Simple Minimal Informationally Complete POVMs for Qudits

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The expectation-value representation (EVR) expresses the pure or mixed state of a quantum system entirely in terms of real numbers which can be directly measured in experiments. The number of required measurement is *minimal.* It is shown that a positive operator-valued measure (POVM) is associated with each EVR. Consequently, each EVR gives rise to a *minimal informationally complete* POVM: it is not possible to have a POVM with fewer outcomes which could describe all possible quantum states. The resulting POVM is among the *most efficient* ones since no redundant information is acquired when using it for state reconstruction.

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I. INTRODUCTION

The very idea to implement information on quantum systems and to subsequently process it requires to initially *prepare* a particular quantum state $\hat{\rho}_{in}$, to *verify* the preparation procedure and to *identify* the final state $\hat{\rho}_{out}$ produced by the desired quantum mechanical dynamics. Since a single unknown quantum state cannot be determined unambiguously be a single measurement, and no copies of the state can be made, one needs to resort to repeated measurements on hopefully identical states. Both the verification of a state and its identification are instances of *state reconstruction* or *estimation.* It is obviously desirable to set up the quantum dynamics in such a way that only a small number of known final states is possible allowone to extraxt the desired information with very high probability from a small number of runs as in Shor's algorithm, for example.

The useful measure for the reliability of a measurement procedure to determine an unknown quantum state is the given by its *fidelity* F, the mean overlap of the reconstructed state with the exact state. To obtain perfect fidelity, one usually requires an infinite number of copies of the unknown state. This might sound unrealistic from an experimental point of view but in theoretical terms $F = 1$ says that the expectation values of the operators mesured form a complete set for all density matrices of a particular size. In the following, the focus will be exactly on sets of operators which allow, in principle, perfect reconstruction of an unknown state described by a density operator $\hat{\rho}$ in a d-dimensional Hilbert space $\mathcal{H}^d$. Many such bases for hermitean operators are known already, and in some cases they have been combined into what is called *minimal informationally complete positive operator-valued measures* (cf. below) which is interesting from a conceptual point of view.

The purpose of this contribution is to elaborate on the link between state reconstruction, minimal complete sets of hermitean operators, and positive operator-valued measures. Firstly, it aims to explain that the so-called *expectation-value representation* of quantum states in finite-dimensional Hilbert spaces can be transformed in a positve operator-valued measure which, contrary to others, can be given analytically. Secondly, this approach will be generalized to show that any set of d^2 linearly independent operators on $\mathcal{H}^d$ can be transformed into a minimal informationally complete positive operator-valued measure. In a sense, this result characterizes all possible such measures, and hence all non-redundant experimental procedures to do state reconstruction.

II. POSITIVE OPERATOR-VALUED MEASURES

A. POVMs

Positive operator-valued measures provide the most general characterization of observable quantities compatible with the fundamental principles of quantum mechanics. Here is a brief discussion of their properties as far as they are relevant for the results developed below; at the same time, the notation used for POVMs will be introduced.

Consider a quantum system capable of residing in d states

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$|\psi_n\rangle$, $n = 1 \ldots d$, which form an orthonormal basis of the d-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$. A hermitean operator $\hat{E} = \hat{E}^\dagger$ is called *positive,* $\hat{E} \geq 0$, if its expectation values in all states $|\psi_n\rangle$ do not take negative values,

$$\langle\psi_n|\hat{E}|\psi_n\rangle \geq 0\,, \quad n = 1\ldots d\,. \tag{1}$$

The density matrices $\hat{\rho}$ used to describe mixed states of the quantum system provide well-known examples of positive operators, $\hat{\rho} \geq 0$. A collection of positive operators $\hat{E}_\alpha, \alpha \in A$, where A may be a discrete or continuous set of labels, qualifies as a *positive operator-valued measure*, or POVM for short, if its elements sum up to the identity in $\mathcal{H}^d$,

$$\sum_{\alpha\in A} \hat{E}_\alpha = \hat{\mathbb{I}}\,. \tag{2}$$

If α is a continuous label, the symbol $\sum$ is understood to denote an appopriate integration over A. Taking the expectation value of this equation in any normalized state $|\psi\rangle$, one finds that the discrete or continuous set of positive numbers $p_\alpha = \langle\psi|\hat{E}_\alpha|\psi\rangle, \alpha \in A$, sum up to one; the numbers p_α having the properties of a probability distribution suggests to think of the operators $\hat{E}_\alpha, \alpha \in A$, as defining an "operator-valued" *measure.*

