AXIOMATISABILITY PROBLEMS FOR S-SYSTEMS

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1. *Introduction*

To a given type of algebraic systems $\mathscr{C}$ there corresponds at least one first order language L. One can then ask whether a property P, defined for members of $\mathscr{C}$, is expressible in the language L. In other words, is there a set of sentences Π such that a member $\mathscr{M}$ of $\mathscr{C}$ has property P if and only if all sentences in Π are true in $\mathscr{M}$. If the set Π exists we say that P is *definable* in L. Further, $\mathscr{D}$ is *axiomatisable* in L and Π *axiomatises* $\mathscr{D}$, where $\mathscr{D}$ is the subclass of $\mathscr{C}$ whose members have property P.

In this paper we are concerned with the following problems. Given a monoid S, what conditions must S satisfy for the class of flat S-systems to be axiomatisable or for the class of projective S-systems to be axiomatisable? The corresponding questions for modules over a ring R have been fully answered by Eklof and Sabbagh in [4].

The relevant algebraic definitions are given in Section 2, where we also outline some of the basic semigroup terms that we shall use. We do assume some knowledge of model theory, including the construction of ultraproducts. As far as possible we follow the notation and terminology of [7] for semigroup theory and [2] for model theory. We adopt the convention that an ordinal is the set of all smaller ordinals.

If S is a monoid (R a ring with a 1) we let $\mathscr{P}$ denote the class of projective S-systems (R-modules) and $\mathscr{F}$ the class of flat S-systems (R-modules). A monoid or a ring is said to be *perfect* if $\mathscr{P} = \mathscr{F}$.

In Theorem 3.1 we give necessary and sufficient conditions on a monoid S for the class $\mathscr{F}$ to be axiomatisable. Eklof and Sabbagh show that for a (unitary) ring R, $\mathscr{P}$ is axiomatisable if and only if $\mathscr{F}$ is axiomatisable and R is perfect. To prove this they draw on a general result which is not true of S-systems. We show that for a monoid S, if $\mathscr{P}$ is axiomatisable, then $\mathscr{F}$ is axiomatisable and S satisfies M_R, the descending chain condition for principal right ideals. A ring satisfies M_R if and only if it is perfect [1], but M_R is not enough to give perfection for a monoid. It is shown in [5] that a monoid S is perfect if and only if it satisfies M_R and condition A, which asserts that every S-system satisfies the ascending chain condition for cyclic S-subsystems. We prove in Proposition 4.3 that if $\mathscr{P}$ is axiomatisable, then a monoid S satisfies A if and only if it satisfies M^L, the ascending chain condition for principal left ideals. This enables us to deduce that in some fairly general cases, for example if S is regular, $\mathscr{P}$ is axiomatisable if and only if $\mathscr{F}$ is axiomatisable and S is perfect.

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2. *Preliminaries*

Throughout this paper S will denote a given monoid and R a given ring with unity. A set A is a *left S-system* if there is a map $\xi: S \times A \to A$ satisfying

$$\xi(1, a) = a$$

and

$$\xi(st, a) = \xi(s, \xi(t, a))$$

for any element a of A and any elements s, t of S. For $\xi(s, a)$ we write sa and we refer to left S-systems simply as *S-systems*. One has the obvious definitions of an *S-subsystem* and an *S-homomorphism*. We assume the reader to be familiar with the elementary definitions and results concerning modules over a ring R. We shall refer to left R-modules simply as *R-modules*.

For a monoid S the relation $\mathscr{R}$ is defined by $a\mathscr{R}b$ if and only if $aS = bS$, where $a, b \in S$. That is, $a\mathscr{R}b$ if and only if a and b generate the same principal right ideal of S. The relation $\mathscr{L}$ is defined dually. Then $\mathscr{R}$ is a left congruence and $\mathscr{L}$ is a right congruence; $\mathscr{R}$ and $\mathscr{L}$ are two of *Green's relations* on S. For an element a of S the $\mathscr{R}$-class of a (respectively $\mathscr{L}$-class of a) is simply the equivalence class of $\mathscr{R}$ (respectively $\mathscr{L}$) to which a belongs.

An element a of a monoid S is *regular* if $a = axa$ for some $x \in S$. A monoid S is *regular* if all its elements are regular or equivalently, every $\mathscr{R}$-class and every $\mathscr{L}$-class of S contains an idempotent. Further details of the semigroup theory we use can be found in [7].

