

Variants of semigroups - the case study of finite full transformation monoids

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Prime suspects



Mr. Shady Corleone



Violet Moon
(special undercover agent)

Now seriously... co-authors



I.D.



James East
(*U. of Western Sydney*)

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where θ is a fixed function $Y \rightarrow X$. For $Y = X$, this is exactly a variant of \mathcal{T}_X .

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A more accessible account of her results may be found in the monograph of **Ganyushkin & Mazorchuk** *Classical Finite Transformation Semigroups* (Springer, 2009).

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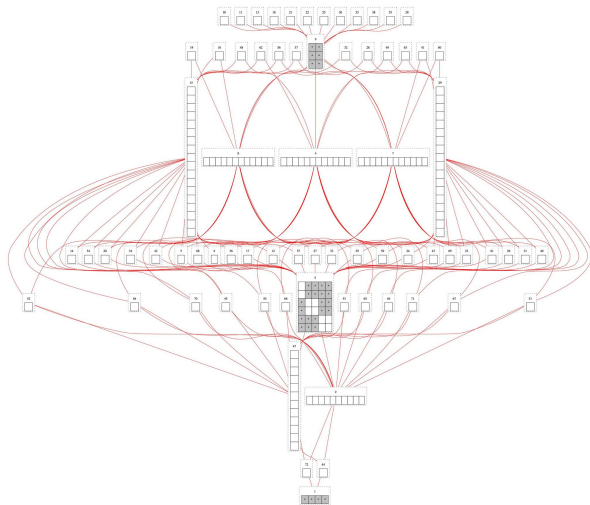
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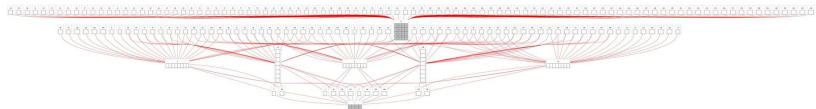
A WORD OF CAUTION: If S is a regular semigroup, S^a is **not regular** in general! However, for regular S and arbitrary $a \in S$, $\text{Reg}(S^a)$ is always a subsemigroup of S^a (Khan & Lawson).

A word of caution, you said...?



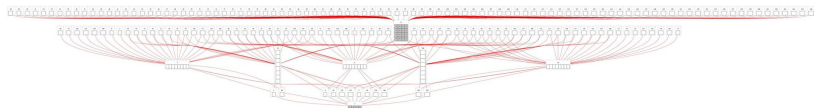
Egg-box picture of \mathcal{T}_4^a for $a = [1, 2, 3, 3]$

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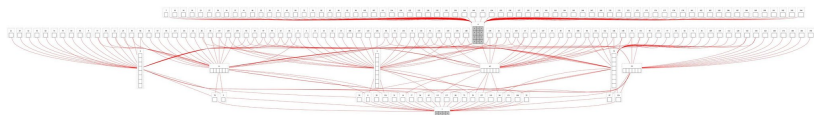


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Egg-box picture of \mathcal{T}_4^a for $a = [1, 1, 1, 4]$

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Green's relations: $\mathcal{R}^a, \mathcal{L}^a, \mathcal{H}^a, \mathcal{D}^a$

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Group \mathcal{H} -classes vs group \mathcal{H}^a -classes (in P)

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x	Is H_x a group \mathcal{H} -class of \mathcal{T}_4 ?	Is H_x a group \mathcal{H}^a -class of \mathcal{T}_4^a ?
$[1, 1, 3, 3]$	Yes	Yes
$[4, 2, 2, 4]$	Yes	No
$[2, 4, 2, 4]$	No	Yes
$[1, 3, 1, 3]$	No	No

