The Brauer Project: Enumeration of idempotents in partition monoids

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York Semigroup
26 June 2014

Happy birthday, Igor!

Outline

1. Transformation semigroups
2. Partition monoids
3. Brauer monoids
4. Partition monoids
5. Diagram algebras

1. Transformation Semigroups

Let

- n be a positive integer
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- What are the idempotents and what can we do with them?

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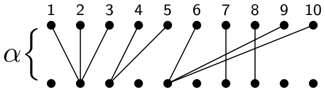

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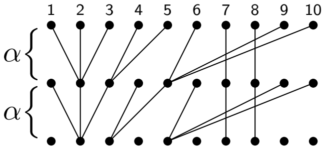

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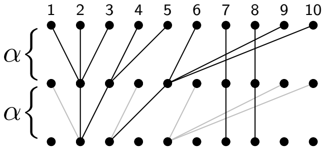

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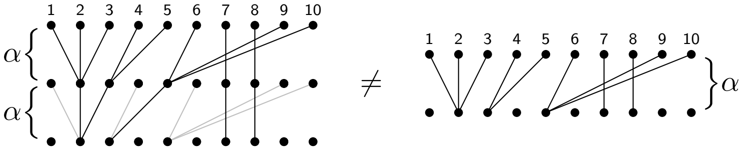

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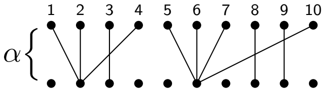

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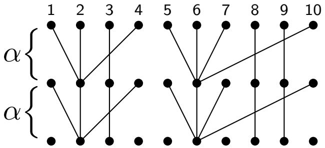

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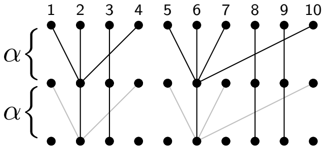

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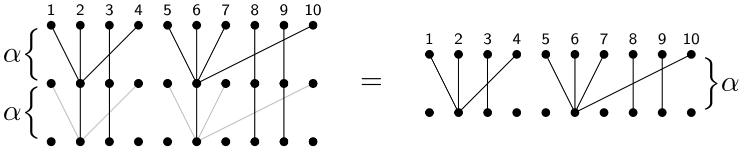

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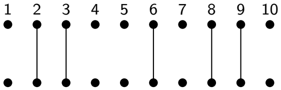

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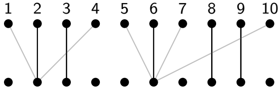

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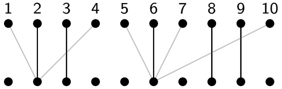

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- Sum over k.

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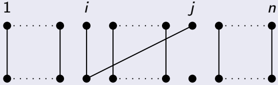
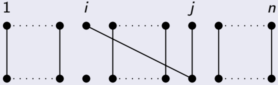
- $\mathcal{T}_n \setminus \mathcal{S}_n = \langle e_{ij}, e_{ji} : 1 \leq i < j \leq n\rangle$.

$e_{ij} =$ 

$e_{ji} =$ 

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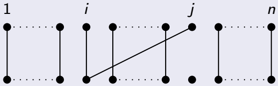
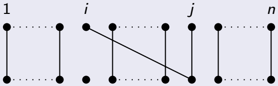
$e_{ij} =$ 

$e_{ji} =$ 

Theorem (Howie, 1978)

- $\text{rank}(\mathcal{T}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{T}_n \setminus \mathcal{S}_n) = \binom{n}{2} = \frac{n(n-1)}{2}$.

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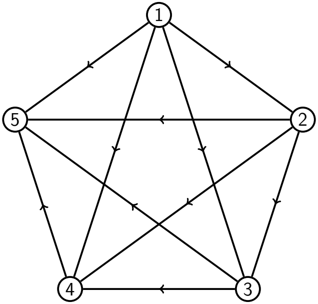

- $\mathcal{T}_5 \setminus \mathcal{S}_5 \neq \langle e_{12}, e_{13}, e_{14}, e_{15}, e_{23}, e_{24}, e_{25}, e_{34}, e_{35}, e_{45} \rangle$.

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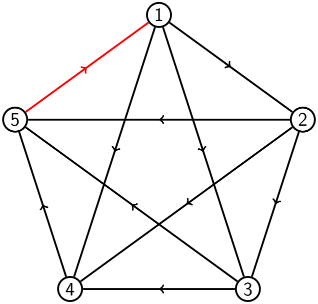

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Theorem

- $\text{rank}(I_r(\mathcal{T}_n)) = \text{idrank}(I_r(\mathcal{T}_n)) = S(n,r)$ if $2 \leq r \leq n-1$.

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- Defining relations (E, 2010, 2013).