For a monoid S we denote by L_S the first order language with equality, which has no constant or relation symbols and which has a unary function symbol ρ_s for each element s of S. We write sx for $\rho_s(x)$ and we regard S-systems as L_S-structures in the obvious way.

For any elements s, t of S, we denote by $\psi_{s,t}$ the sentence

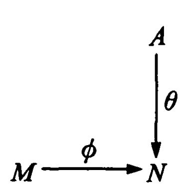
$$(\forall x)((st)x = s(tx))$$

of L_S. Put

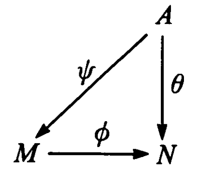
$$\Sigma = \{(\forall x)(1x = x)\} \cup \{\psi_{s,t} : s, t \in S\}.$$

Clearly an L_S-structure $\mathscr{M}$ is an S-system if and only if $\mathscr{M}$ is a model of Σ. Thus Σ axiomatises the class of S-systems in the language L_S. It follows from the Completeness Theorem that if $Th(S)$ is the theory of S-systems, that is, the set of all sentences of L_S true in all S-systems, then $Th(S)$ is the deductive closure of Σ.

An S-system A is *projective* if given any diagram of S-systems and S-homomorphisms



where $\phi: M \to N$ is onto, there exists an S-homomorphism $\psi: A \to M$ such that



is commutative. Projective R-modules are defined in the corresponding way.

In the category of S-systems, the coproduct of S-systems A_i, $i \in I$, is their disjoint union. We use the symbol $\dot{\cup}$ to denote a disjoint union.

Given an S-system A, $a \in A$ and $e \in S$ where e is idempotent, we say that a is *e-cancellable* if $ea = a$ and $sa = ta$ implies that $se = te$, for any $s, t \in S$.

PROPOSITION 2.1 [3]. *An S-system A is projective if and only if A is isomorphic to a coproduct of idempotent generated principal left ideals of S.*

COROLLARY 2.2 [3]. *An S-system A is projective if and only if* $A = \dot{\bigcup}\{Sa_i : i \in I\}$, *where for each* $i \in I$, a_i *is* e_i*-cancellable for some idempotent* e_i *of S.*

The definition of a flat S-system may be given in terms of tensor products, but we state here the equivalent definition in terms of elements of S [9], which is more useful for our purposes.

An S-system A is *flat* if, given any $s, t \in S$ and $a, b \in A$ with $sa = tb$, there exist $s', t' \in S$ and $c \in A$ such that $ss' = tt'$, $a = s'c$, $b = t'c$; moreover, if $a = b$ we may insist that $s' = t'$. One may take an additive version of this condition to define a flat R-module over a ring R.

For a monoid S (ring R) we denote by $\mathscr{P}$ the class of projective S-systems (projective R-modules) and by $\mathscr{F}$ the class of flat S-systems (flat R-modules). It follows from Corollary 2.2 that for a monoid S, $\mathscr{P} \subseteq \mathscr{F}$; the same result is true for rings. If $\mathscr{P} = \mathscr{F}$ then the monoid or ring is said to be (left) *perfect*.

THEOREM 2.3 [5]. *A monoid S is perfect if and only if it satisfies* M_R *and condition A.*

As noted in the introduction, a ring is perfect if and only if it satisfies M_R [1].

To prove our results we rely heavily on the use of ultraproducts, and in particular Łoś's theorem.

THEOREM 2.4 (Łoś, see [2]). *Let L be a first order language and let* $\mathscr{C}$ *be a class of L-structures. Then if* $\mathscr{C}$ *is axiomatisable,* $\mathscr{C}$ *is closed under ultraproducts.*

3. *Axiomatisability of* $\mathscr{F}$

For any elements s, t of a monoid S, we define $R(s, t)$ and $r(s, t)$ as follows:

$$R(s,t) = \{(u,v) \in S \times S : su = tv\},$$

$$r(s,t) = \{u \in S : su = tu\}.$$

Clearly $R(s, t) = \varnothing$ or $R(s, t)$ is a right S-subsystem of the right S-system $S \times S$. Similarly, $r(s, t) = \varnothing$ or $r(s, t)$ is a right ideal of S.

