Ends of Semigroups

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Ends of Graphs

Let Γ be a graph

a ray is an infinite path

$$\bullet \;\text{———}\; \bullet \;\text{———}\; \bullet \;\text{———}\; \cdots$$

Two rays are equivalent if there exists infinitely many disjoint paths between them

$$\begin{array}{ccccccc}
\bullet & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \cdots \\
\wr & & \wr & & \wr & & \\
\wr & & \wr & & \wr & & \\
\bullet & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \cdots
\end{array}$$

Equivalence classes of rays are called the ends of a graph.

Ends of Groups

The ends of a group wrt a generating set are the ends of the corresponding Cayley graph.

1. The number of ends of a group is invariant under change of finite generating sets.
2. A finitely generated group has 1, 2 or $2^{\aleph_0}$ ends.
3. The number ends of a group and a subgroup of finite index are the same.
4. If a group has > 1 end then it is a HNN extension with finite base or a free product with finite amalgamation.

Ends of a Digraph

Let Γ be a digraph

an out-ray is a infinite directed path of the form

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

an in-ray is a infinite directed path of the form

$$\bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \cdots$$

a ray is either an in-ray or an out-ray.

A ray x is greater than a ray y, written $y \preccurlyeq x$, if there are infinitely many disjoint directed paths from x to y

$$\begin{tikzcd}
x\bullet \arrow[r] \arrow[d, squiggly] & \bullet \arrow[r] \arrow[d, squiggly] & \bullet \arrow[r] \arrow[d, squiggly] & \cdots \\
y\bullet & \bullet \arrow[l] & \bullet \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Two rays are equivalent if $x \preccurlyeq y$ and $y \preccurlyeq x$. The equivalences classes are the ends of a digraph.

$\preccurlyeq$ gives a partial order on the set of ends.

This definition was introduced by Zuther and generalises the notion of ends for graphs.

Ends of Semigroups

The left/right ends of a semigroup wrt a generating set are the ends of the corresponding left/right Cayley digraph.

Lemma

Let S be a semigroup and let A and B be finite generating sets for S. Then the end poset of $Cay_r(S, A)$ is isomorphic the end poset of $Cay_r(S, B)$.

Left and Right Ends

Example

The semigroup $L_n \times \mathbb{Z} \times \mathbb{Z} \times R_m$ *has* n *right ends and* m *left ends.*

Theorem

Let S *be an infinite cancellative semigroup then* S *has* $1, 2$ *or infinitely many ends and if* S *has* 2 *ends then* S *is a group.*

Cancellative Semigroups

Theorem

Let S be a cancellative semigroup which is not a group. Then the following are equivalent:

1. S has 1 right end.
2. S has 1 left end.
3. For any finite gernerating set A of S there exists $a \in A$ such that $\langle a \rangle \cong (\mathbb{N}, +)$ and there exists $K \in \mathbb{N}$ such that for all $s \in S$ there exists $i, j \in \mathbb{N}$ satisfying $d_A(s, a^i), d_A(a^j, s) < K$.
4. S has a presentation of the form $\langle a, u_1, \ldots, u_n | u_i a = a^{\alpha(i)} u_{\beta(i)}, u_i u_j = a^{f(i,j)} u_{g(i,j)} \rangle$.

Semigroups with 1 End: Infinitely many $\mathcal{R}$-classes

Theorem

A finitely generated semigroup S with finitely many $\mathcal{R}$-classes has one end if and only if S has a subsemigroup of finite Rees index isomorphic to a Rees matrix semigroup $\mathcal{M}[G; \{i\}, \{\lambda_1, \lambda_2, \ldots, \lambda_n\}; P]$ where G is a finitely generated group with 1 end.

Semigroups with 1 End: Finitely many $\mathcal{R}$-classes

Lemma

Let S be a finitely generated semigroup. Then S has 1 right end if and only if S is a disjoint union of a finitely generated semigroup T such that $|tS| = \infty$ for all $t \in T$ and with 1 right end and an ideal I satisfying the condition $|iS| < \infty$ for all $i \in I$.

Theorem

Let S be a finitely generated semigroup with no infinite $\mathcal{R}$-class and such that $|sS| = \infty$ for all $s \in S$ and A be a finite generating set. If S has 1 end then for any ray $\mathbf{r} = 1 \to r_1 \to \dots$ there exists $N, M \in \mathbb{N}$ such that for all $x \in S$ there exists $i \in \mathbb{N}$ with $d_A(r_i, x), d_A(x, r_{i+M}) \leq N$.

Definitions of Index

Definition

Let S be a semigroup and T be a subsemigroup of S. The Rees index of T in S is $|S \setminus T| + 1$.

Definition

Let S be a semigroup and T be a subsemigroup of S. For $x, y \in S$ we will say that $x\mathcal{R}^T y$ if $xT^1 = yT^1$, and $x\mathcal{L}^T y$ if $T^1 x = T^1 y$. The intersection $\mathcal{H}^T = \mathcal{R}^T \cap \mathcal{L}^T$ is an equivalence relation on S. The Green index of T in S is defined to be $|(S \setminus T)/\mathcal{H}^T| + 1$.

Indices

Theorem

Let S be a semigroup and let T be a subsemigroup of finite Rees index then S has the same end poset as T.

Example

The semigroup $\mathbb{Z} \times \mathbb{Z} \times mon\langle a | a^2 = a \rangle$ has $\mathbb{Z} \times \mathbb{Z}$ as a subgroup of finite Green index.

Theorem

Let S be a cancellative semigroup and let T be a subsemigroup of finite Green index then S has the same end poset as T.

Residual Finiteness of 1-ended Semigroups

Proposition

Let S be a finitely generated semigroup with an infinite $\mathcal{R}$-class and 1 right end. Then S is residually finite if and only if its maximal subgroups are residually finite.

Lemma

Let S be a finitely generated semigroup with no infinite $\mathcal{R}$-class, such that $|xS| = \infty$ for all $x \in S$ and with 1 right end. Then S is residually finite.