Theorem

(E1) $e_{ij}^2 = e_{ij} = e_{ji}e_{ij}$

(E2) $e_{ij}e_{kl} = e_{kl}e_{ij}$

(E3) $e_{ik}e_{jk} = e_{ik}$

(E4) $e_{ij}e_{ik} = e_{ik}e_{ij} = e_{jk}e_{ij}$

(E5) $e_{ki}e_{ij}e_{jk} = e_{ik}e_{kj}e_{ji}e_{ik}$

(E6) $e_{ki}e_{ij}e_{jk}e_{kl} = e_{ik}e_{kl}e_{li}e_{ij}e_{jl}$

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- $\langle E(\mathcal{T}_X)\rangle$ for infinite X (Howie, 1966).

Theorem

- $$\langle E(\mathcal{T}_X)\rangle = \{1\} \cup (\mathcal{T}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}}) \cup \{\alpha \in \mathcal{T}_X : s(\alpha) = d(\alpha) = c(\alpha) \geq \aleph_0\}$$

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Theorem

- Every non-invertible square matrix over a field is a product of idempotent matrices.

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- Diagram semigroups. . .

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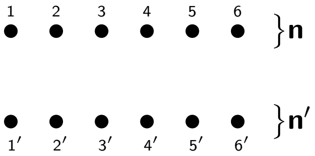

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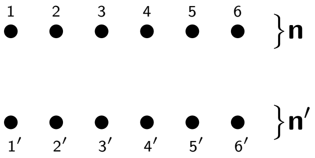

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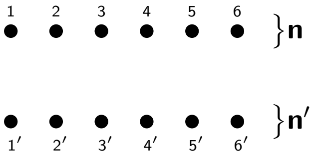
- Eg: $\alpha = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 6'\}, \{2', 3'\}, \{5'\}\Big\} \in \mathcal{P}_6$



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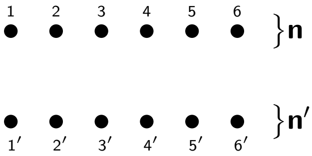

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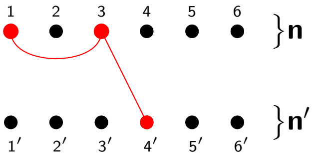

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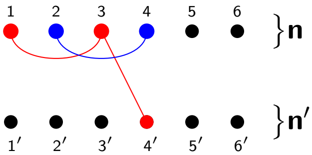

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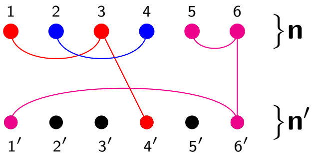

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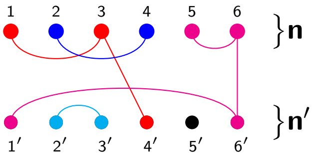

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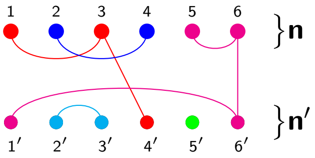

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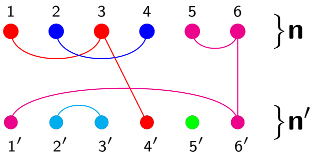

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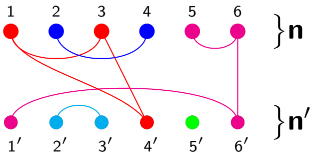

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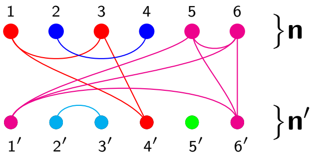

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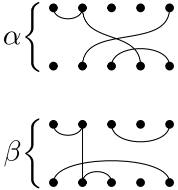
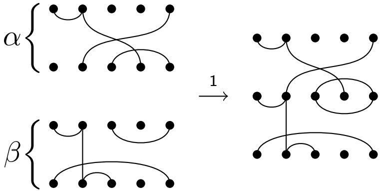
Let $\alpha, \beta \in \mathcal{P}_n$.



Let $\alpha, \beta \in \mathcal{P}_n$. To calculate $\alpha\beta$:

(1) connect bottom of α to top of β,



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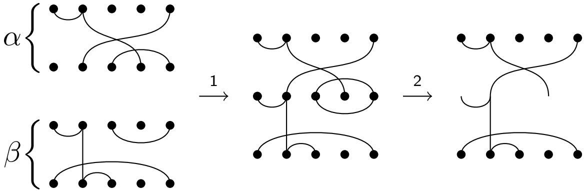

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Let $\alpha, \beta \in \mathcal{P}_n$. To calculate $\alpha\beta$:

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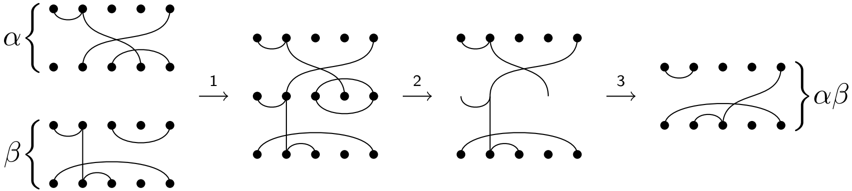
(3) smooth out resulting graph to obtain $\alpha\beta$.