THEOREM 3.1. *The following conditions are equivalent for a monoid S:*

(i) *the class $\mathscr{F}$ is axiomatisable;*

(ii) *every ultraproduct of flat S-systems is flat;*

(iii) *every ultrapower of S is flat;*

(iv) *for any $s, t \in S$, $R(s,t) = \varnothing$ or $R(s,t)$ is finitely generated as a right S-system and $r(s,t) = \varnothing$ or $r(s,t)$ is a finitely generated right ideal of S.*

Proof. (i) $\Rightarrow$ (ii) This follows from Theorem 2.4.

(ii) $\Rightarrow$ (iii) This is obvious since S is flat as a (left) S-system.

(iii) $\Rightarrow$ (iv) Let $s, t \in S$ and suppose that $R(s,t) \neq \varnothing$. Then $R(s,t)$ is an S-subsystem of the right S-system $S \times S$. We suppose that $R(s,t)$ is not finitely generated.

Let $\{(u_\beta, v_\beta) : \beta < \gamma\}$ be a generating subset of $R(s,t)$ of cardinality γ. By assumption, γ is infinite and so γ is a limit ordinal. We may suppose that for any $\beta < \gamma$, (u_β, v_β) is not in the right S-subsystem generated by the preceding elements (u_τ, v_τ), that is, $(u_\beta, v_\beta) \notin \bigcup_{\tau < \beta} (u_\tau, v_\tau) S$.

Let D be a uniform ultrafilter on γ, that is, D is an ultrafilter on γ such that all sets in D have cardinality γ. The existence of such a D is shown in [2]. Put $\mathscr{U} = S^\gamma / D$. By assumption, $\mathscr{U}$ is a flat S-system.

Define elements $\bar{a}$, $\bar{b}$ of $\mathscr{U}$ by $\bar{a} = (u_\beta)_D$, $\bar{b} = (v_\beta)_D$. Since $su_\beta = tv_\beta$ for all $\beta < \gamma$, clearly $s\bar{a} = t\bar{b}$. By the flatness of $\mathscr{U}$ there exist $s', t' \in S$ and $\bar{c} \in \mathscr{U}$ with $ss' = tt'$, $\bar{a} = s'\bar{c}$ and $\bar{b} = t'\bar{c}$. Let $\bar{c} = (w_\beta)_D$.

From $ss' = tt'$ we have $(s', t') \in R(s,t)$ and so $(s', t') = (u_\sigma, v_\sigma) h$ for some $\sigma < \gamma$ and $h \in S$. Since $\bar{a} = s'\bar{c}$ and $\bar{b} = t'\bar{c}$ there exist sets T_1, T_2 in D such that $u_\beta = s'w_\beta$ for all $\beta \in T_1$ and $v_\beta = t'w_\beta$ for all $\beta \in T_2$. Using the facts that $T_1 \cap T_2 \in D$ and D is uniform, $T_1 \cap T_2$ contains an ordinal $\alpha \geqslant \sigma + 1$. Then

$$(u_\alpha, v_\alpha) = (s'w_\alpha, t'w_\alpha) = (s', t') w_\alpha = (u_\sigma, v_\sigma) h w_\alpha$$

and so $(u_\alpha, v_\alpha) \in (u_\sigma, v_\sigma) S$, a contradiction. Thus $R(s,t)$ is finitely generated.

A similar argument shows that if $s, t \in S$ and $r(s,t) \neq \varnothing$, then $r(s,t)$ is finitely generated.

(iv) $\Rightarrow$ (i) We show that $\mathscr{F}$ is axiomatisable by giving explicitly a set of sentences that axiomatises $\mathscr{F}$.

For any element ρ of $S \times S$ with $R(\rho) \neq \varnothing$, we choose and fix a finite set of generators $(u_{\rho 1}, v_{\rho 1}), \ldots, (u_{\rho, n(\rho)}, v_{\rho, n(\rho)})$ of $R(\rho)$. If $r(\rho) \neq \varnothing$, choose and fix a set of generators $w_{\rho 1}, \ldots, w_{\rho, m(\rho)}$ of $r(\rho)$. For $\rho \in S \times S$ where $\rho = (s,t)$, define sentences ϕ_ρ, ξ_ρ of L_S as follows:

if $R(\rho) = \varnothing$, ϕ_ρ is $(\forall x)(\forall y)(sx \neq ty)$,
if $R(\rho) \neq \varnothing$, ϕ_ρ is

$$(\forall x)(\forall y)\Big(sx = ty \rightarrow (\exists z)\Big(\bigvee_{i=1}^{n(\rho)} (x = u_{\rho,i} z \wedge y = v_{\rho,i} z)\Big)\Big),$$

if $r(\rho) = \varnothing$, ξ_ρ is $(\forall x)(sx \neq tx)$,
if $r(\rho) \neq \varnothing$, ξ_ρ is

$$(\forall x)\Big(sx = tx \rightarrow (\exists z)\Big(\bigvee_{i=1}^{m(\rho)} x = w_{\rho,i} z\Big)\Big).$$