The simplest example of a POVm consists of only one element, the identity $\hat{\mathbb{I}}$ in $\mathcal{H}^d$. The completeness relation of the states $|\psi_n\rangle$,

$$\sum_{n=1}^{d} |\psi_n\rangle\langle\psi_n| = \hat{\mathbb{I}} \tag{3}$$

provides an example of a POVM which consist of d orthonormal projectors $\hat{E}_n \equiv |\psi_n\rangle\langle\psi_n|$, which are positive since $\langle\psi_m|\hat{E}_n|\psi_m\rangle = \delta_{mn} \geq 0$.

It is possible,however, that a POVM contains a number D *larger than the* the dimension d of the underlying Hilbert space. Here is a POVM defined for a qubit with Hilbert space $\mathcal{H}^2$ consisting of *three* operators,

$$\hat{E}_1 = \frac{\sqrt{2}}{1+\sqrt{2}}|-\rangle\langle-|\,, \quad \hat{E}_2 = \frac{\sqrt{2}}{1+\sqrt{2}}\left(|-\rangle - |+\rangle\right)\left(\langle-| - \langle+|\right)\,, \tag{4}$$

and $\hat{E}_3 = \hat{\mathbb{I}} - \hat{E}_1 - \hat{E}_2$; the states $|\pm\rangle$ are eigenstates of the z-component of a spin 1/2. Clearly, the elements of the POVM cannot be orthonormal projections for $D > d$, since $\mathcal{H}^d$ cannot accomodate more than d orthogonal states but this is not required anyway. It is possible in principle to implement the POVM just defined experimentally, and it has the following interesting property. Imagine that you are being asked to find out whether you have been sent the state $|+\rangle$ or the state $(1/\sqrt{2})(|-\rangle + |+\rangle)$. Using the above POVM to perform a mesurement on the unknown state $|?\rangle$, you will find an outcome corresponding to one of the three operators given above. In the first case, associated with $\hat{E}_1$, you know that the state provided cannot have been $|+\rangle$ since $\langle+|\hat{E}_1|+\rangle = 0$; similarly, you know that the unknown state must have been $|+\rangle$ if the measurement outcome corresponds to $\hat{E}_2$ since only this state has a non-zero component "along" $\hat{E}_2$. If the third outcome occurs, no nothing can be said about $|?\rangle$. If one were to perform an measurement with any two *orthonormal* projections, no conclusions about $|?\rangle$ could be drawn from a single run. By invoking an appropriately constructed POVM, however, it is possible to extract the desired information with some probability even from a single experiment.

The final example of a POVM has uncountably many elements: define

$$\hat{E}_{\mathbf{n}} = \sqrt{\frac{2\pi}{2d+1}}\,|\mathbf{n}\rangle\langle\mathbf{n}|\,, \tag{5}$$

where $|\mathbf{n}\rangle$ is a coherent state of a spin 1/2, labeled with the vector $\mathbf{n}$, poining from the origin in $\mathbb{R}^3$ to the point $P_{\mathbf{n}}$ on the unit sphere $\mathcal{S}$. The overcompleteness relation of the coherent states implies that these operators are a indeed a POVM,

$$\int_{\mathcal{S}} \hat{E}_{\mathbf{n}}\, d\mathbf{n} = \sqrt{\frac{2\pi}{2d+1}} \int_{\mathcal{S}} |\mathbf{n}\rangle\langle\mathbf{n}|\, d\mathbf{n} = \hat{\mathbb{I}}\,. \tag{6}$$

B. Minimal informationally complete POVMs

Some POVMs have the additional property that they are *informationally complete*, which says that it is possible to write each density matrix $\hat{\rho}$ as a linear combination of its elements $\hat{E}_\alpha$, that is,

$$\hat{\rho} = \sum_{\alpha \in A} \rho_\alpha \hat{E}_\alpha \,. \tag{7}$$

There are density matrices which are not multiples of the identity $\hat{\mathbb{I}}$, hence this operator alone does not provide an IC-POVM. Similarly, the projectors $|\psi_n\rangle\langle\psi_n|$ on a complete set of orthonormal states do not allow one to represent each density operator. If one uses the coherent-state POVM defined in (6), however, lead to

$$\hat{\rho} = \int_{\mathcal{S}} \rho_{\mathbf{n}} \, \hat{E}_{\mathbf{n}} \, d\mathbf{n} \,, \tag{8}$$

where $\rho_{\mathbf{n}}$ is the so-called P-symbol of $\hat{\rho}$, a real function on the unit sphere, taking values between zero and one.