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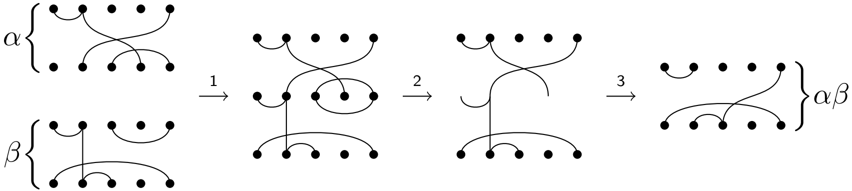

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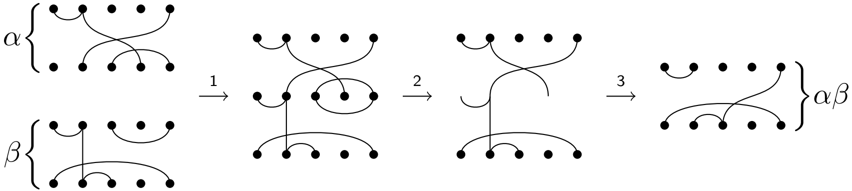

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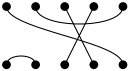
- What can we say about idempotents (etc) of $\mathcal{P}_n$?

- $\mathcal{B}_n = \{\alpha \in \mathcal{P}_n : \text{blocks of } \alpha \text{ have size } 2\}$ — Brauer monoid



$\in \mathcal{B}_5$

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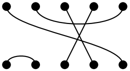

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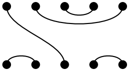

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2. Partition Monoids — Submonoids of $\mathcal{P}_n$

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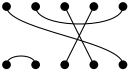 $\in \mathcal{B}_5$

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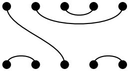 $\in \mathcal{J}_5$

What can we say about idempotents (etc) of $\mathcal{P}_n$? $\mathcal{B}_n$? $\mathcal{J}_n$?

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- $E(\mathcal{P}_n)$ is not a subsemigroup of $\mathcal{P}_n$.

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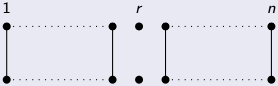
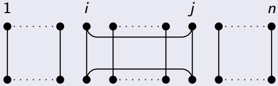
$t_r =$ 

$t_{ij} =$ 

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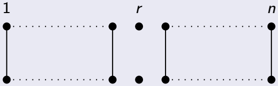
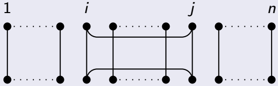
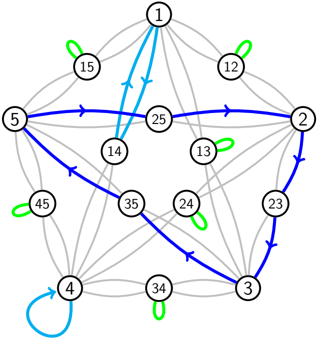
$t_r =$ 

$t_{ij} =$ 

- $\text{rank}(\mathcal{P}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{P}_n \setminus \mathcal{S}_n) = \binom{n+1}{2} = \frac{n(n+1)}{2}$.

- Minimal (idempotent) generating sets (E and Gray, 2014).



- $\mathcal{P}_5 \setminus \mathcal{S}_5 = \langle t_{12}, t_{13}, t_{15}, t_{24}, t_{34}, t_{45}, t_4,$

$$e_{41}, f_{14}, e_{23}, f_{23}, e_{35}, f_{35}, e_{52}, f_{52}\rangle.$$

2. Partition Monoids

- Minimal (idempotent) generating sets (E and Gray, 2014).
- (Idempotent) rank of all ideals (E and Gray, 2014).

Theorem

- $\text{rank}(I_r(\mathcal{P}_n)) = \text{idrank}(I_r(\mathcal{P}_n)) = \sum_{j=r}^{n} \binom{n}{j} S(j,r) B_{n-j}$

  if $0 \leq r \leq n-1$.

- Minimal (idempotent) generating sets (E and Gray, 2014).
- (Idempotent) rank of all ideals (E and Gray, 2014).
- Defining relations (E, 2011).

Theorem

(T0) $t_i^2 = t_i$ (T1) $t_{ij}^2 = t_{ij}$ (T2) $t_{ij}t_kt_{ij} = t_{ij}$

(T3) $t_it_j = t_jt_i$ (T4) $t_{ij}t_{kl} = t_{kl}t_{ij}$ (T5) $t_kt_{ij}t_k = t_k$

(T6) $t_{ij}t_{jk} = t_{jk}t_{ki} = t_{ki}t_{ij}$ (T7) $t_{ij}t_k = t_kt_{ij}$

(T8) $t_kt_{ki}t_it_{ij}t_jt_{jk}t_k = t_kt_{kj}t_jt_{ji}t_it_{ik}t_k$

(T9) $t_kt_{ki}t_it_{ij}t_jt_{jl}t_lt_{lk}t_k = t_kt_{kl}t_lt_{li}t_it_{ij}t_jt_{jk}t_k$

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Theorem

- $\langle E(\mathcal{P}_X)\rangle = \{1\} \cup (\mathcal{P}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}})$
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- What about $E(\mathcal{P}_n)$ itself??