Let $\Pi = \Sigma \cup \{\phi_\rho, \xi_\rho : \rho \in S \times S\}$. We claim that Π axiomatises $\mathscr{F}$.

Suppose first that A is a flat S-system and $\rho \in S \times S$, where $\rho = (s, t)$. If $R(\rho) = \varnothing$ and there exist $a, b \in A$ such that $sa = tb$, then since A is flat, $ss' = tt'$ for some $s', t' \in S$, a contradiction. Thus $A \models \phi_\rho$. If $R(\rho) \neq \varnothing$ and $sa = tb$ where $a, b \in A$, then again using the flatness of A, there are elements s', t' of S and c of A such that $ss' = tt'$, $a = s'c$, $b = t'c$. Now $(s', t') \in R(s, t)$ and so $(s', t') = (u_{\rho,i}, v_{\rho,i})h$ for some $i \in \{1, \ldots, n(\rho)\}$ and some $h \in S$. Thus $a = u_{\rho,i}hc$, $b = v_{\rho,i}hc$ and $hc \in A$. Hence $A \models \phi_\rho$. A similar argument gives that $A \models \xi_\rho$. Then clearly A is a model of Π.

Conversely, if A is a model of Π, then clearly A is an S-system. If $sa = tb$ where $s, t \in S$ and $a, b \in A$, then since $A \models \phi_\rho$, where $\rho = (s, t)$, it follows that $R(\rho)$ cannot be empty and ϕ_ρ is

$$(\forall x)(\forall y)\left(sx = ty \rightarrow (\exists z)\left(\bigvee_{i=1}^{n(\rho)} (x = u_{\rho,i} z \wedge y = v_{\rho,i} z)\right)\right).$$

Hence there is an element c of A with $a = u_{\rho,i}c$ and $b = v_{\rho,i}c$, for some $i \in \{1, \ldots, n(\rho)\}$. By definition of $u_{\rho,i}$, $v_{\rho,i}$, $su_{\rho,i} = tv_{\rho,i}$. In a similar manner one sees that if $sa = ta$ where $s, t \in S$ and $a \in A$, then $ss' = ts'$ and $a = s'c$ for some $s' \in S$ and $c \in A$. Thus A is flat and so Π axiomatises $\mathscr{F}$.

It is easy to construct examples of monoids not satisfying condition (iv) of Theorem 3.1. Let U be a cancellative semigroup with more than one element and V a completely simple semigroup with an infinite number of $\mathscr{R}$-classes. Let 0 be a symbol not occurring in U or V. Then $U \cup V \cup \{0\}$ is a semigroup under the multiplication $*$, where

$$x * y = \begin{cases} xy & \text{if } x, y \in U \text{ or } x, y \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Let S be the monoid obtained by adjoining an identity to $U \cup V \cup \{0\}$.

Let $s, t \in U$ where $s \neq t$. In S, $r(s, t) = V \cup \{0\}$ and it follows that $r(s, t)$ cannot be finitely generated.

4. *Axiomatisability of $\mathscr{P}$*

Eklof and Sabbagh show in [4] that for modules over a ring R, the class $\mathscr{P}$ is axiomatisable if and only if $\mathscr{F}$ is axiomatisable and R is perfect. For a ring to be perfect it is enough that is satisfies M_R [1]. This is a necessary condition for a monoid to be perfect, but as remarked earlier, it is not sufficient. In Proposition 4.3 we show that for a monoid S, if $\mathscr{P}$ is closed under ultraproducts, in particular if $\mathscr{P}$ is axiomatisable, then S satisfies M_R and a number of other conditions that in some special cases immediately give that S is perfect. Before stating the result we require some definitions.

We remind the reader that the relation $\mathscr{L}^*$ on a semigroup S is defined by the rule that for $a, b \in S$, $a\mathscr{L}^*b$ if and only if a, b are related by Green's relation $\mathscr{L}$ in some oversemigroup of S.

