Equations and logic on words

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Overview

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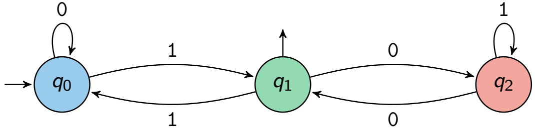

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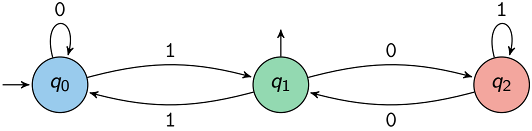

  Answer yes iff A accepts w.
- Solution 2: a homomorphism $\varphi\colon \{0,1\}^* \to S_3$ defined by

$$0 \mapsto (1\,2), \quad 1 \mapsto (0\,1).$$

  Answer yes iff the permutation $\varphi(w)$ sends 0 to 1.

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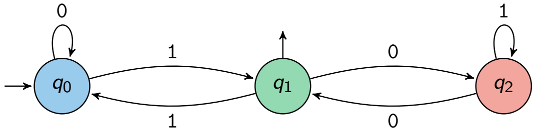

Answer yes iff A accepts w.

- Solution 3: an MSO sentence φ:

$\exists Q_0 \exists Q_1 \exists Q_2 (Q_0(\texttt{first}) \wedge Q_1(\texttt{last}) \wedge$

$\forall x [\texttt{0}(x) \wedge Q_0(x) \rightarrow Q_0(\texttt{S}x)] \wedge [\texttt{1}(x) \wedge Q_0(x) \rightarrow Q_1(\texttt{S}x)] \wedge \ldots).$

Answer yes iff w satisfies the formula φ.

Regular languages

Regular languages are subsets $L \subseteq \Sigma^*$ which are ...

- recognizable by a finite automaton;
- invariant under a finite index monoid congruence;
- definable by a monadic second order sentence.

Myhill-Nerode 1958; Büchi 1960

Monoids and finite index congruences

- A monoid is a set M equipped with an associative binary operation and a unit.
- The set Σ^* of finite words is a free monoid.
  - multiplication is concatenation;
  - unit is the empty word ϵ;
- A congruence on M is an equivalence relation θ which respects multiplication.
  - The quotient M/θ is again a monoid;
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- Any language $L \subseteq \Sigma^*$ has an associated syntactic congruence, θ_L, i.e., the finest congruence under which L is invariant:

$$w \in L \text{ and } w \theta_L w' \text{ implies } w' \in L.$$

- L is called regular iff θ_L has finite index.

Logic on words

- **Syntax.** Monadic Second Order (MSO) logic over $<$, Σ.
  - Basic propositional connectives: $\wedge$, $\neg$.
  - Quantification over first-order variables x, y, $\ldots$ and monadic second-order variables P, Q, $\ldots$.
  - Relational signature: $x < y$, $\mathtt{a}(x)$ for $a \in \Sigma$.

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- **Semantics.** A word $w = a_1 \dots a_n$ gives a structure W.
  - The underlying set of W is $\{1, \dots, n\}$.
  - The natural linear order $<^W$ interprets the binary predicate $<$.
  - For every letter $a \in \Sigma$, $\mathtt{a}^W := \{i \in \{1, \dots, n\} : a_i = a\}$.

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- For a sentence φ, $L_\varphi := \{w \in \Sigma^* \mid w \models \varphi\}$.
- A language L is regular iff $L = L_\varphi$ for some φ in MSO.

- Shortcuts such as $\mathtt{S}(x)$, `first`, `last`, $\subseteq$, ... are MSO-definable.

Logic on words: examples

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Question. Does such an equivalent first order formula exist for φ?

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Duality

Key insight. The connection between MSO logic on words and monoids is an instance of Stone-Jónsson-Tarski duality.

Algebra	**Space**
Lindenbaum algebra of a logic	Canonical model
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Profinite monoids and their clopens

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- Also: a limit of finite monoids with the discrete topology.

- A subset of a profinite monoid is clopen iff it is recognizable, i.e., invariant under a finite index *topological* congruence.

Duality and profinite monoids

- There are natural division operators on the Boolean algebra of clopen sets of a profinite monoid:

$$K \backslash L = \{m \mid mK \subseteq L\}, \quad L/K = \{m \mid Km \subseteq L\}.$$

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Under this duality...

- the free profinite monoid is dual to the residuated Boolean algebra of all regular languages;
- quotients of the free profinite monoid correspond to subalgebras of regular languages that are ideals for division.

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Logic and monoids

A language $L \subseteq \Sigma^*$ is MSO-definable

if, and only if,

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A congruence θ on Σ^* is called aperiodic if Σ^*/θ does not have non-trivial subgroups.