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```
n=1 partitions=2 idempots=2
n=2 partitions=15 idempots=12
n=3 partitions=203 idempots=114
n=4 partitions=4140 idempots=1512
n=5 partitions=115975 idempots=25826
n=6 partitions=4213597 idempots=541254
n=7 partitions=190899322 idempots=13479500
```

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```
for i in [2..8] do
 Print(NrIdempotents(PartitionMonoid(i)), n);
od;!

2 12 114 1512 25826 541254 13479500 389855014
```

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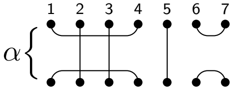

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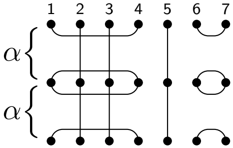

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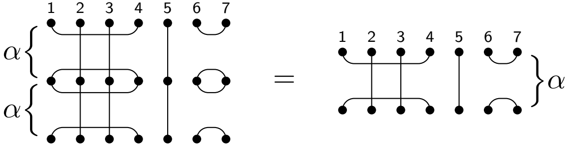

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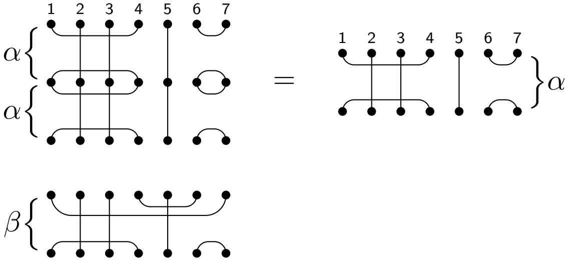

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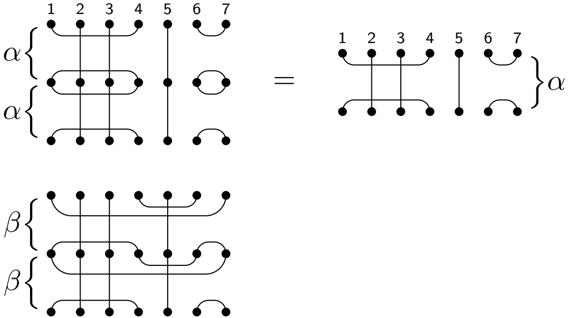

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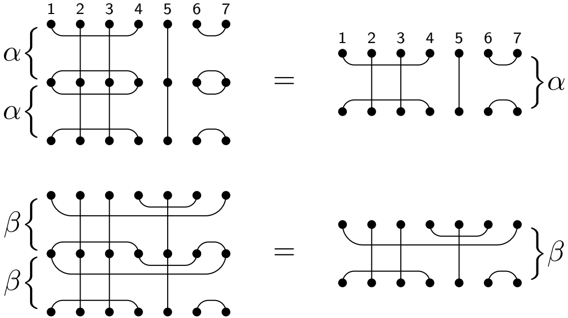

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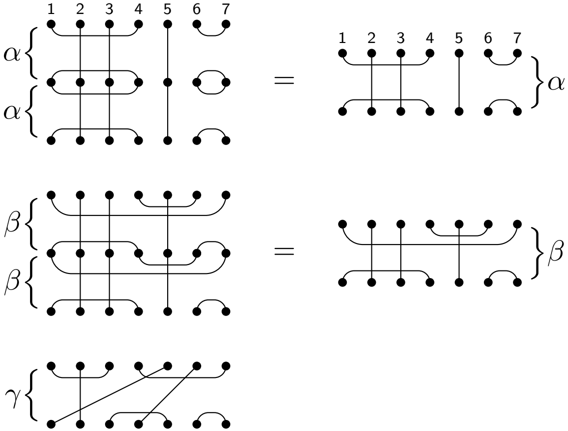

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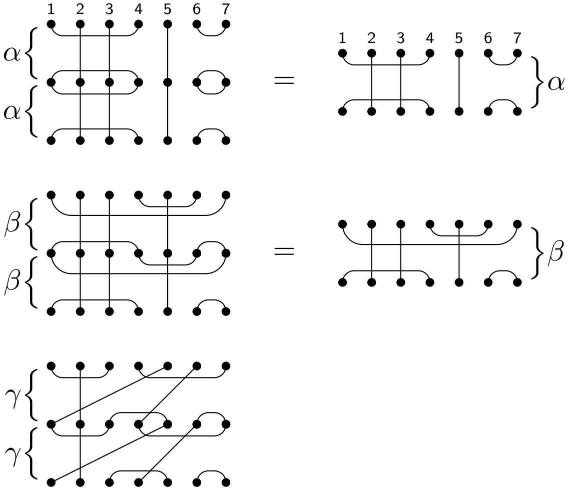

