The word problem for free idempotent-generated semigroups: an overview and elaboration for $\mathcal{T}_n$

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Idempotents in a semigroup

Question

How to record (without dealing with the entire semigroup) sufficient information about the structure of idempotents in a semigroup?

Answer (Nambooripad, 1980s): Biordered sets!

Biordered set (of S) $=$ partial algebra $\mathcal{E}_S = (E(S), \cdot)$ obtained by retaining products of basic pairs (e, f):

$$\{ef, fe\} \cap \{e, f\} \neq \varnothing.$$

Induced quasi-orders:

$$e \leq_\ell f \text{ if and only if } ef = e, \qquad e \leq_r f \text{ if and only if } ef = f,$$

$$\leq = \leq_\ell \cap \leq_r \text{ – this is the usual Rees order.}$$

$\mathrm{IG}(\mathcal{E})$

Nambooripad, Easdown (1980s): Biordered sets of semigroups have a finite axiomatisation. Thus we can speak about abstract biordered sets.

Also: There is a largest / free-est / most general idempotent-generated semigroup with a prescribed biordered set $\mathcal{E}$.

This is the free idempotent-generated semigroup over $\mathcal{E}$:

$$\mathrm{IG}(\mathcal{E}) = \langle \overline{E} \mid \overline{e}\overline{f} = \overline{e \cdot f} \text{ whenever } \{e, f\} \text{ is a basic pair in } \mathcal{E} \rangle.$$

$$\begin{tikzcd}
\overline{\mathcal{E}} \arrow[r, "\cong"] \arrow[d, "\subseteq"'] & \mathcal{E} \arrow[r, "\phi"] & \mathcal{E}_S \arrow[r, "\subseteq"] & S \\
\mathrm{IG}(\mathcal{E}) \arrow[urrr, "\Psi_\phi"'] & & &
\end{tikzcd}$$

Basic properties of $\mathsf{IG}(\mathcal{E})$

Assume we have fixed a homomorphism $\Psi : \mathsf{IG}(\mathcal{E}) \to S$ extending the map $\overline{e} \mapsto e$, $e \in E(S)$.

(IG1) For any $e \in E$, Ψ maps the $\mathscr{D}$-class of $\overline{e}$ in $\mathsf{IG}(\mathcal{E})$ precisely onto the $\mathscr{D}$-class of e in $S' = \langle E(S) \rangle$.

(IG2) In fact, Ψ maps the $\mathscr{R}$-class of $\overline{e}$ onto the $\mathscr{R}$-class of e, and the $\mathscr{L}$-class of $\overline{e}$ onto the $\mathscr{L}$-class of e.

(IG3) Hence, the restriction of Ψ to $H_{\overline{e}}$ in $\mathsf{IG}(\mathcal{E})$ is a surjective group homomorphism onto H_e in S'.

This third property was (partially) responsible for spawning

Conjecture (Folklore, 80s)

Maximal subgroups of free idempotent-generated semigroups must always be free.

(Spectacular) failure of the freeness conjecture

Brittenham, Margolis, Meakin (2009): A 73-element semigroup S generated by its 37 idempotents (arising from a combinatorial design) such that $\mathrm{IG}(\mathcal{E}_S)$ contains $\mathbb{Z} \times \mathbb{Z}$ as a subgroup

Gray, Ruškuc (2012): Quite the opposite of the conjecture is true – for any group G there is a suitable semigroup S such that G arises as a maximal subgroup in $\mathrm{IG}(\mathcal{E}_S)$

IgD, Ruškuc (2013): For finitely presented G, (the biorder of) a finite band S will do

Computing the maximal subgroups (1)

Brittenham, Margolis, Meakin (2009): The maximal subgroup $H_{\overline{e}}$ in $\mathrm{IG}(\mathcal{E})$ = the fundamental group of the GH-complex of the $\mathscr{D}$-class $D = D_e$ in $S' = \langle E \rangle$:

Vertices: The $\mathscr{R}$- and the $\mathscr{L}$-classes in D

Edges: (R, L) such that $R \cap L$ contains an idempotent (so edges correspond to idempotents in D)

2-cells: singular squares = 4-cycles $e \mathrel{\mathscr{R}} e' \mathrel{\mathscr{L}} f' \mathrel{\mathscr{R}} f \mathrel{\mathscr{L}} e$ such that $(\exists h \in E)$ with

- either $eh = e'$, $fh = f'$, $he = e$, $hf = f$ ("left-right"), or
- $he = f$, $he' = f'$, $eh = e$, $fh = f$. ("up-down").