Minimal informationally complete POVMs, or MIC-POVMs, are POVMs which contain the *least* number of elements such that (7) holds for all operators $\hat{\rho}$. This requirement is equivalent to saying that the operators $\hat{E}_\alpha$ should form a (minimal) *basis* of the vector space of hermitean operators acting on a Hilbert space $\mathcal{H}^d$. Counting the number of real parameters necessary to parameterize all such operators, conveniently represented as hermitean matrices of size $(d \times d)$, one concludes that a MIC-POVMs must contain d^2 (linearly independent) elements.

Not every set of d^2 operators spanning all hermitean operators on $\mathcal{H}^d$ is a POVM. To see this, let us look at the simple example of a spin 1/2. Any spin observable $\hat{A}$ can be written as

$$\hat{A} = A_0 \hat{\mathbb{I}} + \mathbf{A} \cdot \hat{\sigma} \,,$$

with a real number A_0 and a real three-component vector $\mathbf{A}$, and $\hat{\sigma}$ is a operator with Pauli matrices as components. However, the four operators $(\hat{\mathbb{I}}, \hat{\sigma})$ do not constitute a POVM since the expectation value of each spin component in normalized states ranges form -1 to $+1$. All is not lost: the three indefinite operators turn positive by adding the identity:

$$0 \leq \langle\psi| \left(\hat{\mathbb{I}} + \hat{\sigma}_i\right) |\psi\rangle \leq 2 \,, \quad i = x, y, z \,.$$

Using this idea one easily constructs MIC-POVMs for a spin-1/2:

$$\hat{E}_\alpha = \frac{1}{4}\left(\hat{\mathbb{I}} + \mathbf{n}_\alpha \cdot \hat{\sigma}_\alpha\right) \geq 0 \,, \quad \alpha = 1 \ldots 4, \quad \text{where } \sum_{\alpha=1}^{4} \mathbf{n}_\alpha = 0 \,; \tag{9}$$

the four unit vectors $\mathbf{n}_\alpha$ must not lie in a plane. Note that the expressions in the brackets are proportional to the projectors on the states $|\mathbf{n}_\alpha\rangle$ implying that one can also write $\hat{E}_\alpha = |\mathbf{n}_\alpha\rangle\langle\mathbf{n}_\alpha|/2$.

For qudits living in $\mathcal{H}^d$, one can use the following approach to ascertain the existence of MIC-POVMs. Consider any set of d^2 linearly independent non-negative operators $\hat{F}_\alpha \geq 0$, say, satisfying the relation

$$\sum_{\alpha=1}^{d^2} \hat{F}_\alpha = \hat{G} > 0 \,.$$

Since $\hat{G}$ is strictly positive, it has a unique, strictly positive square root $\hat{G}^{\frac{1}{2}}$, with an inverse $\hat{G}^{-\frac{1}{2}}$, enjoying these properties as well. Thus, one map the original operators to new one according to $\hat{F}_\alpha \to \hat{E}_\alpha = \hat{G}^{-\frac{1}{2}} \hat{F}_\alpha \hat{G}^{-\frac{1}{2}}$. This invertible transformation preserves positivity, hermiticity and the rank of the original operators; in addition, the new operators satisfy the desired relation

$$\sum_{\alpha=1}^{d^2} \hat{E}_\alpha = \hat{\mathbb{I}} \,,$$

hence giving rise to a MIC-POVM.

