An Introduction to Graph Expansions

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15th June 2010

Inverse Semigroups

Definition

An element $a' \in S$ is an *inverse* of $a \in S$ if $a = aa'a$ and $a' = a'aa'$. If each element of S has exactly one inverse in S, then S is an *inverse semigroup*.

Definition

For $a, b \in S$,

$$a \mathcal{R} b \Leftrightarrow a = bt \text{ and } b = as \text{ for some } s, t \in S$$

and

$$\begin{aligned} a \, \sigma \, b &\Leftrightarrow ea = eb \text{ for some } e \in E(S) \\ &\Leftrightarrow af = bf \text{ for some } f \in E(S). \end{aligned}$$

Left Restriction and Weakly Left Ample Semigroups

Definition

Suppose S is a semigroup and E a set of idempotents of S. Let $a, b \in S$. Then $a \widetilde{\mathcal{R}}_E\, b$ if and only if for all $e \in E$,

$$ea = a \text{ if and only if } eb = b.$$

Definition

A semigroup S is *left restriction* (formerly known as *weakly left E-ample*) if the following hold:

1) E is a subsemilattice of S;
2) Every element $a \in S$ is $\widetilde{\mathcal{R}}_E$-related to an idempotent in E (idempotent denoted by a^+);
3) $\widetilde{\mathcal{R}}_E$ is a left congruence;
4) For all $a \in S$ and $e \in E$,

$$ae = (ae)^+ a \text{ (the \textit{left ample condition}).}$$

Proper Left Restriction and Weakly Left Ample Semigroups

Let S be a left restriction semigroup with distinguished semilattice E. Then for $a, b \in S$,

$$a \, \sigma_E \, b \Leftrightarrow ea = eb \text{ for some } e \in E.$$

Definition

A left restriction semigroup is *proper* if and only if $\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota$.

Left Ample Semigroups

Definition

Let S be a semigroup and let $a, b \in S$. Then $a \mathcal{R}^* b$ if and only if for all $x, y \in S^1$,

$$xa = ya \Leftrightarrow xb = yb.$$

Proposition

Let $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}$ be the relations defined above on a semigroup S. Then

$$\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E.$$

Left Ample Semigroups

Definition

A semigroup S is *left ample* (formally known as *left type A*) if the following hold:

1) $E(S)$ is a subsemilattice of S;
2) Every element $a \in S$ is $\mathcal{R}^*$-related to an idempotent in $E(S)$ (idempotent denoted by a^+);
3) For all $a \in S$ and $e \in E(S)$,

$$ae = (ae)^+ a.$$

Definition

A left ample semigroup is *proper* if and only if $\mathcal{R}^* \cap \sigma = \iota$.

The Szendrei Expansion

Definition

Let M be a monoid and let $\mathcal{P}_1^f(M)$ denote the finite subsets of M that contain the identity.

The *Szendrei expansion* of M is

$$Sz(M) = \{(A, g) : A \in \mathcal{P}_f^1(M), g \in A\}.$$

For $(A, g), (B, h) \in Sz(M)$,

$$(A, g)(B, h) = (A \cup gB, gh).$$

The Szendrei Expansion

Proposition (Hollings)

Let M be an arbitrary monoid. Then $Sz(M)$ is a proper left restriction monoid with distinguished semilattice

$$E = \{(A, 1) : A \in \mathcal{P}_1^f(M)\}.$$

Proposition (Fountain, Gomes)

If M is a unipotent monoid, $Sz(M)$ is a weakly left ample monoid.

Proposition (Fountain, Gomes, Gould)

If M is a right cancellative monoid, $Sz(M)$ is a left ample monoid.

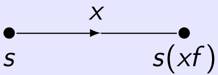
Proposition (Birget & Rhodes, Szendrei)

If M is a group, $Sz(M)$ is an inverse monoid.

The Graph Expansion

Monoid presentation (X, f, S), where X is a set, S a monoid and $f : X \to S$ such that $< Xf >= S$.

Let $\Gamma = \Gamma(X, f, S)$ be the Cayley graph of (X, f, S), which has *vertices* $V(\Gamma) = S$ and *edges* $E(\Gamma)$. For $s \in S$ and $x \in X$ an edge is given by $(s, x, s(xf))$.



The Graph Expansion

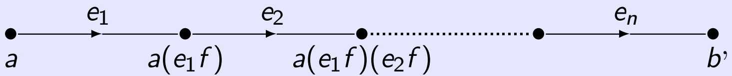
Δ is a *subgraph* of the Cayley graph Γ if Δ is a graph such that

- $V(\Delta) \subseteq V(\Gamma)$;
- $E(\Delta) \subseteq E(\Gamma)$;
- the initial and terminal vertices of an edge in Δ are vertices of Γ.

Union of finite subgraphs Δ and Σ: subgraph created by taking vertices $V(\Delta \cup \Sigma) = V(\Delta) \cup V(\Sigma)$ and edges $E(\Delta \cup \Sigma) = E(\Delta) \cup E(\Sigma)$.