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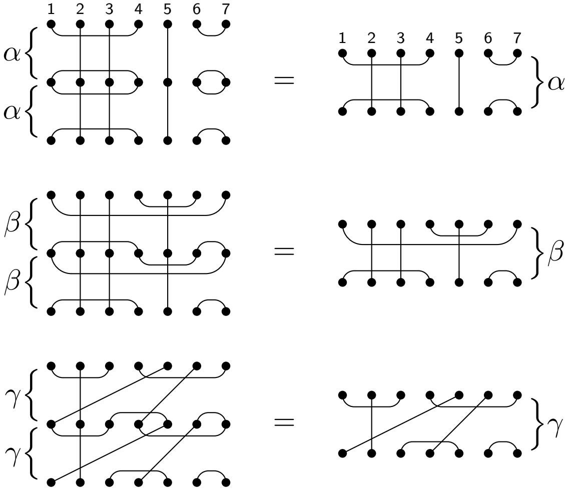

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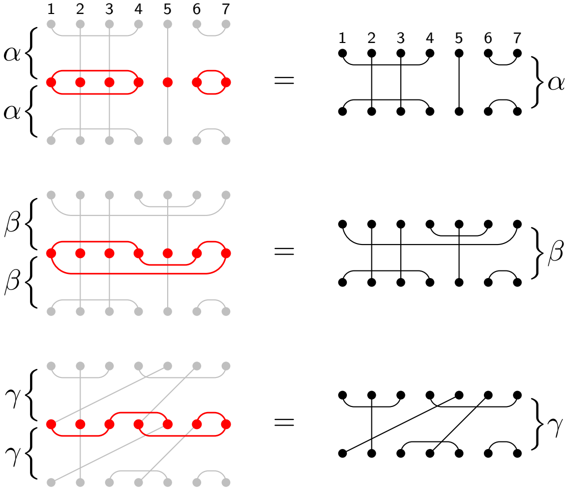

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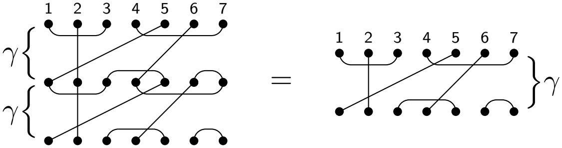

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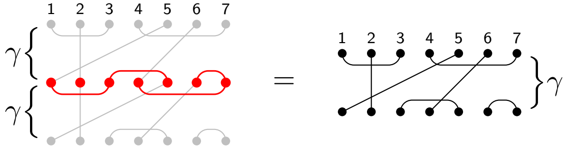

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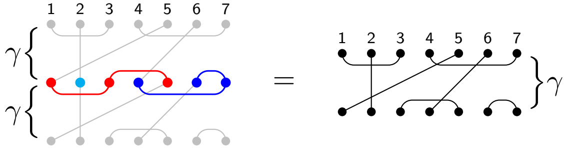

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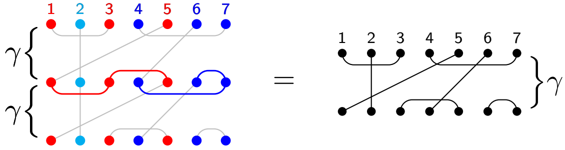

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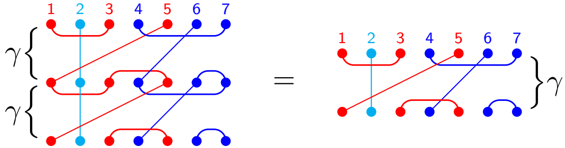

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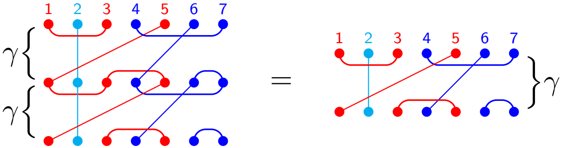

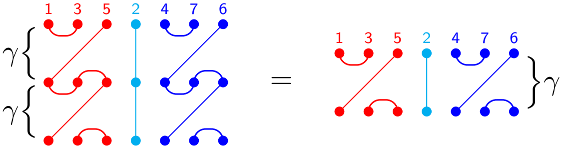

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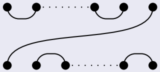
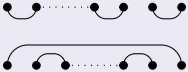
(1) 

(2) 

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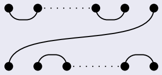
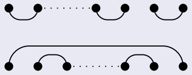
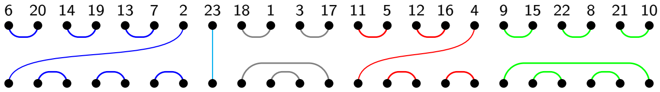
(1) 

(2) 



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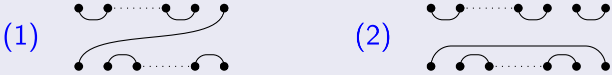

