UNIVERSITY OF YORK

BA, BSc and MMath Examinations 2006
MATHEMATICS
Analysis I

Time Allowed: $1\frac{1}{2}$ hours.

Answer all parts of Question 1 and two of the remaining three questions.
Question 1 carries 40 marks; questions 2, 3 and 4 carry 25 marks each.
Please write your answers in ink; pencil is acceptable for graphs and diagrams. Do not use red ink.
Standard calculators will be provided.
The marking scheme shown on each question is indicative only.

1 (of 4). (a) Suppose S is a non-empty subset of $\mathbb{R}$. Define the terms *supremum* and *infimum*, as applied to S. Under what circumstances do the supremum and infimum exist?

For each of the following sets, write down the supremum and infimum, if they exist. *You are not required to justify your answers.*

$$S_1 = \{x \in \mathbb{R} : (x-1)^2 < 3\}; \qquad S_2 = \{x \in \mathbb{R} : x^3 \leq 2\}.$$

[8]

(b) State the definition of *convergence* of a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ to a limit $a \in \mathbb{R}$, and use this definition to show that

$$\frac{2n}{n+3} \to 2 \text{ as } n \to \infty. \qquad \textbf{[10]}$$

(c) Determine the limits as $n \to \infty$ of the following sequences. *Make free use of standard combination rules and standard limits, briefly indicating which results you are using; "ε" arguments are **not** required.*

$$a_n = \frac{2-n+n^2}{n^2+3}; \qquad b_n = \frac{3^n - n}{4^n + \sqrt{2}}; \qquad c_n = \frac{n! + n}{(n+1)^n}.$$

[6]

(d) Determine which of the following series converge:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{x^n}$ ($x \neq 0$ fixed);

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(n+1)}$. **[6]**

 continued on next page

1 (of 4) *cont.*

(e) What is meant by the *radius of convergence* of a power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

(including the cases 0 and ∞)? Find the radius of convergence of the following series:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$;

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^{n/2}}$. **[10]**

2 (of 4). Suppose S is a bounded, non-empty subset of $\mathbb{R}$. Define the terms *upper bound* and *lower bound*, as applied to S. **[4]**

Let

$$-S = \{-x : x \in S\}.$$

Show that b is an upper bound for S if and only if $-b$ is a lower bound for $-S$, and that $\inf(-S) = -\sup(S)$. Deduce that $\sup(-S) = -\inf(S)$. **[14]**

State without proof the *Archimedean property* (or *axiom of Archimedes*) and use it to show that

$$\inf\{1/n : n \in \mathbb{N}\} = 0.$$

What is the supremum of this set? Justify your answer. **[7]**

3 (of 4). *Throughout this question, make free use of the standard combination rules and the basic fact that* $1/n \to 0$ *as* $n \to \infty$.

State without proof the *Sandwich Theorem*. **[3]**

Show by induction that $2^n > n$ for all $n \in \mathbb{N}$, and deduce that $1/2^n \to 0$ as $n \to \infty$. **[6]**

By expanding the numerator and denominator into n factors, show that if $n \geq 3$ then

$$\frac{2^n}{n!} \leq \frac{4}{n}.$$

Deduce that $2^n/n! \to 0$ as $n \to \infty$. **[8]**

Apply a similar technique to show that $(n!)^2/(2n)! \to 0$ as $n \to \infty$. **[8]**

4 (of 4). *Throughout this question, make free use of any standard facts about convergence of geometric series.*

State without proof the *comparison test*. **[4]**

Suppose $(a_n)_{n\in\mathbb{N}}$ is a sequence such that for every $x \in \mathbb{R}$ with $|x| < 1$, the sequence $(a_n x^n)_{n\in\mathbb{N}}$ is bounded; say $|a_n x^n| \leq C_x$ for all $n \in \mathbb{N}$.

Suppose $0 \leq x < 1$. By considering the factorisation and estimate

$$|a_n x^n| = |a_n x^{n/2}| x^{n/2} \leq C_{\sqrt{x}} x^{n/2},$$

or otherwise, show that the series

$$\sum_{n=1}^{\infty} a_n x^n$$

is absolutely convergent. **[10]**

Show that the same is true if $-1 < x < 0$. Express these results in terms of the radius of convergence of a power series. **[11]**

End of examination.