Gray, Ruškuc (2012): A presentation for the group $H_{\overline{e}}$ via the Reidemester-Schreier theory for substructures of monoids
☞ turns out to be a specific instance of the above for a particular spanning tree of the GH-complex

Computing the maximal subgroups (2)

S	max. subgroups	who & when
$\mathbb{T}_n$	$\mathbb{S}_r$ $r \leq n-2$	Gray, Ruškuc (2012, PLMS)
$\mathbb{PT}_n$	$\mathbb{S}_r$ $r \leq n-2$	IgD (2013, Comm. Alg.)
$\mathcal{M}_n(\mathbb{F})$	$\mathrm{GL}_r(\mathbb{F})$ $r < n/3$	IgD, Gray (2014, TrAMS)
$\mathrm{End}(F_n(G))$	$G \wr \mathbb{S}_r$ $r \leq n-2$	Yang, IgD, Gould (2015, J. Algebra)

A first stab at the WP for IG($\mathcal{E}$)

IgD, Gray, Ruškuc (2017):

- There is an algorithm which, given $w \in E^+$ recognises whether w represents a regular element of IG($\mathcal{E}$).
- Given $u, v \in E^+$ representing regular elements of IG($\mathcal{E}$), the question whether $u = v$ entirely boils down to the WP for the maximal subgroups.
- There is a finite (20-element) band S such that all max. subgroups of IG($\mathcal{E}_S$) are either trivial or products of two free groups (so they have decidable WP), and yet the WP is undecidable (by using the Mikhailova construction).

So, what is the WP for IG($\mathcal{E}$) really all about?
☞ Yang, IgD, Gould (2019, Adv. Math.)
& IgD (2021, Israel J. Math.)

Words representing regular elements

Assume that $\mathbf{w} \in E^+$ represents a regular element $\overline{\mathbf{w}}$ of $\mathsf{IG}(\mathcal{E})$. (By [DGR 17] this can be algorithmically tested.) Then it can be "coordinatised" within its (regular) $\mathscr{D}$-class D as

$$(i, g, \lambda),$$

where i, λ record the $\mathscr{R}$- and the $\mathscr{L}$-class of $\overline{\mathbf{w}}$, and g is a (group) word in the generators of the maximal subgroup in D.

- [YDG 19]: There is an algorithm for computing $\mathbf{w} \to (i, g, \lambda)$.

General situation

In general, for $\mathbf{w} \in E^+$, the element $\overline{\mathbf{w}} \in \mathrm{IG}(\mathcal{E})$ need to to be regular. However, then we can consider the notion of a

- minimal r-factorisation: A coarsest factorisation

$$\mathbf{w} = \mathbf{w}_1 \dots \mathbf{w}_k$$

  into pieces representing regular elements

Theorem (Yang, IgD, Gould, 2019)

Assume $\overline{\mathbf{u}} = \overline{\mathbf{v}}$ holds in $\mathrm{IG}(\mathcal{E})$, and that $\mathbf{u} = \mathbf{u}_1 \dots \mathbf{u}_k$ and $\mathbf{v} = \mathbf{v}_1 \dots \mathbf{v}_r$ are minimal r-factorisations. Then $k = r$ and we have

- $\overline{\mathbf{u}_i} \mathscr{D} \overline{\mathbf{v}_i}$ *for all $1 \leq i \leq k$, and, furthermore*
- $\overline{\mathbf{u}_1} \mathscr{R} \overline{\mathbf{v}_1}$ *and* $\overline{\mathbf{u}_k} \mathscr{L} \overline{\mathbf{v}_k}$.