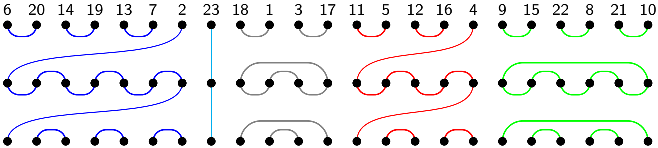

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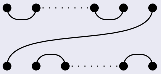
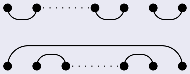
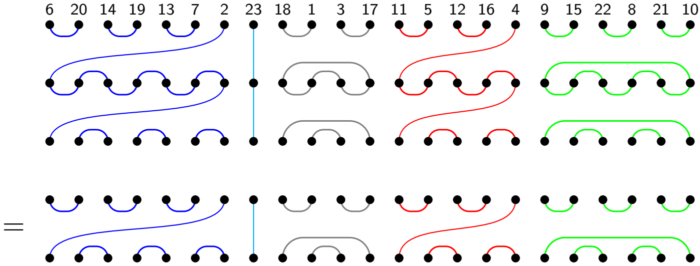
(1) 

(2) 



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Theorem (DEEFHHL)

The numbers $e_n = |E(\mathcal{B}_n)|$ satisfy

- $e_0 = 1$,
- $e_n = a_1 e_{n-1} + a_2 e_{n-2} + \cdots + a_n e_0$

where $a_{2i} = \binom{n-1}{2i-1}(2i-1)!$ and $a_{2i+1} = \binom{n-1}{2i}(2i+1)!$.

3. Brauer Monoids

Theorem (DEEFHHL)

$$|E(D_r(\mathcal{B}_n))| = \sum_{\substack{\mu \vdash n \\ \mu_1+\mu_3+\cdots=r}} \frac{n!}{\mu_1! \cdot \mu_2! \cdots \mu_n! \cdot 2^{\mu_2} \cdot 4^{\mu_4} \cdots (2k)^{\mu_{2k}}}$$

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The numbers $e_{nr} = |E(D_r(\mathcal{B}_n))|$ satisfy

- $e_{nn} = 1,\ e_{n0} = \begin{cases} (n-1)!!^2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$
- $e_{nr} = a_1 e_{n-1,r-1} + a_2 e_{n-2,r} + a_3 e_{n-3,r-1} + a_4 e_{n-4,r} + \cdots$

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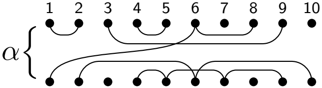

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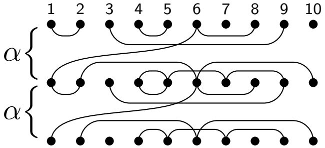

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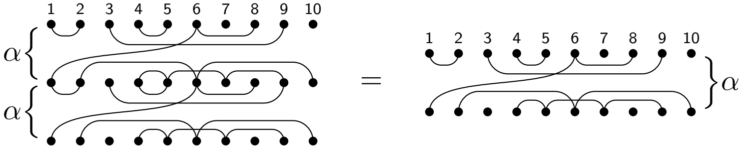

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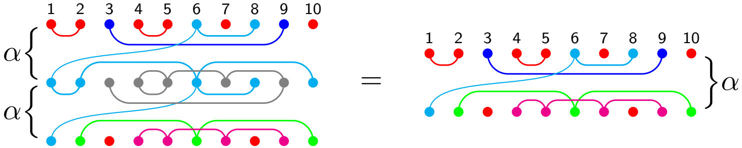

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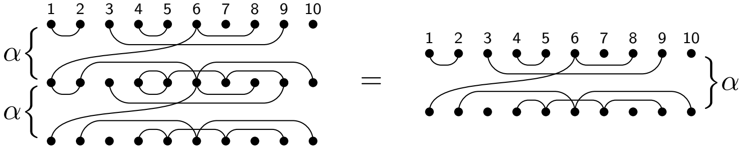

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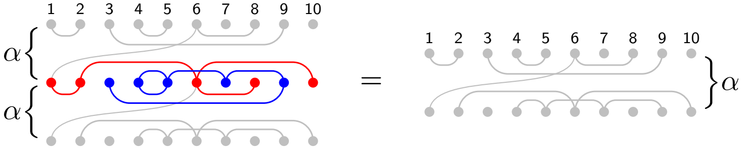

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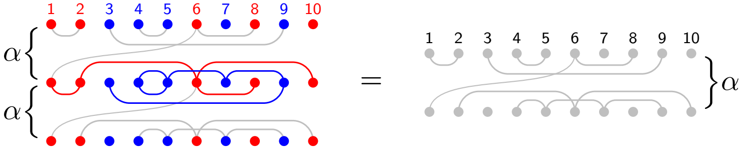

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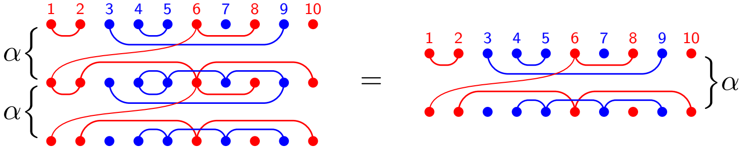

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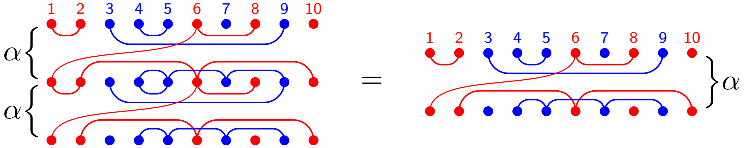