Schützenberger 1965; McNaughton, Papert 1971

ω

In a finite monoid, any element x has a unique idempotent, x^ω, in its orbit $\{x, x^2, x^3, \ldots\}$.

Fact. A finite monoid is aperiodic iff it validates the equation

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The quotient of the free profinite monoid obtained by enforcing $x^\omega = x^\omega x$ is the free pro-aperiodic monoid.
This is the dual space of the residuated algebra of FO-definable languages (instance of Eilenberg-Reiterman).

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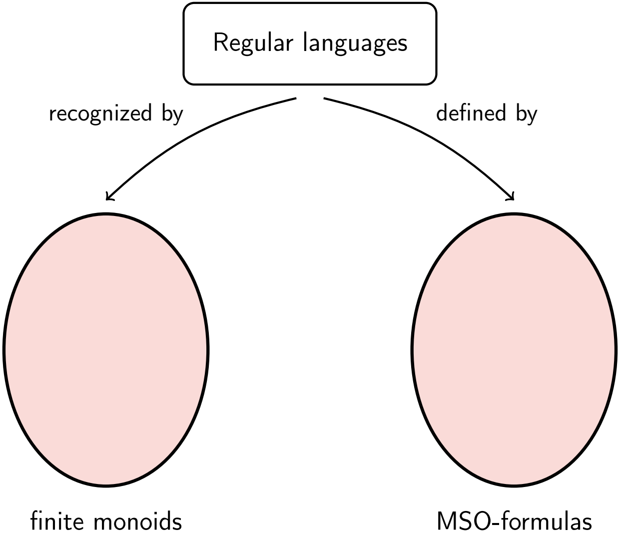
Question. Does an equivalent first order formula exist for φ?

No, because:

- any quotient under which L_φ is invariant must contain a subgroup $\mathbb{Z}_2$; or:
- for any generator a of the free profinite monoid, we have $a^\omega \in \widehat{L_\varphi}$ and $a^\omega a \notin \widehat{L_\varphi}$, so L_φ 'falsifies' the equation $x^\omega = x^\omega x$.

Monoids and logic



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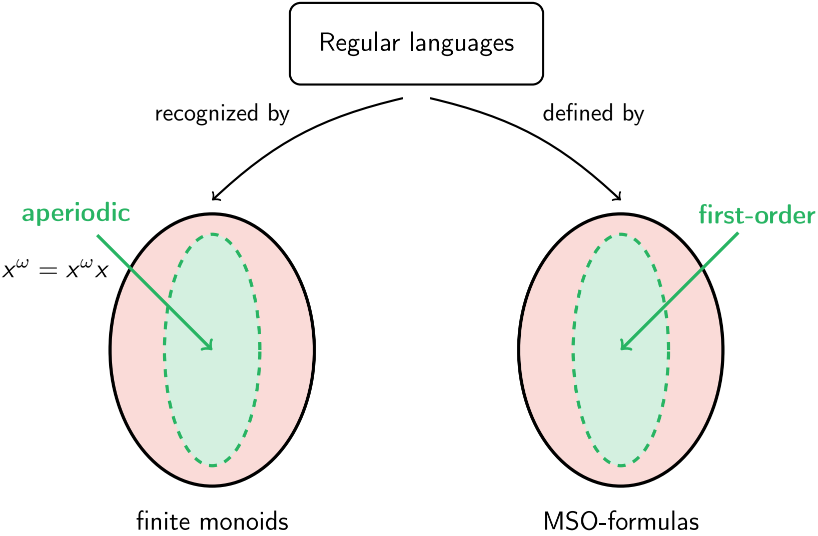

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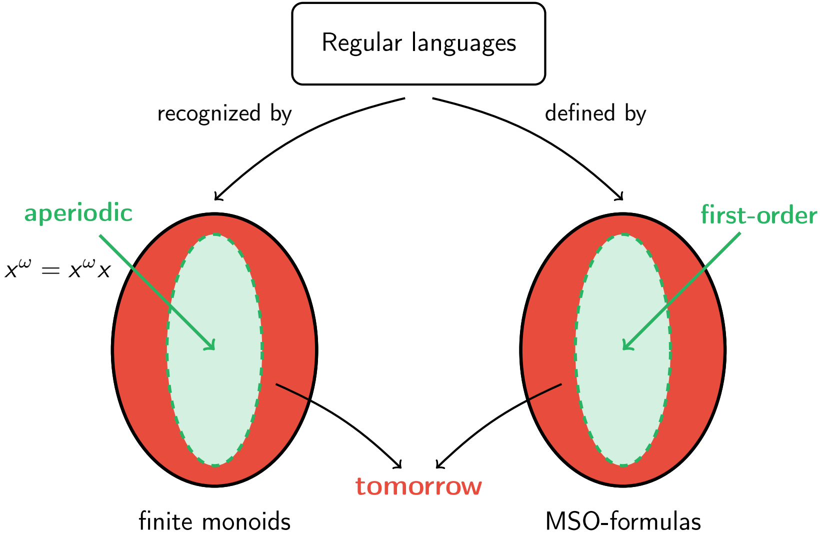

The free profinite aperiodic monoid

Theorem.