So, we have an invariant: $\overline{\mathbf{w}} \to \mathscr{D}$-fingerprint $(D_1, \dots, D_k)$ of $\overline{\mathbf{w}}$

The moral of the story

The WP for $\mathrm{IG}(\mathcal{E})$ (for finite $\mathcal{E}$) comes down to comparing elements of the form

$$(i_1, g_1, \lambda_1)(i_2, g_2, \lambda_2) \dots (i_k, g_k, \lambda_k)$$

of a given $\mathscr{D}$-fingerprint $(D_1, \dots, D_k)$.

Aaand... Action!

Let D be a regular $\mathscr{D}$-class of $\mathrm{IG}(\mathcal{E})$, with index sets I, Λ and maximal subgroup G. Then the idempotents from $\overline{E}$ exercise partial left and right actions on I and Λ respectively:

$$\overline{e} \cdot i = i' \quad \text{if and only if} \quad \overline{e}(i, g, \lambda) = (i', b_{\overline{e},i,i'} g, \lambda)$$
$$\lambda \cdot \overline{e} = \lambda' \quad \text{if and only if} \quad (i, g, \lambda)\overline{e} = (i, g a_{\overline{e},\lambda,\lambda'}, \lambda')$$

(The coefficients a, b depend solely on the displayed indices, and are easily expressed in terms of the generators of G.)

Contact graphs $\mathcal{A}(D_1, D_2)$

D_p $(p = 1, 2)$ – regular D-classes with index sets I_p, Λ_p & max. subgroups G_p.

Vertices: $\Lambda_1 \times I_2$

Edges: $(\lambda, i) \longrightarrow (\mu, j)$ such that $\lambda = \mu \cdot \overline{e}$ and $\overline{e} \cdot i = j$

Group labels: $(a, b^{-1}) \in G_1 \times G_2$ where $a = a_{\overline{e},\lambda,\mu}$ and $b = b_{\overline{e},i,j}$

Label of a walk: the product of edges along the walk (and edges can be travesed backwards, when we take the inverse of the label)

Vertex group $W_{(\lambda,i)}$: the subgroup of $G_1 \times G_2$ consisting of the labels of all closed walks based at (λ, i)

It's all about new relations

Assume we have the following data:

- groups $G_1, \ldots, G_m$ $(m \geq 2)$,
- relations $\rho_k \subseteq G_k \times G_{k+1}$ $(\leq k < m)$,
- elements $a_k, b_k \in G_k$ $(1 \leq k \leq m)$.

From these, we construct a new relation $\rho \subseteq G_1 \times G_m$ by defining: $(g, h) \in \rho$ if and only if $\exists x_r \in G_r$ $(2 \leq r \leq m)$ such that

$$\begin{aligned} (a_1^{-1} g b_1, x_2) &\in \rho_1, \\ (a_2^{-1} x_2 b_2, x_3) &\in \rho_2, \\ &\vdots \\ (a_{m-1}^{-1} x_{m-1} b_{m-1}, x_m) &\in \rho_{m-1}, \\ a_m^{-1} x_m b_m &= h. \end{aligned}$$

Clearly, ρ induces a map $\varphi_\rho : \mathcal{P}(G_1) \to \mathcal{P}(G_m)$.

The map θ

Now let

$$\begin{aligned} \mathbf{x} &= (i_1, a_1, \lambda_1) \dots (i_m, a_m, \lambda_m) \\ \mathbf{y} &= (j_1, b_1, \mu_1) \dots (j_m, b_m, \mu_m) \end{aligned}$$

be two elements of $\mathrm{IG}(\mathcal{E})$ of $\mathscr{D}$-fingerprint $(D_1, \dots, D_m)$.
Let G_k be the max. subgroup in D_k $(1 \leq k \leq m)$ and

$$\rho_k = \begin{cases} W_{(\lambda_k, i_{k+1})}(g_k, h_k) & \text{if } \exists \text{ a walk } (\lambda_k, i_{k+1}) \rightsquigarrow (\mu_k, j_{k+1}), \\ \varnothing & \text{otherwise.} \end{cases}$$

where $W_{(\lambda_k, i_{k+1})}$ is the vertex group of $\mathcal{A}(D_k, D_{k+1})$ at (λ_k, i_{k+1}), and (g_k, h_k) is the label of any walk $(\lambda_k, i_{k+1}) \rightsquigarrow (\mu_k, j_{k+1})$.