LEMMA 4.1 [6]. *Let S be a monoid and let $a, b \in S$. Then the following conditions are equivalent:*

(i) $a\mathscr{L}^*b$;

(ii) *for all $x, y \in S$, $ax = ay$ if and only if $bx = by$;*

(iii) *there is an S-isomorphism $\phi: aS \rightarrow bS$ with $\phi(a) = b$.*

Given a monoid S, we define a preorder $\leqslant_l$ by $a \leqslant_l b$ if and only if $bx = by$ implies that $ax = ay$, for any elements x, y of S. The equivalence relation associated with $\leqslant_l$ is clearly $\mathscr{L}^*$, that is, $a\mathscr{L}^*b$ if and only if $a \leqslant_l b$ and $b \leqslant_l a$. if $a, b \in S$ it is easy to see that if $Sa \subseteq Sb$ then $a \leqslant_l b$ and if b is regular then $Sa \subseteq Sb$ if and only if $a \leqslant_l b$. Thus if $a, b \in S$ are regular, then $a\mathscr{L}^*b$ if and only if $a\mathscr{L}b$.

We define a condition M^{L^*} on S by analogy with M^L, the ascending chain condition for principal left ideals. We say that S satisfies M^{L^*} if there are no strictly increasing chains of the form

$$a_1 \leqslant_l a_2 \leqslant_l \ldots,$$

where $a_1, a_2, \ldots$ are elements of S. Thus if S is regular, then S satisfies M^L if and only if S satisfies M^{L^*}.

We recall that a monoid S satisfies condition A if every S-system satisfies the ascending chain condition on cyclic S-subsystems. Clearly, if S satisfies A then S satisfies M^L. The converse is not true—it is shown in [8] that S may satisfy M^L and M_R but not A. We show below that if $\mathscr{P}$ is axiomatisable, then S satisfies A if and only if S satisfies M^L. To prove this we use an equivalent characterisation of A, given in [8].

LEMMA 4.2 [8]. *A monoid S satisfies A if and only if for any elements $a_1, a_2, \ldots$ of S, there exists $n \in \mathbb{N}$ such that for any $i \in \mathbb{N}$, $i \geqslant n$, there exists $j_i \in \mathbb{N}$, $j_i \geqslant i+1$, with*

$$Sa_i a_{i+1} \ldots a_{j_i} = Sa_{i+1} \ldots a_{j_i}.$$

PROPOSITION 4.3. *Let S be a monoid such that every ultrapower of S is projective. Then*

(i) *the class $\mathscr{F}$ is axiomatisable;*

(ii) *S satisfies M_R;*

(iii) *S satisfies M^{L^*};*

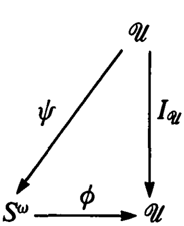
(iv) *S satisfies the ascending chain condition for idempotent generated principal left ideals;*

(v) *if S satisfies M^L, then S satisfies A.*

Proof. (i) Since every ultrapower of S is flat, this statement is immediate from Theorem 3.1.

(ii) If $a_1 S \supseteq b_2 S \supseteq b_3 S \supseteq \ldots$ is a decreasing sequence of principal right ideals of S, then $b_2 = a_1 a_2$, $b_3 = b_2 a_3 = a_1 a_2 a_3, \ldots$ for some elements $a_2, a_3, \ldots$ in S.

Let D be a non-principal ultrafilter over ω and put $\mathscr{U} = S^\omega/D$. By assumption, $\mathscr{U}$ is projective and so there is an S-homomorphism $\psi: \mathscr{U} \to S^\omega$ such that



is a commutative diagram, where $\phi: S^\omega \to \mathscr{U}$ is the canonical mapping.

Define elements $\bar{u}_i$ of $\mathscr{U}$, $i \in \mathbb{N}$, by

$$\bar{u}_i = (1, 1, \ldots, 1, a_i, a_i a_{i+1}, a_i a_{i+1} a_{i+2}, \ldots)_D,$$

where the entry a_i is the ith coordinate. Then for any $i, j \in \mathbb{N}$ with $i < j$ we have $\bar{u}_i = a_i \ldots a_{j-1} \bar{u}_j$.