The Graph Expansion

There is a *path* between $a, b \in V(\Delta)$, where a is the initial vertex, if there is a sequence of edges $e_1, e_2, ..., e_n$ such that



where $b = a(e_1f)(e_2f)...(e_nf)$.

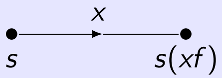
A subgraph is said to be *a-rooted* if there is a path from the vertex $a \in S$ to any other vertex in the subgraph.

The Graph Expansion

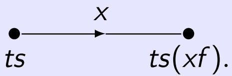
The action of a monoid S on a subgraph Δ is defined by

$$t \cdot v = tv$$

for $t \in S$ and $v \in V(\Delta)$. An edge, $(s, x, s(xf))$ say, in the subgraph becomes $(ts, x, ts(xf))$, i.e. the subgraph



becomes



The Graph Expansion

Let Γ_f be the set of finite 1-rooted subgraphs of Γ. Then a *graph expansion* is defined by

$$M = M(X, f, S) = \{(\Delta, s) : \Delta \in \Gamma_f, s \in \Delta\},$$

with binary operation

$$(\Delta, s)(\Sigma, t) = (\Delta \cup s\Sigma, st)$$

for $(\Delta, s), (\Sigma, t) \in M$.

The Graph Expansion

Proposition (Gomes)

Let (X, f, S) be a monoid presentation. Then $M = M(X, f, S)$ is a proper left restriction monoid, where $(A, a)^+ = (A, 1)$ for $(A, a) \in M$.

Proposition (Gomes, Gould)

A graph expansion $M(X, f, S)$ is a weakly left ample monoid if and only if S is a unipotent monoid.

Proposition (Gould)

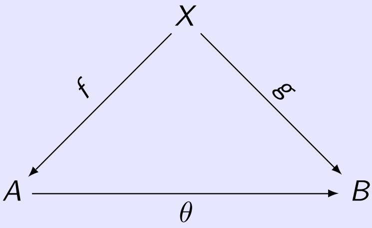
A graph expansion $M(X, f, S)$ is a left ample monoid if and only if S is right cancellative.

The Category $\mathbf{A}(X)$

Let X be a set and $\mathcal{A}$ a class of algebras of a given fixed type.

$\mathbf{A}(X)$: category with objects pairs (f, A) where $A \in \mathcal{A}, f : X \to A$ and $< Xf >= A$.

A morphism in $\mathbf{A}(X)$ from (f, A) to (g, B) is a homomorphism $\theta : A \to B$ such that the following diagram commutes:



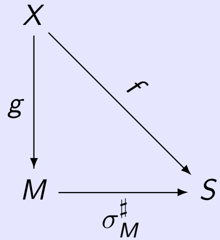
The Category $\mathbf{PLR}(X)$

$\mathbf{PLR}(X)$: category where $\mathcal{P}LR$ is the class of proper left restriction monoids.

The Category $\mathbf{PLR}(X, f, S)$

Let (X, f, S) be the monoid presentation of a fixed monoid S.

An object (g, M) of $\mathbf{PLR}(X)$ is an object of $\mathbf{PLR}(X, f, S)$ if the following diagram commutes:



where $\sigma_M^\sharp$ is a homomorphism with kernel σ_M.

The Category $\mathbf{PLR}(X, f, S)$

Facts:

- (f, S) is an object in $\mathbf{PLR}(X, f, S)$;
- If (g, M) is an object in the category, then $\sigma_M^\sharp : M \to S$ is the unique morphism in $Mor((g, M), (f, S))$;
- (f, S) is a terminal object in $\mathbf{PLR}(X, f, S)$.

The Category $\mathbf{PLR}(X, f, S)$

Proposition

Let (X, f, S) be a monoid presentation of a monoid S. Putting $M = M(X, f, S)$ we have

- $M =< X\tau_M >$, *where* $\tau_M : X \to M$ *is defined by* $x\tau_M = (\Gamma_x, xf)$;
- (τ_M, M) *is an initial object in* $\mathbf{PLA}(X, f, S)$.

Theorem

Let X be a set, $\iota : X \to X^*$ be the canonical embedding and $M = (X, \iota, X^*)$.
Then $\tau_M : X \to M$ is an embedding and M is the free left restriction monoid on $X\tau_M$.

Varieties Work

Definition

Let V be a variety of monoids and N a left restriction monoid. Then N has a proper cover over V if N has a proper cover M such that $M/\sigma \in V$.

If V is a variety of monoids, then the class of left restriction monoids having proper covers over V is a variety of left restriction monoids, where the variety is determined by

$$\Sigma = \{\bar{u}^{+}\bar{v} \equiv \bar{v}^{+}\bar{u} : \bar{u} \equiv \bar{v} \text{ is a law in } V\}.$$

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