Then the associated mapping φ_ρ is denoted $(\cdot, \mathbf{x}, \mathbf{y})\theta$.
It can be calculated in terms of standard computational tasks within group theory.

The WP for $\mathrm{IG}(\mathcal{E})$ ($\mathcal{E}$ finite)

Theorem (IgD, 2021)

$\mathbf{x} = \mathbf{y}$ *holds in* $\mathrm{IG}(\mathcal{E})$ *if and only if* $i_1 = j_1$, $\lambda_m = \mu_m$, *and*

$$1 \in (\{1\}, \mathbf{x}, \mathbf{y})\theta.$$

($m = 2$: the membership problem for a certain subgroup of $G_1 \times G_2$)

Theorem

Let $\mathbf{x}, \mathbf{y} \in \mathrm{IG}(\mathcal{E})$. If these elements are not of the same $\mathscr{D}$-fingerprint, they cannot be $\mathscr{J}$-related. Otherwise, if they are, we have:

(i) $\mathbf{x} \mathscr{R} \mathbf{y}$ *if and only if* $i_1 = j_1$ *and* $(\{1\}, \mathbf{x}, \mathbf{y})\theta \neq \varnothing$;

(ii) $\mathbf{x} \mathscr{L} \mathbf{y}$ *if and only if* $\lambda_m = \mu_m$ *and* $1 \in (G_1, \mathbf{x}, \mathbf{y})\theta$;

(iii) $\mathbf{x} \mathscr{D} \mathbf{y}$ *if and only if* $(G_1, \mathbf{x}, \mathbf{y})\theta \neq \varnothing$.

Also, $\mathscr{D} = \mathscr{J}$ + Sch-group of $\mathbf{x} \cong (G_1, \mathbf{x}, \mathbf{x})\theta / (\{1\}, \mathbf{x}, \mathbf{x})\theta$.

$\mathrm{IG}(\mathcal{E}_{\mathcal{T}_n})$ facts

- $\langle E(\mathcal{T}_n)\rangle = (\mathcal{T}_n \setminus \mathbb{S}_n) \cup \{\mathrm{id}_n\}$
- $\mathscr{D}$-classes form a chain $D_n, D_{n-1}, \ldots D_1$ (classified by rank)
- maximal subgroup in $\overline{D}_m$ (in $\mathrm{IG}(\mathcal{E}_{\mathcal{T}_n})$) is
  - $m = n$: trivial
  - $m = n - 1$: free of rank $\binom{n}{2} - 1$
  - $m \leq n - 2$: $\mathbb{S}_m$
- a typical element of $\overline{D}_m$ is of the form

$$(P, g, A)$$

P – a partition of $[1, n]$ into m classes; A – a subset of $[1, n]$ of size m; g – an element of the max. subgroup (see above)

Contact graph $\mathcal{A}(\overline{D}_m, \overline{D}_r)$ in $\mathrm{IG}(\mathcal{E}_{\mathcal{T}_n})$

Vertices: Subset-partition pairs (A, P)

Edges: $(A, P) \longrightarrow (B, Q)$ labelled by e exists iff

- $\ker e$ separates B (with $A = Be$), and
- $\operatorname{im} e$ saturates P (with the classes of Q being unions of $(\ker e)$-classes mapping into the same P-class)

$\mathcal{P}$ separates X = every $\mathcal{P}$-class contains max 1 element of X

X saturates $\mathcal{P}$ = every $\mathcal{P}$-class contains at least 1 element of X

Lemma

For $(P, g, A) \in \overline{D}_m$ and $(P', g', A') \in \overline{D}_r$ the product $(P, g, A)(P', g', A')$ is regular if and only if either (1) $m \geq r$ and A saturates P', or (2) $m \leq r$ and P' separates A.