III. MINIMAL POVMS FROM THE EXPECTATION-VALUE REPRESENTATION

A. The Expectation-Value Representation of Quantum Mechanics

If you randomly pick d^2 points $\mathbf{n}_n, n = 1 \dots d^2$, on the unit sphere, then the projection operators $\hat{Q}_n = |\mathbf{n}_n\rangle\langle\mathbf{n}_n|$ on the associated coherent states $|\mathbf{n}_n\rangle$ are (in almost all cases) linearly independent . This means that they provide a basis for the hermitean operators on $\mathcal{H}^d$,

$$\hat{A} = \frac{1}{d}\sum_{n=1}^{d^2} A^n \hat{Q}_n \, ,$$

with unique real coefficients A^n. The trace of the product of two operators on $\mathcal{H}^d$ has all the properties of a scalar product which can be used to introduce the basis *dual* to the projectors $\hat{Q}_n$ by the requirement

$$\frac{1}{d}\mathrm{Tr}\left[\hat{Q}^{n'}\hat{Q}_n\right] = \delta_{n'n} \quad , \quad n, n' = 1. \dots d^2 \, . \tag{10}$$

The hermitean operators $\hat{Q}^n$ provide a basis for observables just as the original ones,

$$\hat{A} = \frac{1}{d}\sum_{n=1}^{d^2} A_n \hat{Q}^n \, ,$$

with a second set of real expansion coefficients A_n. Having defined the dual basis via (10), one finds that the expansion coefficients in one basis are given by the scalar product of the operator at hand with the corresponding element of the dual basis,

$$A^n = \mathrm{Tr}\left[\hat{Q}^n\hat{A}\right] \; , \quad \text{and} \quad A_n = \mathrm{Tr}\left[\hat{A}\hat{Q}_n\right] \quad , \quad n = 1. \dots d^2 \, . \tag{11}$$

The coefficients A_n have an interesting property: recalling that the $\hat{Q}_n$ are projections, the second set of equations in (11) takes the form

$$A_n = \langle\mathbf{n}_n|\hat{A}|\mathbf{n}_n\rangle \, . \tag{12}$$

This means that any operator $\hat{A}$ is determined entirely by its expectation values in d^2 appropriate coherent states. If applied to a density matrix $\hat{\rho}$ of a qudit, i.e. a spin $(d-1)/2$, this result says that it is possible to parameterize the quantum state in terms of d^2 numbers $p_n = \langle\mathbf{n}_n|\hat{\rho}|\mathbf{n}_n\rangle$, each of which corresponds to a probability which can be measured with an ordinary Stern-Gerlach apparatus.[1] When expressed in this way, $\hat{\rho}$ is said to be given in the *expectation-value representation* which provides a faithful representation of the quantum system involving only measurable quantities.

B. Obstacles

Let us now explore the link between the expectation-value representation and MIC-POVMs. Being positive sei-definite, the d^2 operators $\hat{Q}_n$ are promising candidates for a minimal informationally POVM. The only remaining condition is that they must add up to the identity. Being linearly independent, one can expand the identity in terms of the projectors,

$$\hat{\mathbb{I}} = \frac{1}{d}\sum_{n=1}^{d^2} \mathbb{I}^n \hat{Q}_n \, . \tag{13}$$

If all coefficients $\mathbb{I}^n = \mathrm{Tr}[\hat{Q}^n]$ were positive, it would be a simple matter to define a POVM: simply use the rescaled projectors $\mathbb{I}^n\hat{Q}_n$ as its elements. However, the assumption that all $\mathbb{I}^n$ are positive is inconsistent whatever the choice of the directions $\mathbf{n}_n$. Taking the trace over both sides of (13) and recalling that $\mathrm{Tr}[\hat{Q}_n] = 1$ leads to the 'summation rule'

$$d^2 = \sum_{n=1}^{d^2} \mathbb{I}^n . \tag{14}$$

If one multiplies Eq. (13) with $\hat{Q}_m$ and takes the trace, one sees that the Gram matrix links the two sets of cefficients of the identity,

$$\hat{\mathbb{I}}_m = \sum_{n=1}^{d^2} \mathfrak{G}_{mn} \hat{\mathbb{I}}^n , \quad m = 1 \ldots d^2 .$$

Recalling that all $\hat{\mathbb{I}}_m$ are equal to unity, one finds from these equations that

$$1 = \mathfrak{G}_{mm} \hat{\mathbb{I}}^m + \sum_{n \neq m}^{d^2} \mathfrak{G}_{mn} \hat{\mathbb{I}}^n > \hat{\mathbb{I}}^m , \quad m = 1 \ldots d^2 ,$$

since the diagonal elements of $\mathfrak{G}$ equal one while all its other elements are positive and the coefficients $\hat{\mathbb{I}}^m$ have been assumed to positive. This leads to a contradiction: summing the d^2 inequality gives $d^2 > \sum_m \hat{\mathbb{I}}^m$, contradicing the sum rule (14).