1. (a) The supremum of S is the least upper bound of S, if S is bounded above. The infimum is the greatest lower bound of S, if S is bounded below. [4 Marks]

$$\sup(S_1) = 1 + \sqrt{3}; \qquad \inf(S_1) = 1 - \sqrt{3}$$

$$\sup(S_2) = 2^{1/3}; \qquad \inf(S_2) \text{ does not exist}$$ [4 Marks]

(b) a_n converges to a as $n \to \infty$ if for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that if $n > N_\varepsilon$ then $|a_n - a| < \varepsilon$. [4 Marks]
If $a_n = 2n/(n+3)$ and $a = 1$ then we calculate:

$$\begin{aligned} |a_n - a| < \varepsilon &\iff |2n/(n+3) - 2| < \varepsilon \\ &\iff |-6/(n+3)| < \varepsilon \\ &\iff 6/(n+3) < \varepsilon \\ &\iff n > 6/\varepsilon - 3 \end{aligned}$$

Given $\varepsilon > 0$, let N_ε be some natural number greater than $6/\varepsilon - 3$. If $n > N_\varepsilon$ then $n > 6/\varepsilon - 3$ so by the above calculation (all steps being "if and only if") it follows that $|a_n - a| < \varepsilon$, showing that $a_n \to a$ as $n \to \infty$. [6 Marks]

(c)

$$a_n = \frac{2 - n + n^2}{n^2 + 3} = \frac{2/n^2 - 1/n + 1}{1 + 3/n^2} \to \frac{0 - 0 + 1}{1 + 0} = 1$$

using standard combination rules and $1/n, 1/n^2 \to 0$.

$$b_n = \frac{3^n - n}{4^n + \sqrt{2}} = \frac{(3/4)^n - n/4^n}{1 + \sqrt{2}/4^n} \to \frac{0 - 0}{1 + 0} = 0$$

using combination rules, $(3/4)^n, (1/4)^n \to 0$ since $|3/4| < 1, |1/4| < 1$ and "exponential beats power".

$$c_n = \frac{n! + n}{(n+1)^n} = \frac{n!}{(n+1)^n} + \frac{n}{(n+1)^n} \to 0 + 0 = 0$$

using combination rules and "n^n beats factorial". [6 Marks]

(d)

(i) Ratio test:

$$\left| \frac{(n+1)!}{x^{n+1}} \frac{x^n}{n!} \right| = \frac{n+1}{|x|}$$

This tends to $+\infty$ for any $x \neq 0$. so the series diverges. [3 Marks]

(ii) $1/\log(n+1)$ decreases and tends to zero, so by Leibniz's alternating series test, the series converges. 3 Marks

(e) The radius of convergence of the given power series is 0 if it converges only when $x = x_0$, or ∞ if it converges for all $x \in \mathbb{R}$. Otherwise, it is the number $R > 0$ with the property that the series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$. 4 Marks

In both cases we use the ratio test.

(i)
$$\left| \frac{(x+2)^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{(x+2)^n} \right| = |x+2| \sqrt{n/(n+1)} \to |x+2|$$
This show that we have convergence if $|x+2| < 1$ and divergence if $|x+2| > 1$; that is, $R = 1$. 3 Marks

(ii)
$$\left| \frac{(x+1)^{n+1}}{(n+1)^{(n+1)/2}} \frac{n^{n/2}}{(x+1)^n} \right| = \frac{|x+1|}{\sqrt{n+1}} \left(\frac{n}{n+1} \right)^{n/2} \to 0$$
using the fact that $n/(n+1) < 1$. Since the limits is < 1 for all x, we have convergence for all x; that is, $R = \infty$. 3 Marks

Remarks. *Everything in this question is supposed to be straightforward and, if not identical to examples they've already seen, certainly similar. Full credit, of course, for equally correct solutions, e.g. root test in very last part.*

Total: 40 Marks

2. An upper bound of S is a number $b \in \mathbb{R}$ such that for all $x \in S$, $x \leq b$. A lower bound is a number $a \in \mathbb{R}$ such that for all $x \in S$, $a \leq x$. 4 Marks

b is an upper bound for $S \iff$ for all $x \in S$, $x \leq b \iff$ for all $x \in S$, $-x \geq -b$ $\iff$ for all $y \in -S$, $-b \leq y \iff -b$ is a lower bound for $-S$. 4 Marks

Since $\sup(S)$ is an upper bound for S, $-\sup(S)$ is a lower bound for $-S$. If $a > -\sup(S)$ then $-a < \sup(S)$, so $-a$ is not an upper bound for S; it follows that a is not a lower bound for $-S$. This shows that $-\sup(S)$ is the greatest lower bound of $-S$; that is, $\inf(-S) = -\sup(S)$. 6 Marks

To see that $\sup(-S) = -\inf(S)$, it is possible to use a similar argument; but it is easier to observe that $-(-S) = S$ so, appying the above result to $(-S)$, we

have $\inf(S) = \inf(-(-S)) = -\sup(-S)$, or equivalently $\sup(-S) = -\inf(S)$. 4 Marks