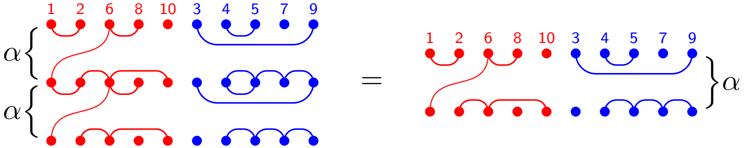

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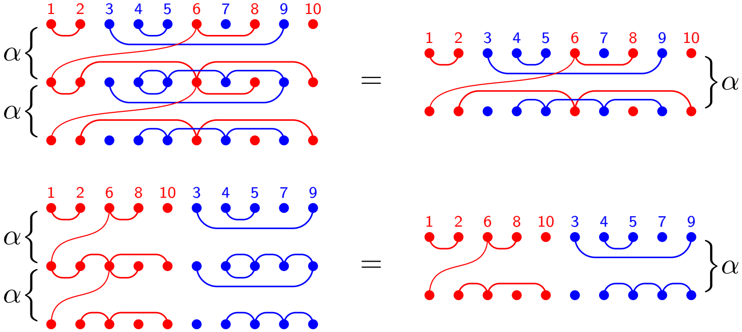

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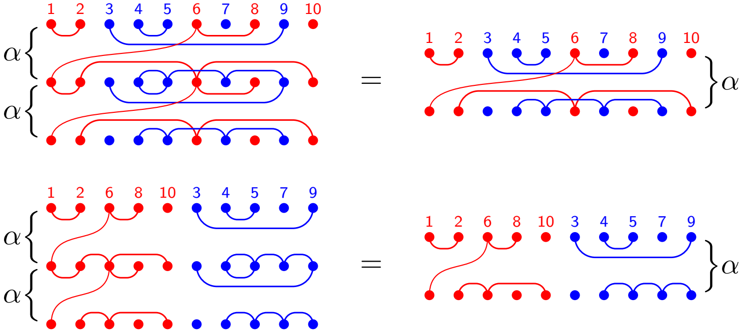

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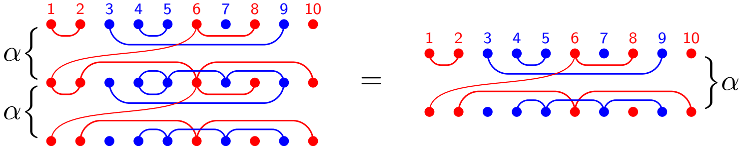

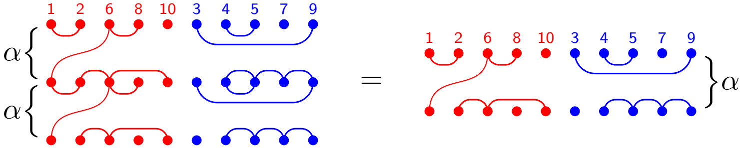

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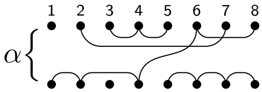

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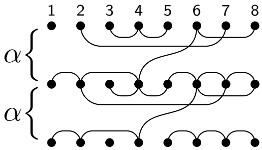

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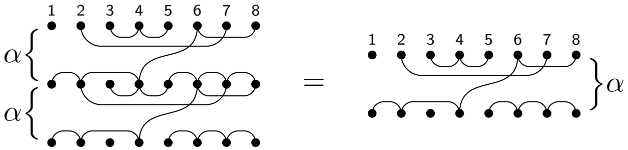

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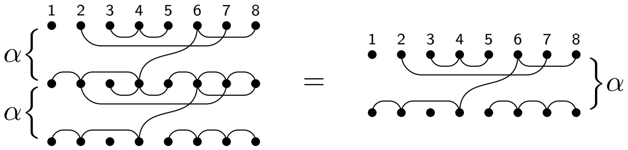

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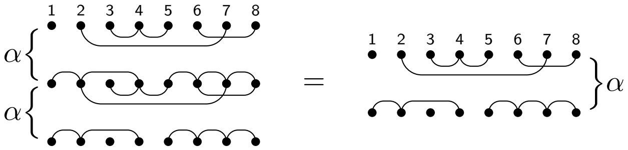

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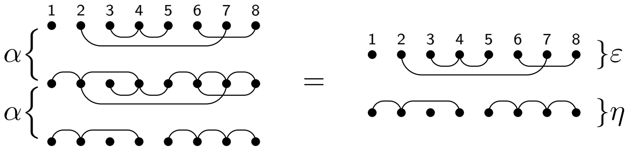

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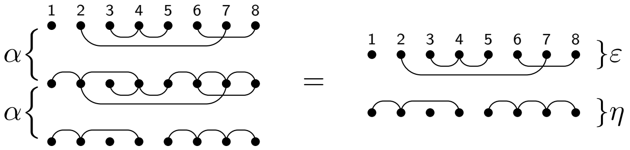