In view of this result, it might be a good idea to expand the identity in the *dual* basis,

$$\hat{\mathbb{I}} = \frac{1}{d} \sum_{n=1}^{d^2} \hat{Q}^n , \tag{15}$$

the coefficients being equal to one, $\mathbb{I}_n = \langle \mathbf{n}_n | \hat{\mathbb{I}} | \mathbf{n}_n \rangle \equiv 1$. However, this relation does again *not* constitute a POVM because not all of the operators $\hat{Q}^n$ are positive, as will be shown now. The elements of Gram matrix $\mathfrak{G}$ of the basis $\hat{Q}_n$, defined by pairwise scalar products of basis elements, are positive,

$$\mathfrak{G}_{nn'} = \mathrm{Tr} \left[\hat{Q}_n \hat{Q}_{n'} \right] = |\langle \mathbf{n}_n | \mathbf{n}_{n'} \rangle|^2 > 0 , \quad n, n' = 1 \ldots . d^2 .$$

Therefore, the inverse $\mathfrak{G}^{-1}$ of the Gram matrix, known to exist for linearly independent $\hat{Q}_n$, must have at least one *negative* entry in each row, otherwise, the off-diagonal elements of the product $\mathfrak{G}^{-1}\mathfrak{G}$ could not vanish as is necessary. Since a second expression for the matrix elements of $\mathfrak{G}^{-1}$ is given by the scalar products of the elements of the dual basis, there will be at least one pair of indices, n_0, n_0', say, such that

$$\mathfrak{G}^{n_0 n_0'} = \mathrm{Tr} \left[\hat{Q}^{n_0} \hat{Q}^{n_0'} \right] < 0 . \tag{16}$$

This relation is incompatible with all operators $\hat{Q}^n, n = 1 \ldots d^2$, being positive semi-definite: the eigenvalues of a product of two positive semi-definite operators are real and non-negative implying that the trace in (16) cannot take a negative value. Thus, one of the two operators in (16) cannot be positive semi-definite, and the identity in (15) is *not* given as a sum of only non-negative operators.

C. Constructing new MIC-POVMs

In the following, the minimal informationally complete sets $\{\hat{Q}_n, n = 1 \ldots d^2\}$ and $\{\hat{Q}^n, n = 1 \ldots d^2\}$ will be used to construct MIC-POVMs.

1. FSC-construction

The method described by FSC to construct a POVM out of any set of positive operators can be applied to any sum of the projection operators $\hat{Q}_n$ with positive coefficients,

$$\hat{S} = \sum_{n=1}^{d^2} \alpha_n \hat{Q}_n , \quad \alpha_n > 0 . \tag{17}$$

The following argument shows that the hermitean operator $\hat{S}$ is *strictly* positive. Defined as the sum of positive semi-definite operators, the expectation value of $\hat{S}$ in any state $|\psi\rangle$ is cleary non-negative, $\langle\psi|\hat{S}|\psi\rangle \geq 0$. However, $\hat{S}$ having a zero eigenvalue would lead to a contradiction: assume that there is a normaliable state $|\psi_0\rangle$ which $\hat{S}$ annihilates, $\hat{S}|\psi_0\rangle = 0$, and expand the projector $\hat{S}_0 = |\psi_0\rangle\langle\psi_0|$ in terms of the basis $\hat{Q}^n$. The sum of its non-negative expansion coefficients

$$S_n = \mathrm{Tr}[\hat{S}_0\hat{Q}_n] = |\langle\psi_0|\mathbf{n}_n\rangle|^2$$

would vanish since

$$\sum_{n=1}^{d^2}\alpha_n S_n = \langle\psi_0|\sum_{n=1}^{d^2}\alpha_n\hat{Q}_n|\psi_0\rangle = \langle\psi_0|\left(\hat{S}|\psi_0\rangle\right) = 0\,,$$

which is only possible if each term S_n of the sum vanishes individually. Hence, $\hat{S}_0$ is zero and $|\psi_0\rangle$ cannot be a normalizable state leaving us with $\hat{S} > 0$. Consequently, $\hat{S}$ has a unique square root and an inverse, which is all one needs to complete the FSC-construction. Explicitly, the resulting family of MIC-POVMs is given by

$$\{\hat{E}_n = \alpha_n\hat{S}^{-\frac{1}{2}}\hat{Q}_n\hat{S}^{-\frac{1}{2}}\,, \alpha_n > 0\,, n = 1\ldots d^2\}\,,$$

where, unfortunately, no analytic expressions for the square roots are available if $d > 4$.