For $i \in \mathbb{N}$ let $\psi(\bar{u}_i) = (c_i^1, c_i^2, \ldots)$. Then for any $i, j, k \in \mathbb{N}$ with $i < j$,

$$c_i^k = a_i a_{i+1} \ldots a_{j-1} c_j^k$$

and so $c_i^k S \subseteq a_i a_{i+1} \ldots a_{j-1} S$.

From $\bar{u}_1 = \phi\psi(\bar{u}_1)$ we have

$$(a_1, a_1 a_2, a_1 a_2 a_3, \ldots)_D = (c_1^1, c_1^2, c_1^3, \ldots)_D.$$

Thus there exists $n \in \mathbb{N}$ with $a_1 a_2 \ldots a_n = c_1^n$. Then for any $m \geqslant n$,

$$a_1 \ldots a_m S \subseteq a_1 \ldots a_n S = c_1^n S \subseteq a_1 \ldots a_m S.$$

Hence $b_n S = b_{n+1} S = \ldots$ and so S satisfies M_R.

(iii) If $a_1, a_2, \ldots$ are elements of S such that

$$a_1 \leqslant_l a_2 \leqslant_l \ldots,$$

then by definition of $\leqslant_l$, $R(a_1, a_1) \supseteq R(a_2, a_2) \supseteq \ldots$. Note that for any $i \in \mathbb{N}$, $R(a_i, a_i) \neq \varnothing$ as $(1, 1) \in R(a_i, a_i)$; thus by (i), $R(a_i, a_i)$ is finitely generated. It is an easy consequence of (ii) that any right S-system satisfies the descending chain condition for cyclic S-subsystems. Thus $S \times S$ satisfies the descending chain condition for cyclic S-subsystems and so by [5, Lemma 6], $S \times S$ satisfies the descending chain condition for finitely generated S-subsystems. It follows that for some n, $R(a_n, a_n) = R(a_{n+1}, a_{n+1}) = \ldots$ and so $a_n, a_{n+1}, \ldots$ are all $\mathscr{L}^*$-related.

(iv) If $Se_1 \subseteq Se_2 \subseteq \ldots$ is an ascending chain of principal left ideals of S, where $e_i^2 = e_i$ for each i, then clearly $e_1 \leqslant_l e_2 \leqslant_l \ldots$ and so there exists $n \in \mathbb{N}$ such that $e_n \mathscr{L}^* e_{n+1}$. Hence $e_n \mathscr{L} e_{n+1}$ and $Se_n = Se_{n+1}$.

(v) Suppose now that S satisfies M^L and $a_1, a_2, \ldots$ are elements of S. Define D, $\mathscr{U}$ and $\bar{u}_i$, $i \in \mathbb{N}$, as in the proof of (ii). For $i \in \mathbb{N}$, $\bar{u}_i = a_i \bar{u}_{i+1}$ and since $\mathscr{U}$ is projective, this gives that $\bar{u}_i = d_i \bar{v}$, $i \in \mathbb{N}$, for some elements d_i of S and some e-cancellable element $\bar{v}$ of $\mathscr{U}$, where e is idempotent. So for any $i \in \mathbb{N}$, $d_i \bar{v} = a_i d_{i+1} \bar{v}$, giving $d_i e = a_i d_{i+1} e$. Hence $Sd_1 e \subseteq Sd_2 e \subseteq \ldots$ and by assumption, $Sd_n e = Sd_{n+1} e = \ldots$ for some $n \in \mathbb{N}$ and it follows that $S\bar{u}_n = S\bar{u}_{n+1} = \ldots$. Now let $i \geqslant n$, so that $\bar{u}_{i+1} = c\bar{u}_i$ for some $c \in S$. Since D is non-principal there exists $j_i \in \mathbb{N}$ with $j_i \geqslant i+1$ such that $a_{i+1} \ldots a_{j_i} = c a_i a_{i+1} \ldots a_{j_i}$ and so

$$Sa_{i+1} \ldots a_{j_i} = Sa_i a_{i+1} \ldots a_{j_i},$$

that is, S satisfies condition A.

Corollary 4.4. *Let S satisfy M^L. Then $\mathscr{P}$ is axiomatisable if and only if $\mathscr{F}$ is axiomatisable and S is perfect.*

Proof. If $\mathscr{P}$ is axiomatisable then it is closed under ultraproducts so certainly every ultrapower of S is projective. By Proposition 4.3, $\mathscr{F}$ is axiomatisable, S satisfies M_R and since S satisfies M^L, S satisfies A. From Theorem 2.3, S is perfect. The converse is clear.