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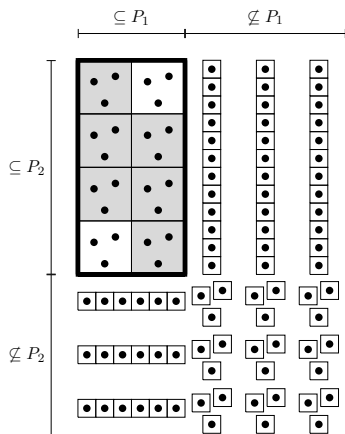
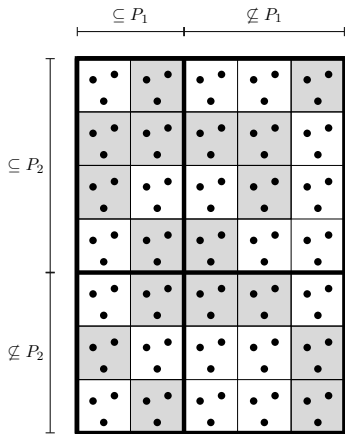
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- ▶ Since all constant maps trivially belong to P , D_1 is preserved, and remains a right zero band.
- ▶ For $2 \leq m \leq r$, the class D_r separates into a single regular chunk $D_r \cap P$ and a number of non-regular pieces, as seen on the following picture...

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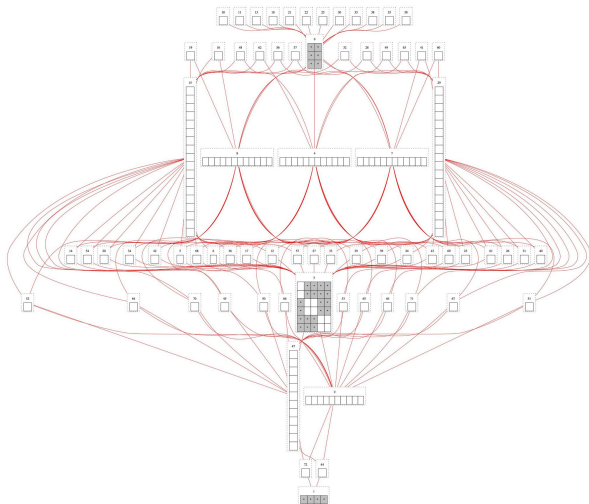
Order of the \mathcal{D}^a -classes

Let $f, g \in \mathcal{T}_X$. Then $D_f^a \leq D_g^a$ in \mathcal{T}_X^a if and only if one of the following holds:

- ▶ $f = g$,
- ▶ $\text{rank}(f) \leq \text{rank}(aga)$,
- ▶ $\text{im}(f) \subseteq \text{im}(ag)$,
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The maximal \mathcal{D}^a -classes are those of the form $D_f^a = \{f\}$ where $\text{rank}(f) > r$.

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Consequently, M is the unique minimal (with respect to containment or size) generating set of \mathcal{T}_X^a , and

$$\text{rank}(\mathcal{T}_X^a) = |M| = \sum_{m=r+1}^n S(n, m) \binom{n}{m} m!,$$

where $S(n, m)$ denotes the Stirling number of the second kind.

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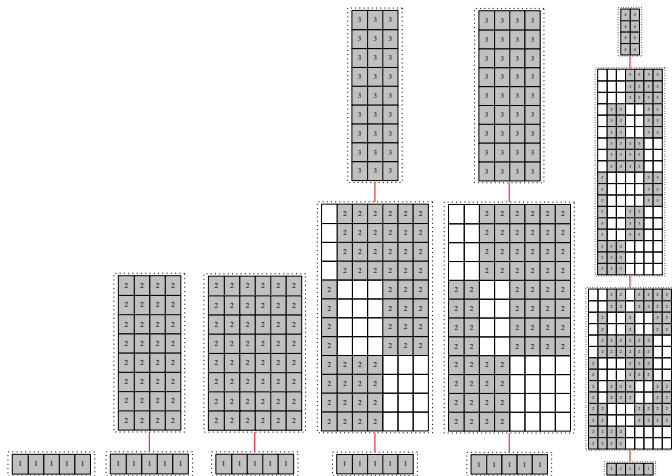
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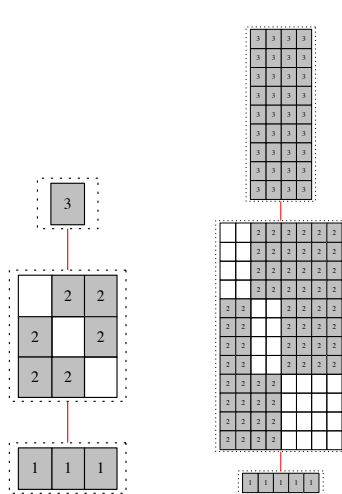
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- ▶ The 'crown': A maximal \mathcal{D}^a -class $D_f^a = \{f\}$ sits above D_r^a if and only if $\text{rank}(afa) = r$. The number of such \mathcal{D}^a -classes is equal to $(n^{n-r} - r^{n-r})r!\Lambda$.