2. *Analytic MIC-POVMs*

The expansions (13) and (15) did not define POVms since neither the expansion coefficients $\mathbb{I}^n$ on the one hand nor the elements of the basis $\hat{Q}^n$ on the other hand are non-negative. It will be shown now that minor modifications are sufficient in order to obtain MIC-POVMs.

Rewrite (13) in the form

$$\hat{\mathbb{I}} = \frac{1}{d}\sum_{n_+=1}^{N_+}\mathbb{I}^{n_+}\hat{Q}_{n_+} - \frac{1}{d}\sum_{n_-=1}^{N_-}|\mathbb{I}^{n_-}|\,\hat{Q}_{n_-}\,, \quad N^+ + N^- = d^2\,, \tag{18}$$

rearranging the sum in such a way that the sums contain terms with non-negative and negative coefficients, $\mathbb{I}^{n_+} \geq 0$ and $\mathbb{I}^{n_-} < 0$, respectively. Adding on both sides a (C/d)-fold multiple of the identity, with $C = \sum_{n_-}\mathbb{I}^{n_-} < 0$, one findsM

check next equation

$$(1 - C)\,\hat{\mathbb{I}} = \frac{1}{d}\sum_{n_+=1}^{N_+}\mathbb{I}^{n_+}\hat{Q}_{n_+} + \frac{1}{d}\sum_{n_-=1}^{N_-}|\mathbb{I}^{n_-}|\,(\hat{\mathbb{I}} - \hat{Q}_{n_-})\,. \tag{19}$$

This can be written as

$$\hat{\mathbb{I}} = \sum_{n_+=1}^{N_+}\hat{E}_{n_+} + \sum_{n_-=1}^{N_-}\hat{E}_{n_-} \equiv \sum_{n=1}^{d^2}\hat{E}_n\,, \tag{20}$$

where

redefin things without modulus?

$$\hat{E}_{n_+} = \frac{\mathbb{I}^{n_+}}{d + C}\,\hat{Q}_{n_+}\,, \quad \hat{E}_{n_-} = \frac{|\mathbb{I}^{n_-}|}{d + C}\,(\hat{\mathbb{I}} - \hat{Q}_{n_+})\,, \tag{21}$$

are d^2 positive semi-definite operators each being either of rank one or of rank $(d - 1)$. In view of (20), they form a MIC-POVM. The physical interpretation of this POVM is as follows: when it is measured there are d^2 possible outcomes, N_+ of which correspond to finding the particle in the coherent state $|\mathbf{n}_{n_+}\rangle$, while the remaining N_- cases correspond to finding the state in a $(d-1)$-dimensional subspace orthogonal to the states $|\mathbf{n}_{n_-}\rangle$. The case of a qubit is special since both the operators $\hat{E}_{n_+}$ and $\hat{E}_{n_-}$are of rank one. It will be shown below that the present construction leads to other (or the same set of) POVMs as introduced above.

Not surprisingly, a similar modification enables one to find a MIC-POVM from the dual basis. The second expansion of the identity can be rewritten as

$$\left(1-\frac{\tilde{C}}{d}\right)\hat{\mathbb{I}}=\frac{1}{d}\sum_{n_+=1}^{\tilde{N}_+}\hat{Q}^{n_+}+\frac{1}{d}\sum_{n_-=1}^{\tilde{N}_-}\left(\hat{Q}^{n_-}-q^{n_-}\hat{\mathbb{I}}\right), \tag{22}$$

where $q^{n_-}<0$ denotes the smallest eigenvalue of $\hat{Q}^{n_-}$ and the number $\tilde{C}$ is their sum,

$$\tilde{C}=\sum_{n_-=1}^{\tilde{N}_-}q^{n_-}<0\,. \tag{23}$$

By construction, all of the operators

$$\hat{\varepsilon}^{n_+}=\frac{1}{d-\tilde{C}}\hat{Q}^{n_+}\,,\quad \hat{\varepsilon}^{n_-}=\frac{1}{d-\tilde{C}}(\hat{Q}^{n_-}-q^{n_-}\hat{\mathbb{I}})\,, \tag{24}$$

are positive semi-definite which is obvious for the first ones and the second set has been shifted appropriately, so that finally

$$\hat{\mathbb{I}}=\sum_{n_+=1}^{\tilde{N}_+}\hat{\varepsilon}^{n_+}+\sum_{n_-=1}^{\tilde{N}-}\hat{\varepsilon}^{n_-}\equiv\sum_{n=1}^{d^2}\hat{\varepsilon}^{n}\,. \tag{25}$$