COROLLARY 4.5. *Suppose that the relations $\mathscr{L}^*$, $\mathscr{L}$ coincide for the monoid S. Then $\mathscr{P}$ is axiomatisable if and only if $\mathscr{F}$ is axiomatisable and S is perfect.*

Proof. We need only show that if $\mathscr{P}$ is axiomatisable, then S satisfies M^L.

If $\mathscr{P}$ is axiomatisable and $Sa_1 \subseteq Sa_2 \subseteq \dots$ is an ascending chain of principal left ideals of S, then certainly $a_1 \leqslant_l \leqslant a_2 \leqslant_l \dots$ and so from Proposition 4.3(iii), $a_n \mathscr{L}^* a_{n+1} \mathscr{L}^* \dots$ for some $n \in \mathbb{N}$. Thus $a_n \mathscr{L} a_{n+1} \mathscr{L} \dots$ and $Sa_n = Sa_{n+1} = \dots$.

COROLLARY 4.6. *If S is a regular monoid, then $\mathscr{P}$ is axiomatisable if and only if $\mathscr{F}$ is axiomatisable and S is perfect.*

For our next corollary we need a subsidiary lemma.

LEMMA 4.7. *If S is a left cancellative monoid and S satisfies M_R, then S is a group.*

Proof. For any $a \in S$ we have $aS \supseteq a^2S \supseteq \dots$ and so $a^n = a^{n+1}t$ for some $n \in \mathbb{N}$ and $t \in S$. Since S is left cancellative it follows that a is a unit. Hence S is a group.

COROLLARY 4.8. *If the monoid S is a semilattice of left cancellative monoids, then $\mathscr{P}$ is axiomatisable if and only if $\mathscr{F}$ is axiomatisable and S is perfect.*

Proof. We suppose that $\mathscr{P}$ is axiomatisable and that S is a semilattice Y of left cancellative monoids S_α, $\alpha \in Y$, with identities e_α. Note that e_α is the only idempotent of S_α.

If $\alpha \in Y$ and $a, b \in S_\alpha$, then $a \mathscr{L}^* b$. For if $x \in S_\beta$, $y \in S_\gamma$ and $ax = ay$, then $\alpha\beta = \alpha\gamma$ and so $(e_{\alpha\beta} a)(xe_{\alpha\beta}) = (e_{\alpha\beta} e)(ye_{\alpha\beta})$. Since $S_{\alpha\beta}$ is left cancellative, $xe_{\alpha\beta} = ye_{\alpha\beta}$ and so $bxe_{\alpha\beta} = bye_{\alpha\beta}$, giving $bx = by$.

We show that each S_α satisfies M_R. For if $\alpha \in Y$ and $a_1, a_2, \dots$ are elements of S_α such that $a_1 S_\alpha \supseteq a_2 S_\alpha \supseteq \dots$, then $a_1 S \supseteq a_2 S \supseteq \dots$. But by Proposition 4.3, S satisfies M_R and so $a_n S = a_{n+1} S = \dots$ for some $n \in \mathbb{N}$. Thus for $i \in \mathbb{N}$, $a_n = a_{n+i} h_i$ for some $h_i \in S$, say $h_i \in S_{\beta_i}$. Then for each $i \in \mathbb{N}$, $\alpha \leqslant \beta_i$, which gives that $h_i e_\alpha \in S_\alpha$ and $a_n = a_{n+i}(h_i e_\alpha)$. Hence $a_n S_\alpha = a_{n+1} S_\alpha = \dots$. Lemma 4.7 now gives that S is regular and so the result follows from Corollary 4.6.

COROLLARY 4.9. *If S is a commutative monoid, then $\mathscr{P}$ is axiomatisable if and only if $\mathscr{F}$ is axiomatisable and S is perfect.*

