\partial_\xi^\nu((-\xi)^\mu e^{\alpha\xi^*-\alpha^*\xi})=\sum_{s=0}^{\nu}\binom{\nu}{s}\frac{\partial(-\xi)^\mu}{\partial\xi^s}\,\frac{\partial^{\nu-s}e^{\alpha\xi^*-\alpha^*\xi}}{\partial\xi^{\nu-s}}$$

$$=\sum_{s=0}^{\nu}\binom{\nu}{s}\frac{\mu!(-)^s}{(\mu-s)!}(-\xi)^{\mu-s}(-\alpha^*)^{\nu-s}e^{\alpha\xi^*-\alpha^*\xi}; \tag{A6}$$

due to $1/\Gamma(-k)=0,k=0,1,2,\ldots$, there are no contributions to the sum if s exceeds μ. Now that the derivatives have been evaluated, one can set $\xi=\xi^*=0$ in the resulting expression: the terms with nonzero powers of ξ or ξ^* vanish, and the sums simplify according to

$$(-\xi)^{\mu-s}\rightarrow\delta_{\mu s}\quad\text{and}\quad(-\xi^*)^{n-\nu}\rightarrow\delta_{n\nu}\,. \tag{A7}$$

The left-hand-side of (64) becomes

$$\sum_{n+m=0}^{N}\,(-)^m b_{nm}\sum_{s=0}^{s_0}\,s!\binom{m}{s}\binom{n}{s}\alpha^{m-s}\alpha^{*\,n-s}\,, \tag{A8}$$

where $s_0=\min(m,n)$. Note that the term with $s=0$ in this expression is identical to the right-hand side of (A3) which implies that the equality (62) is satisfied if

$$\sum_{n+m=0}^{N}\,(-)^m b_{nm}\sum_{s=1}^{s_0}\,s!\binom{m}{s}\binom{n}{s}\alpha^{m-s}\alpha^{*\,n-s}=0 \tag{A9}$$

holds for all complex numbers α. This equation does not restrict the coefficients $b_{n0},0\leqslant n\leqslant N$, and $b_{0m},0\leqslant m\leqslant N$: if either m or n are equal to zero, the sum over s is empty since $s_0=0$. However, *all* other coefficients must vanish as can be seen in the following way. Writing $\alpha=r\exp[i\varphi]$, Eq. (A9) turns into a sum of terms multiplying phase factors $\exp[i(m-n)\varphi]\equiv\exp[ik\varphi]$, $k=0,1,2,\ldots,N-1$. Each of these terms must vanish individually due to the linear independence of the exponentials. Their coefficients, in turn, are power series in r which can be shown to vanish identically only if $b_{1N}=0$ for $\exp[i(N-1)\varphi]$, $b_{2N}=0$, which implies that $b_{1\,N-2}=0$ for $\exp[i(N-2)\varphi]$, etc. Taking into account that $b_{nm}=(-)^{m+n}b^*_{nm}$, the coefficients B_ξ of *Lüders* operators finally read

$$B_\xi=\left(\sum_{n=0}^{N} b_{n0}\partial_\xi^n+\sum_{m=0}^{N} b_{0m}\partial_{\xi*}^m\right)\delta(\xi)=\sum_{n=0}^{N}(b_{n0}\partial_\xi^n+(-)^n b^*_{n0}\partial_{\xi*}^n)\,\delta(\xi). \tag{A10}$$

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