The free profinite aperiodic monoid

$=$

The topological monoid of ultrafilters of FO-definable languages

$=$

The topological monoid of $\equiv_{FO}$-classes of pseudo-finite words.

G. & Steinberg STACS 2017

Pseudo-finite words

- By a pseudo-finite word we mean a first-order structure $(W, <, (a^W)_{a \in \Sigma})$ that is a model of the theory of finite words.
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Ultrafilters and pseudo-finite words

- An ultrafilter $\mathcal{U}$ of FO-definable languages uniquely determines an $\equiv_{FO}$-class $[W]$ of pseudo-finite words.
- This is a homeomorphism between the ultrafilter space and the space of types.
- There is a natural topological monoid multiplication on types:

$$\text{if } W \equiv W' \text{ then } VW \equiv VW' \text{ and } WV \equiv W'V.$$

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G. & Steinberg STACS 2017

An application: the aperiodic ω-word problem

Decision problem. Given two terms in $\cdot$ and $()^\omega$, are they equal in every finite aperiodic monoid?

An application: the aperiodic ω-word problem

Decision problem. Given two terms in $\cdot$ and $()^\omega$, are they equal in the free profinite aperiodic monoid?

Realizing ω-words as ω-saturated models

- A countable model is ω-saturated if it realizes all the complete types over a finite parameter set.
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- Crucially, substitutions of ω-saturated words into ω-saturated words are again ω-saturated.
- Thus, any ω-term can be realized as an ω-saturated word.
- Using the uniqueness of countable ω-saturated models, equality of ω-terms reduces to isomorphism of these words, which we know is decidable.

Hüschenbett & Kufleitner STACS 2013;

G. & Steinberg STACS 2017

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- A T-structure A is existentially closed* if any existential sentence that becomes true in some T-structure extending A already holds in A.
- This property is often first order definable:
  - Linear orders without endpoints: density;
  - Boolean algebras: atomless;
  - Heyting algebras: mimick fields, use uniform interpolation.

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Model companion

A first order theory T^* which captures the existentially closed models for a universal theory T is called a model companion of T.

Theorem.
The theory T^*, if it exists, is the unique theory such that:

1. T and T^* believe the same universal sentences;

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Robinson, 1963

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   T and T^* are co-theories

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   T^* is model complete

Robinson, 1963

Model companions and languages

Theorem.

The first order theory T^* of an algebra for word languages, $\mathcal{P}(\omega)$,

is the model companion of

a theory T of algebras for a linear temporal logic.

Ghilardi & G. JSL 2017

Proof idea: set-up

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- Enrich the Boolean algebra $\mathcal{P}(\omega)$ with temporal operators:
  - $\mathbf{X}a := \{t \in \omega \mid t+1 \in a\}$,
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Theorem. The theory T^* of $\mathcal{P}(\omega)$ is the model companion of T.

i.e., T^* is model complete and T^* is a co-theory of T.

Proof idea: co-theories

- Need to show: any equation of the form $t(\overline{p}) = \top$ that is valid in $\mathcal{P}(\omega)$ is valid in all T-structures.
- The theory T axiomatizes linear temporal logic on $\mathbf{X}$, $\mathbf{F}$, $\mathbf{I}$:
  - Boolean algebra axioms, $\mathbf{X}$ is a homomorphism, $\mathbf{F}a$ is the least fix point of the function $x \mapsto a \vee \mathbf{X}x$.
  - $\mathbf{I}$ is an atom and $\mathbf{I} \leq \mathbf{F}a$ whenever $a \neq \bot$.

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- If $t(\overline{p}) \neq \top$ in some T-structure A, consider its dual space X.
- By carefully using filtration-type techniques, we may read off from X a valuation $\overline{p} \to \mathcal{P}(\omega)$ which invalidates $t(\overline{p}) = \top$.

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- Build an automaton A for Φ.
- Describe the automaton A with an existential first order formula φ' in the temporal algebra $\mathcal{P}(\omega)$.
- **Conclusion.** $\mathcal{P}(\omega)$ believes that any first order formula φ is equivalent to an existential formula φ'.

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Theorem.

The first order theory T^* of an algebra for tree languages, $\mathcal{P}(2^*)$,

is the model companion of

a theory T of algebras for a fair computation tree logic.

Ghilardi & G. LICS 2016

The future

- From FO to MSO
- Model companions for more logics
- Using ordered spaces