Reg(\mathcal{T}_X^a) – examples



Egg-box diagrams of the regular subsemigroups $P = \text{Reg}(\mathcal{T}_5^a)$ in the cases
 (from left to right): $a = [1, 1, 1, 1, 1]$, $a = [1, 2, 2, 2, 2]$, $a = [1, 1, 2, 2, 2]$,
 $a = [1, 2, 3, 3, 3]$, $a = [1, 2, 2, 3, 3]$, $a = [1, 2, 3, 4, 4]$.

Do you see what I am seeing???



Egg-box diagrams of \mathcal{T}_3 (left) and $\text{Reg}(\mathcal{T}_5^a)$ for $a = [1, 2, 2, 3, 3]$ (right).

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Fact:

$$\text{Reg}(\mathcal{T}(X, A)) = \mathcal{T}(X, A) \cap P_2$$

$$\text{Reg}(\mathcal{T}(X, \alpha)) = \mathcal{T}(X, \alpha) \cap P_1$$

Structure Theorem – Part 1

$$\psi : f \mapsto (fa, af)$$

is a well-defined embedding of $\text{Reg}(\mathcal{T}_X^a)$ into the direct product $\text{Reg}(\mathcal{T}(X, A)) \times \text{Reg}(\mathcal{T}(X, \alpha))$.

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Thus $\text{Reg}(\mathcal{T}_X^a)$ is a subdirect product of $\text{Reg}(\mathcal{T}_X^a)$ and $\text{Reg}(\mathcal{T}(X, \alpha))$.

Structure Theorem – Part 2

The maps

$$\phi_1 : \text{Reg}(\mathcal{T}(X, A)) \rightarrow \mathcal{T}_A : g \mapsto g|_A$$

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Further, the induced map $\phi = \psi_1\phi_1 = \psi_2\phi_2 = \text{Reg}(\mathcal{T}_X^a) \rightarrow \mathcal{T}_A$ is an epimorphism that is **'group / non-group preserving'**.

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$$\text{rank}(P) = \text{rank}(D) + \text{rank}(P : D) = r^{n-r} + 1$$

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- ▶ We obtain a pleasing generalisation of celebrated Howie's Theorem:

$$\mathcal{E}_X^a = \langle E_a(\mathcal{T}_X^a) \rangle_a = E_a(D) \cup (P \setminus D).$$

The idempotent generated subsemigroup $\langle E_a(\mathcal{T}_X^a) \rangle_a$



$$\text{rank}(\mathcal{E}_X^a) = \text{idrank}(\mathcal{E}_X^a) = r^{n-r} + \rho_r,$$

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- ▶ The number of idempotent generating sets of \mathcal{E}_X^a of the minimal possible size is

$$[(r-1)^{n-r} \Lambda]^{\rho_r} \Lambda! S(r^{n-r}, \Lambda) \sum_{\Gamma \in \mathbb{T}_r} \frac{1}{\lambda_1^{d_\Gamma^+(1)} \cdots \lambda_r^{d_\Gamma^+(r)}}.$$

where \mathbb{T}_r is the set of all strongly connected tournaments on r vertices.

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$$\text{rank}(I_m^a) = \text{idrank}(I_m^a) = \begin{cases} m^{n-r} S(r, m) & \text{if } 1 < m < r \\ n & \text{if } m = 1. \end{cases}$$

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These **sandwich semigroups** generalise the variants.

- ▶ applicable to functions, matrices, diagrams, . . .

THANK YOU!

Questions and comments to:

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Further information may be found at:

<http://people.dmu.ac.uk/~dockie>