It is not difficult to see that the two MIC-POVMs just constructed are not dual to each other, hence intrinsically different: taking the scalar products within each basis one has immediately

$$\mathrm{Tr}\left[\hat{E}_n\hat{E}_{n'}\right]\geq 0\,,\quad \text{and}\quad \mathrm{Tr}[\hat{\varepsilon}^n\hat{\varepsilon}^{n'}]\geq 0\,,\quad n,n'=1\ldots N\,, \tag{26}$$

which says that both sets of operators lead to Gram matrices with non-negative entries only making it impossible that they would be inverses of each other.

D. Generalization

Having gained some experience with the construction of MIC-POVMs, one generalize the FCS-approach described above. Effectively, it is possible to relax the condition of having d^2 non-negative operators and avoid the appearance of the analytically inaccessible square root of an operator. To couch the result to be shown now in positive terms: *every set of d^2 linearly independent hermitean operators acting on $\mathcal{H}^d$ can be used to define a MIC-POVM.*

Consider d^2 hermitean operators $\hat{\kappa}_n$ on $\mathcal{H}^d$ bounded by the inequalites

$$\kappa_n^-\leq\hat{\kappa}_n\leq\kappa_n^+\,,\quad n=1\ldots d\,,$$

where the paira of numbers $-\infty<\kappa_n^\pm<\infty$ are, respectively, the maximal and the minimal eigenvalue of $\hat{\kappa}_n$, not both of which can be equal to zero simultaneously. Upon shifting and rescaling according to

$$\hat{K}_n=\frac{1}{\kappa_n^+-\kappa_n^-}\left(\hat{\kappa}_n-\kappa_n^-\hat{\mathbb{I}}\right)\,,\quad n=1\ldots d\,,$$

one obtains non-negative operators bounded by zero and one,

$$0\leq\hat{K}_n\leq 1\,,\quad n=1\ldots d\,,$$

as is necessary for the elements of a POVM. The conditions

$$\frac{1}{d}\,\mathrm{Tr}\left[\hat{K}_n\hat{K}^{n'}\right]=\delta_n^{n'}$$

define a unique dual set of d^2 operators $\hat{K}^n$. Hence, it is straighforward to write down the expansion of the identity,

$$\hat{\mathbb{I}}=\frac{1}{d}\sum_{n=1}^{d^2}\mathbb{I}^n\hat{K}_n\,,\quad \mathbb{I}^n=\mathrm{Tr}\left[\hat{\mathbb{I}}\hat{K}^n\right]\,, \tag{27}$$

where, as before, some of the coefficients inevitably will be negative. Using the same remedy as above, one, namely to effectively replace the operators $\hat{K}_{n_-}$ with negative coefficients by $(\hat{\mathbb{I}} - \hat{K}_{n_-})$, also being bounded by zero and one, one produces a set of d^2 non-negative operators summing up to the identity

$$\hat{\mathbb{I}} = \sum_{n_+=1}^{N_+} \hat{\epsilon}_{n_+} + \sum_{n_-=1}^{N_-} \hat{\epsilon}_{n_-} \equiv \sum_{n=1}^{d^2} \hat{\epsilon}_n \,, \tag{28}$$

where

$$\hat{\epsilon}_{n_+} = \frac{\mathbb{I}^{n_+}}{d+C} \hat{K}_{n_+} \,, \quad \hat{\epsilon}_{n_-} = \frac{|\mathbb{I}^{n_-}|}{d+C} (\hat{\mathbb{I}} - \hat{K}_{n_-}) \,, \tag{29}$$

C being defined as before as the sum of the moduli of the negative coefficients. The last two equations are the main result of this paper.

