

Ends of Semigroups

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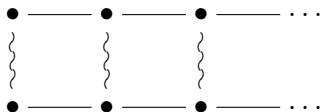
Ends of Graphs

Let Γ be a graph

a **ray** is an infinite path



Two rays are **equivalent** if there exists infinitely many disjoint paths between them



Equivalence classes of rays are called the **ends** of a graph.

Ends of Groups

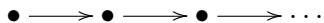
The ends of a group wrt a generating set are the ends of the corresponding Cayley graph.

1. The number of ends of a group is invariant under change of finite generating sets.
2. A finitely generated group has 1, 2 or 2^{\aleph_0} ends.
3. The number ends of a group and a subgroup of finite index are the same.
4. If a group has > 1 end then it is a HNN extension with finite base or a free product with finite amalgamation.

Ends of a Digraph

Let Γ be a digraph

an **out-ray** is a infinite directed path of the form

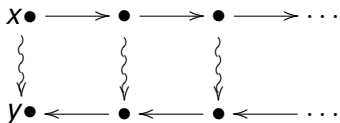


an **in-ray** is a infinite directed path of the form



a **ray** is either an in-ray or an out-ray.

A ray x is **greater** than a ray y , written $y \preceq x$, if there are infinitely many disjoint directed paths from x to y



Two rays are **equivalent** if $x \preceq y$ and $y \preceq x$. The equivalence classes are the **ends of a digraph**.

\preceq gives a partial order on the set of ends.

This definition was introduced by Zuthr and generalises the notion of ends for graphs.

Ends of Semigroups

The **left/right ends** of a semigroup wrt a generating set are the ends of the corresponding left/right Cayley digraph.

Lemma

Let S be a semigroup and let A and B be finite generating sets for S . Then the end poset of $\text{Cay}_r(S, A)$ is isomorphic the end poset of $\text{Cay}_r(S, B)$.

Left and Right Ends

Example

The semigroup $L_n \times \mathbb{Z} \times \mathbb{Z} \times R_m$ has n right ends and m left ends.

Theorem

Let S be an infinite cancellative semigroup then S has 1, 2 or infinitely many ends and if S has 2 ends then S is a group.

Cancellative Semigroups

Theorem

Let S be a cancellative semigroup which is not a group. Then the following are equivalent:

1. S has 1 right end.
2. S has 1 left end.
3. For any finite generating set A of S there exists $a \in A$ such that $\langle a \rangle \cong (\mathbb{N}, +)$ and there exists $K \in \mathbb{N}$ such that for all $s \in S$ there exists $i, j \in \mathbb{N}$ satisfying $d_A(s, a^i), d_A(a^j, s) < K$.
4. S has a presentation of the form $\langle a, u_1, \dots, u_n \mid u_i a = a^{\alpha(i)} u_{\beta(i)}, u_i u_j = a^{f(i,j)} u_{g(i,j)} \rangle$.

Semigroups with 1 End: Infinitely many \mathcal{R} -classes

Theorem

A finitely generated semigroup S with finitely many \mathcal{R} -classes has one end if and only if S has a subsemigroup of finite Rees index isomorphic to a Rees matrix semigroup $\mathcal{M}[G; \{i\}, \{\lambda_1, \lambda_2, \dots, \lambda_n\}; P]$ where G is a finitely generated group with 1 end.

Semigroups with 1 End: Finitely many \mathcal{R} -classes

Lemma

Let S be a finitely generated semigroup. Then S has 1 right end if and only if S is a disjoint union of a finitely generated semigroup T such that $|tS| = \infty$ for all $t \in T$ and with 1 right end and an ideal I satisfying the condition $|iS| < \infty$ for all $i \in I$.

Theorem

Let S be a finitely generated semigroup with no infinite \mathcal{R} -class and such that $|sS| = \infty$ for all $s \in S$ and A be a finite generating set. If S has 1 end then for any ray $\mathbf{r} = 1 \rightarrow r_1 \rightarrow \dots$ there exists $N, M \in \mathbb{N}$ such that for all $x \in S$ there exists $i \in \mathbb{N}$ with $d_A(r_i, x), d_A(x, r_{i+M}) \leq N$.

Definitions of Index

Definition

Let S be a semigroup and T be a subsemigroup of S . The **Rees index** of T in S is $|S \setminus T| + 1$.

Definition

Let S be a semigroup and T be a subsemigroup of S . For $x, y \in S$ we will say that $x\mathcal{R}^T y$ if $xT^1 = yT^1$, and $x\mathcal{L}^T y$ if $T^1x = T^1y$. The intersection $\mathcal{H}^T = \mathcal{R}^T \cap \mathcal{L}^T$ is an equivalence relation on S . The **Green index** of T in S is defined to be $|(S \setminus T)/\mathcal{H}^T| + 1$.

Indices

Theorem

Let S be a semigroup and let T be a subsemigroup of finite Rees index then S has the same end poset as T .

Example

The semigroup $\mathbb{Z} \times \mathbb{Z} \times \text{mon}\langle a \mid a^2 = a \rangle$ has $\mathbb{Z} \times \mathbb{Z}$ as a subgroup of finite Green index.

Theorem

Let S be a cancellative semigroup and let T be a subsemigroup of finite Green index then S has the same end poset as T .

Residual Finiteness of 1-ended Semigroups

Proposition

Let S be a finitely generated semigroup with an infinite \mathcal{R} -class and 1 right end. Then S is residually finite if and only if its maximal subgroups are residually finite.

Lemma

Let S be a finitely generated semigroup with no infinite \mathcal{R} -class, such that $|xS| = \infty$ for all $x \in S$ and with 1 right end. Then S is residually finite.