Proof. If $\mathscr{P}$ is axiomatisable, then S satisfies M_R and M^{L^*}. Let

$$Sa_1 \subseteq Sa_2 \subseteq \dots$$

be an ascending chain of principal left ideals. Then

$$a_1 \leqslant_l a_2 \leqslant_l \dots$$

and so $a_n \mathscr{L}^* a_{n+1} \mathscr{L}^* \dots$ for some $n \in \mathbb{N}$. Put $b_i = a_{n+i-1}$, $i \in \mathbb{N}$, and for each $i \in \mathbb{N}$ suppose that $b_i = t_i b_{i+1}$. Now $t_1 S \supseteq t_1 t_2 S \supseteq \dots$ and so $t_1 \dots t_m S = t_1 \dots t_{m+1} S = \dots$ for some $m \in \mathbb{N}$. Thus for any $i \in \mathbb{N}$ there is $h_i \in S$ with $t_1 \dots t_{m+i} h_i = t_1 \dots t_m$. Hence $t_1 \dots t_{m+i} h_i b_{m+1} = t_1 \dots t_m b_{m+1}$ and so

$$t_{m+1} \dots t_{m+i} h_i t_1 \dots t_m b_{m+1} = t_1 \dots t_m b_{m+1},$$

that is, $t_{m+1} \ldots t_{m+i} h_i b_1 = b_1$. Now $b_1 \mathscr{L}^* b_{m+i+1}$ and S is commutative, so

$$t_{m+1} \ldots t_{m+i} h_i b_{m+i+1} = b_{m+i+1},$$

giving $b_{m+i+1} = h_i t_{m+1} \ldots t_{m+i} b_{m+i+1} = h_i b_{m+1} \in Sb_{m+1}$. Thus $Sb_{m+1} = Sb_{m+2} = \ldots$ and S has M^L. Corollary 4.4 gives the result.

5. *Examples*

We begin by remarking that for any finite monoid $\mathscr{F}$ and $\mathscr{P}$ are axiomatisable classes.

For any group both $\mathscr{F}$ and $\mathscr{P}$ are axiomatisable. This is easy to see, since if G is a group then G is perfect so that $\mathscr{F}$ is axiomatisable if and only if $\mathscr{P}$ is axiomatisable. Let $s, t \in G$. Then $r(s,t) = \varnothing$ or $r(s,t) = G$. For any $u, v \in G$, $su = tv$ if and only if $u = s^{-1}tv$, which is so if and only if $(u,v) \in (s^{-1}t, 1)G$. Thus $R(s,t)$ is cyclic. By Theorem 3.1, $\mathscr{F}$ is an axiomatisable class.

An example of a monoid for which $\mathscr{F}$ is axiomatisable but $\mathscr{P}$ is not is provided by an ω-chain C where $C = \{1 = e_0, e_1, e_2, \ldots\}$ and $e_i e_j = e_{\max\{i,j\}}$.

Suppose that C is as above and let $e_i, e_j \in C$. If $i = j$ then $r(e_i, e_j) = C$. If $i < j$ then $e_i e_k = e_j e_k$ if and only if $k \geqslant j$. Thus $r(e_i, e_j) = e_j C$. Considering $R(e_i, e_j)$, if $i = j$ then put $R = \bigcup \{(e_k, e_l)C : k, l \leqslant i\}$. It is clear that $R \subseteq R(e_i, e_i)$. Conversely, if $e_i e_u = e_i e_v$ then $\max\{i,u\} = \max\{i,v\}$ so that $i \geqslant u$ if and only if $i \geqslant v$. If $i \geqslant u$ then $(e_u, e_v) \in R$. If $i < u$, then $e_i e_u = e_u$ and it follows that $v = u$ and $(e_u, e_v) = (e_i, e_i) e_u \in R$. Thus $R(e_i, e_i) \subseteq R$ and so $R(e_i, e_i)$ is finitely generated. If $i < j$ we claim that $R(e_i, e_j) = (e_j, 1)C$. For $e_i e_j e_k = e_j e_k$ for all $e_k \in C$, so that $(e_j, 1)C \subseteq R(e_i, e_j)$. Conversely, suppose that $e_i e_k = e_j e_h = e_t$. Then $t = \max\{i,k\} = \max\{j,h\}$. If $t = h$ then also $k = h$ and

$$(e_k, e_h) = (e_h, e_h) = (e_j, 1) e_h \in (e_j, 1) C.$$

If $t = j$ then $k = j$ and

$$(e_k, e_h) = (e_j, e_h) = (e_j, 1) e_h \in (e_j, 1) C.$$

Thus $R(e_i, e_j) = (e_j, 1) C$.

We have shown that $\mathscr{F}$ is axiomatisable. Since S satisfies M^L, Corollary 4.4 applies. But C is not perfect as it does not satisfy M_R and so $\mathscr{P}$ is not axiomatisable.

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