1. *State reconstruction in terms of POVMs*

Suppose now that a MIC-POVM $\hat{E}_n, n = 1 \ldots d^2$, has been defined, and an unknown density matrix $\hat{\rho}$ is assumed to be available in an arbitrary large number of copies. The most elegant approch to use the PoVM for state reconstruction passes through the dual set of operators $\hat{E}^n$, defined by the equivalent of the condition (10). Once these operators have been found, and the collection of numbers

$$p_n(\hat{\rho}) = \text{Tr}\left[\hat{\rho}\hat{E}_n\right] \in [0,1] \,, \quad n = 1 \ldots d^2 \,,$$

have been measured, one can write down an explicit formula for the density matrix

$$\hat{\rho} = \frac{1}{d} \sum_{n=1}^{d^2} p_n(\hat{\rho}) \hat{E}^n \,.$$

Formally, this result is very similar to what is known from the expectation-value representation involving discrete P- and Q-symbols. However, it is fundamentally different in the sense that the numbers $p_n(\hat{\rho})$ are 'honest' probabilities emerging from an experiment performed with a *single* apparatus.

check these formulas!

E. Example of a spin 1/2

Let us illustrate the construction of new MIC-POVMs by for a spin-1/2 system with Hilbert space $\mathcal{H}^2$. To make the example as explicit as possible, let us consider three pairwise orthogonal unit vectors $\mathbf{n}_i \,, i = 1,2,3$, in $\mathbb{R}^3$, and

$$\mathbf{n}_4 = \frac{1}{\sqrt{3}} (\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3) \,. \tag{30}$$

These four vectors clearly do not satisfy the condition given in (9), hence the four projection operators $\hat{Q}_n = |\mathbf{n}_\alpha\rangle\langle\mathbf{n}_\alpha| \,, n = 1 \ldots 4$, do not form a POVM. By applying the procedure outlined above one can associate to them a unique MIC-POVM with elements

$$\begin{aligned} \hat{E}_i &= \frac{2}{\sqrt{3}(\sqrt{3}+1)} |\mathbf{n}_i\rangle\langle\mathbf{n}_i| \,, \quad i = 1,2,3 \,, \\ \hat{E}_4 &= \frac{2}{(\sqrt{3}+1)} |-\mathbf{n}_4\rangle\langle-\mathbf{n}_4| \,; \end{aligned}$$

note that the only the forth of the vectors acquires an additional minus sign. This result follows from the fact that one can express the expansion coefficients for any four vectors not on a circle on the unit sphere of the identity in the present situation in the form

$$\mathbb{I}^n = \frac{4}{1 + \mathbf{f}^n \cdot \mathbf{n}_n} \,, \quad n = 1 \ldots 4 \,,$$

where the vector $\mathbf{f}^1 \in \mathbb{R}^3$ is determined by

$$\mathbf{f}^1 = -\frac{\mathbf{n}_2 \wedge \mathbf{n}_3 + \mathbf{n}_3 \wedge \mathbf{n}_4 + \mathbf{n}_4 \wedge \mathbf{n}_2}{(\mathbf{n}_2 \wedge \mathbf{n}_3) \cdot \mathbf{n}_4} \, ,$$

and the other three vectors follow from this relation by cyclic permutation of the indices 1 through 4. A straightforward calculation leads to

$$\begin{aligned} \mathbb{I}^i &= \frac{4}{\sqrt{3}(\sqrt{3}-1)} > 0 \, , \quad i = 1,2,3 \, , \\ \mathbb{I}^4 &= \frac{4}{(1-\sqrt{3})} < 0 \, , \end{aligned}$$

i.e. only one negative coefficient which needs to be eliminated by adding a multiple of the identity to the expension of the identity.

IV. CONCLUSIONS

Starting from the expectation-value representation of quantum mechanics in d-dimensional Hilbert spaces, new simple POVMs with d^2 elements have been defined which are informationally complete. Mathematically speaking, the elements of these POVMs provide a basis in the Hilbert-Schmidt space of operators acting on $\mathcal{H}^d$ while, from a physical point of view, they are suited to reconstruct unknown quantum states if an arbitrarily large number of systems in the same state are available. Repeated measurements with a single such POVM produce (d^2-1) numbers which are in a one-to-one correspondence with a density matrix $\hat{\rho}$. The POVMs obtained here have the important property that they can be written down analytically in finite Hilbert spaces of any dimension. Since any set of d^2 linearly independent operators can be used as a starting point, a wide range of possibilities opens up to construct MIC-POVMs most suited for the application at hand.

More? SICPO

[1] Since a density matrix $\hat{\rho}$ has unit trace, only (d^2-1) expectation values will be necessay in this case.