So, such pairs (A, P') are regular (= uninteresting).

Connected components in $\mathcal{A}(\overline{D}_m, \overline{D}_r)$

The type of (A, P) ($|A| = m$, $|P| = r$): the sequence

$$|A \cap P_1|, \ldots, |A \cap P_r|$$

sorted in a non-increasing order.

Example

$n = 9$, $A = \{1, 3, 5, 7\}$, $P = \{\{1, 2, 6\}, \{3, 5, 7, 9\}, \{4, 8\}\}$.
The type of (A, P) is $(3, 1, 0)$.

When (A, P) and (B, Q) are of the same type, we say they are homeomorphic.

Homeomorphism $(\phi, \psi) : (A, P) \sim (B, Q)$ – a pair of bijections $\phi : A \to B$, $\psi : P \to Q$ such that

$$a_i \in P_j \quad \text{if and only if} \quad a_i\phi \in P_j\psi.$$

Connected components in $\mathcal{A}(\overline{D}_m, \overline{D}_r)$ (cont'd)

(A, P) is stationary if all P-classes containing elements from $[1, n] \setminus A$ are singletons.

Proposition

(A, P) and (B, Q) are connected in $\mathcal{A}(\overline{D}_m, \overline{D}_r)$ iff they are homeomorphic and not stationary.

Remark

Stationary pairs are always isolated vertices.

The degenerate case

Proposition

If $m = n - 1$ or $r = n - 1$ then (A, P) is non-regular in $\mathcal{A}(\overline{D}_m, \overline{D}_r)$ iff it is stationary. $\Longrightarrow$ *The vertex group $W_{(A,P)}$ is trivial.*

Other vertex groups? We don't know. (But also we don't care.)

So, in the rest of the talk assume that $m, r \leq n - 2$.

Group labels of edges

Proposition

Assume there is an edge $(A,P) \longrightarrow (B,Q)$ *in* $\mathcal{A}(\overline{D}_m, \overline{D}_r)$ *induced by* $e \in E$. *Assume, further, that* $A = \{a_1 < \cdots < a_m\}$, $B = \{b_1 < \cdots < b_m\}$, *and that the classes of* P, Q *are enumerated such that* $\min P_1 < \cdots < \min P_r$ *and* $\min Q_1 < \cdots < \min Q_r$. *Then the considered edge is labelled by* $(\pi, \pi') \in \mathbb{S}_m \times \mathbb{S}_r$ *such that*

- $b_{i\pi} e = a_i$ *for all* $1 \leq i \leq m$,
- $P_j e^{-1} = Q_{j\pi'}$ *for all* $1 \leq j \leq r$.

Corollary

The label of every walk $(A,P) \rightsquigarrow (B,Q)$ *is a homeomorphism of its endpoints. In particular, the label of every loop based at* (A,P) *is an auto-homeomorphism of* (A,P).

The main result

Theorem (IgD, 2022)

Let (A, P) be a vertex in $\mathcal{A}(\overline{D}_m, \overline{D}_r)$.

- *If it is stationary, $W_{(A,P)}$ is trivial.*
- *Otherwise, $W_{(A,P)} = \mathrm{AHom}(A, P)$.*

Remarks

- The first component π of an element of $\mathrm{AHom}(A, P)$ is just any $P|_A$-preserving permutation of A.
- There is an easy description whether $(\pi, \pi') \in \mathrm{AHom}(A, P)$.
- Each non-empty "vertical slice" of $\mathrm{AHom}(A, P)$ is a coset of a symmetric group (permuting the P-classes not intersecting A).
- This leads to a nice generating set of $\mathrm{AHom}(A, P)$ within $\mathbb{S}_m \times \mathbb{S}_r$.

Conclusion

Now, all elements are "in place" so that one can, in a more-less straightforward manner, write a GAP code solving the WP for $\mathrm{IG}(\mathcal{E}_{\mathcal{T}_n})$.

Namely, for the "coset representatives" (g_k, h_k) in the WP it suffices to take any homeomorphism $(A_k, P_{k+1}) \sim (B_k, Q_{k+1})$.

Thank you!

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