Archimedes axiom states that the natural numbers are not bounded above; that is, for any $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $n > x$. Clearly 0 is a lower bound of the given set. If a is another lower bound, then $a \leq 1/n$ for all $n \in \mathbb{N}$. In case $a > 0$ this implies $n < 1/a$ for all $n \in \mathbb{N}$, in contradiction to Archimedes. We therefore have $a \leq 0$, so 0 is the greatest lower bound. 5 Marks

Since $1 \leq n$ for all $n \in \mathbb{N}$, $1/n \leq 1$ for all $n \in \mathbb{N}$. This shows that 1 is an upper bound of the given set. But $1 = 1/1$ is also an element of the set, so it is the maximum and, in particular, the supremum. 2 Marks

Remarks. *Candidates typically find formal calculations with bounds difficult, so I've tried to make this question reasonably straightforward. The calculation* $\inf(S) = -\sup(-S)$ *is in the notes, in the context if proving that infima exist. The infimum of* $\{1/n : n \in \mathbb{N}\}$ *is also in the notes, as part of a list of equivalent formulation of the Archimedean proper, but the proof isn't as direct as this.*

Total: 25 Marks

3. The Sandwich Theorem states that if $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are real sequences such that $a_n \leq b_n \leq c_n$ for all n and a_n and c_n both converge to the same limit a as $n \to \infty$, then $b_n \to a$ as $n \to \infty$. 3 Marks

Certainly $2^1 > 1$. If, for some n, $2^n > n$, then

$$2^{n+1} = 2.2^n > 2n = n + n \geq n + 1.$$ 3 Marks

It follows that $2^n >$ for all n. We now have

$$0 < \frac{1}{2^n} < \frac{1}{n}$$

so $1/2^n \to 0$ as $n \to \infty$ by the sandwich theorem. 3 Marks

We have

$$\frac{2^n}{n!} = \frac{2}{1}\frac{2}{2}\frac{2}{3}\frac{2}{4}\cdots\frac{2}{n}$$

The product of the first two terms is 2. The next $n - 3$ terms are all less than 1. The last term is $2/n$. This gives

$$0 < \frac{2^n}{n!} \leq 2\frac{2}{n} = \frac{4}{n}$$

for $n \geq 3$. 5 Marks

The RHS tends to 0, so $2^n/n! \to 0$ as $n \to \infty$ by the sandwich theorem (the cases $n = 1, 2$ not making any difference to the limit). 3 Marks

Similarly, the first n terms in the expansion $(2n)! = 1.2 \ldots (2n)$ cancel with $n!$ to give

$$\frac{(n!)^2}{(2n)!} = \frac{1}{n+1}\frac{2}{n+2} \cdots \frac{n}{2n}$$

and each term on the RHS is $\leq 1/2$. This gives

$$0 < \frac{(n!)^2}{(2n)!} \leq \frac{1}{2^n}$$ 5 Marks

The RHS tends to 0 by an earlier result so $(n!)^2/(2n)! \to 0$ as $n \to \infty$ by the sandwich theorem. 3 Marks

Remarks. *Inequalities like this, and the consequences of the sandwich theorem, are familiar, though mostly in more general contexts such as $x^n/n! \to 0$ for any x. The last inequality was on a problem sheet.*

Total: 25 Marks

4. Suppose $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are series such that $|p_n| \leq |q_n|$ for all n and $\sum_{n=1}^{\infty} q_n$ is absolutely convergent. Then $\sum_{n=1}^{\infty} p_n$ is absolutely convergent. 4 Marks

The suggested factorisation

$$a_n x^n = a_n x^{n/2} x^{n/2} = a_n (\sqrt{x})^n (\sqrt{x})^n$$

yields

$$|a_n x^n| \leq C_{\sqrt{x}} (\sqrt{x})^n$$

Since $0 < x < 1$, we also have $0 < \sqrt{x} < 1$ so the geometric sum $\sum_{n=1}^{\infty} (\sqrt{x})^n$ converges. By the comparison test, $\sum_{n=1}^{\infty} a_n x^n$ is absolutely convergent. 10 Marks

In the case $-1 < x < 0$, we can instead use the factorisation

$$a_n x^n = a_n (-1)^n |x|^{n/2} |x|^{n/2} = a_n (-1)^n (\sqrt{|x|})^n (\sqrt{|x|})^n$$

which leads to

$$|a_n x^n| \leq C_{\sqrt{|x|}} (\sqrt{|x|})^n$$

The comparison test now works in the same way to show that $\sum_{n=1}^{\infty} a_n x^n$ is absolutely convergent, since $0 < \sqrt{|x|} < 1$. 8 Marks

This hows that the power series $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R \geq 1$. 3 Marks

Remarks. *Apart from the comparison test and the convergence of the geometric series, this is unseen. It's quick and easy for the good students.*

Total: 25 Marks