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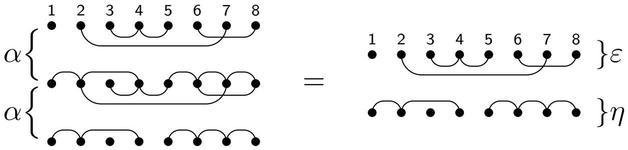

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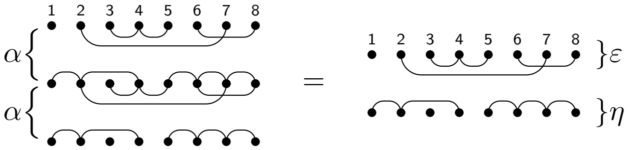

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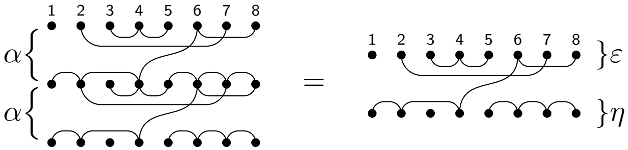

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- $c(k) = \sum_{r,s=1}^{k} (1 + rs) c(k, r, s)$, and
- $$c(n, r, 1) = S(n, r)$$
  $$c(n, 1, s) = S(n, s)$$
  $$c(n, r, s) = s \cdot c(n-1, r-1, s) + r \cdot c(n-1, r, s-1) + rs \cdot c(n-1, r, s) + \sum_{m=1}^{n-2} \binom{n-2}{m} \sum_{a=1}^{r-1} \sum_{b=1}^{s-1} \big(a(s-b) + b(r-a)\big) c(m, a, b) c(n-m-1, r-a, s-b).$$
  if $r, s \geq 2$.

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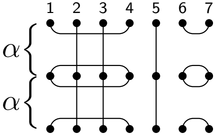
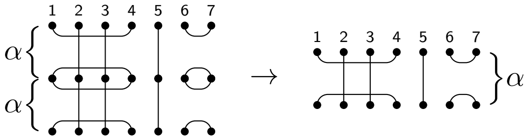
- $m(\alpha,\beta) + m(\alpha\beta,\gamma) = m(\alpha,\beta\gamma) + m(\beta,\gamma)$ so $\star$ is associative.
- $\mathcal{P}_n^\xi$ plays an important role in statistical mechanics and Schur-Weyl duality.

5. Diagram Algebras

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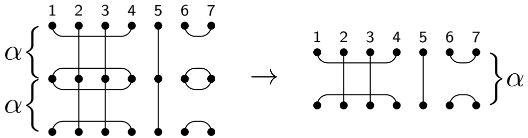

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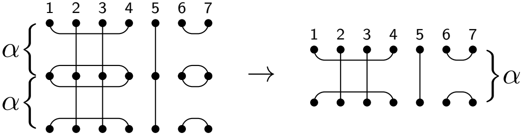

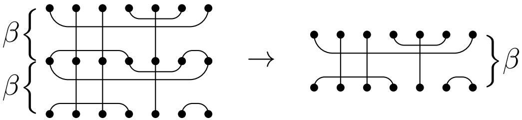

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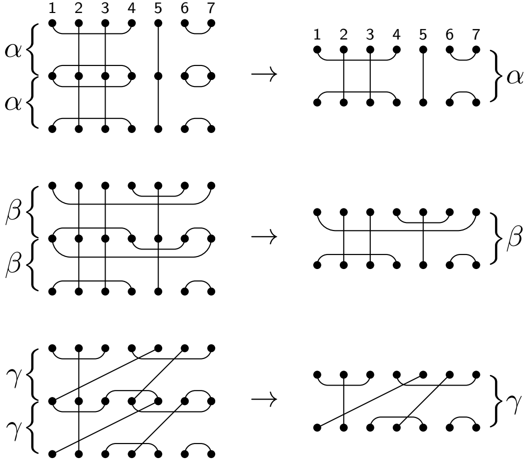

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Theorem (DEEFHHL)

The number of idempotent basis elements of $\mathcal{P}_n^\xi$ is equal to

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where

- $c'(k) = \sum_{r,s=1}^{k} rs \cdot c(k,r,s)$, and
- ξ is not an Mth root of unity with $M \leq n$.

(Similar statements exist when ξ is an Mth root of unity.)

Theorem (DEEFHHL)

The number of idempotent basis elements of $\mathcal{B}_n^\xi$ is equal to

$$\sum_{\mu} \frac{n!}{\mu_1!\mu_3!\cdots\mu_{2k+1}!},$$

where

- $k = \lfloor \frac{n-1}{2} \rfloor$,
- the sum is over all integer partitions $\mu = (1^{\mu_1}, \ldots, n^{\mu_n}) \vdash n$ with $\mu_{2i} = 0$ for $i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, and
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- $|E(\mathcal{B}_X)| = |E(\mathcal{P}_X)| = 2^{|X|}$ if X is infinite

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