FREE IDEMPOTENT GENERATED SEMIGROUPS OVER BANDS AND BIORDERED SETS WITH TRIVIAL PRODUCTS

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Abstract. For any biordered set of idempotents E there is an initial object $\mathrm{IG}(E)$, the *free idempotent generated semigroup over* E, in the category of semigroups generated by a set of idempotents biorder-isomorphic to E. Recent research on $\mathrm{IG}(E)$ has focussed on the behaviour of the maximal subgroups. Inspired by an example of Brittenham, Margolis and Meakin, several proofs have been offered that any group occurs as a maximal subgroup of some $\mathrm{IG}(E)$, the latest being that of Dolinka and Ruškuc, who show that E can be taken to be a band. From a result of Easdown, Sapir and Volkov, periodic elements of any $\mathrm{IG}(E)$ lie in subgroups. However, little else is known of the 'global' properties of $\mathrm{IG}(E)$, other than that it need not be regular, even where E is a semilattice. The aim of this article is to deepen our understanding of the overall structure of $\mathrm{IG}(E)$ in the case where E is a biordered set with trivial products (for example, the biordered set of a poset) or where E is the biordered set of a band B.

Since its introduction by Fountain in the late 1970s, the study of abundant and related semigroups has given rise to a deep and fruitful research area. The class of abundant semigroups extends that of regular semigroups in a natural way and itself is contained in the class of weakly abundant semigroups. Our main results show that (1) if E is a biordered set with trivial products then $\mathrm{IG}(E)$ is abundant and (if E is finite) has solvable word problem, and (2) for *any* band B, the semigroup $\mathrm{IG}(B)$ is weakly abundant and moreover satisfies a natural condition called the *congruence condition*. Further, $\mathrm{IG}(B)$ is abundant for a normal band B for which $\mathrm{IG}(B)$ satisfies a given technical condition, and we give examples of such B. On the other hand, we give an example of a normal band B such that $\mathrm{IG}(B)$ is not abundant.

1. Introduction

Let S be a semigroup with set of idempotents $E = E(S)$. It is easy to see that if idempotents of S commute, then E may be endowed with a partial order under which it becomes a semilattice, that is, every pair of elements has a greatest lower bound, which is just their product in S. For an arbitrary semigroup S, the set E, equipped with the restriction of the quasi-orders $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ defined on S and the restriction of products to elements that are related under $\leq_{\mathcal{R}}, \leq_{\mathcal{L}}$ and their converses, is a partial algebra called a *biordered set* [15]. On the other hand, Easdown [7] shows every biordered set E occurs as $E(S)$ for some semigroup S.

Given a biordered set E, which we can without prejudice take as the set E of idempotents of some semigroup S, there is a free object in the category of idempotent generated semigroups

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that have biordered set of idempotents isomorphic to E. This is called the *free idempotent generated semigroup* over E and is denoted by $\mathrm{IG}(E)$. We obtain $\mathrm{IG}(E)$ by the following presentation:

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\, e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle,$$

where $\overline{E} = \{\bar{e} : e \in E\}$.[1] Note that $\{e, f\} \cap \{ef, fe\} \neq \emptyset$ implies both ef and fe are idempotents of E; in this case both (e, f) and (f, e) are referred to as *basic pairs* and ef and fe as *basic products*. If E has the property that for any basic pair (e, f) we have $ef, fe \in \{e, f\}$ then we say E has *trivial (basic) products*. We note that if Y is a semilattice then the biordered set of Y has trivial products. Clearly, for any biordered set E there is a natural morphism φ from $\mathrm{IG}(E)$ to $\langle E \rangle$, the subsemigroup of S generated by E. In fact, $E(\mathrm{IG}(E)) = \overline{E}$, and the restriction $\varphi|_{\overline{E}} : \overline{E} \to E$ is an isomorphism of biordered sets [7], justifying the presentation above. We refer our readers to [11] for other classical properties of $\mathrm{IG}(E)$.

Given the universal nature of free idempotent generated semigroups, it is natural to enquire into their structure. Early investigations of Pastijn [18] showed that where B is a rectangular band then the corresponding $\mathrm{IG}(B)$ is a completely simple semigroup with free maximal subgroups. Pastijn's result for rectangular bands was extended by McElwee in [17] to the case where the principal ideals of a biordered set are singletons. For more general bands, Pastijn focussed on what in modern terminology is called the *free regular idempotent generated semigroup*. Following the first example of a non-free maximal subgroup of an $\mathrm{IG}(E)$ given by Brittenham, Margolis and Meakin [2], it was proved, first by Gray and Ruškuc [11] and later by the authors [10], that *every* group is a maximal subgroup of $\mathrm{IG}(E)$ for some biordered set E. Dolinka and Ruškuc show that E may be taken to be a *band* (that is, a semigroup of idempotents) [4], reinforcing the signficance of bands in the study of free idempotent generated semigroups. Biordered sets of bands were first characterised by Nambooripad [15] with an alternative description given by Easdown [6]. It is also worth remarking that Dolinka, Gray and Ruškuc have recently shown that every group occurs as a Schützenberger group of a non-regular $\mathcal{D}$-class of some $\mathrm{IG}(E)$ [5].

Whereas a deal of energy has recently been put into the question of the maximal subgroups of free idempotent generated semigroups $\mathrm{IG}(E)$, in contrast, very little is known of the overall structure of semigroups of this form, even in the case where E is the biordered set of idempotents of a band. What can be said is that periodic elements of $\mathrm{IG}(E)$ must lie in subgroups, a result of Easdown, Sapir and Volkov [8], and that $\mathrm{IG}(E)$ need not be regular. Indeed, even for a semilattice Y, the semigroup $\mathrm{IG}(Y)$ need not be regular [2, Example 2]. Regularity is a property of semigroups that can be phrased in terms of Green's relations $\mathcal{R}$ and $\mathcal{L}$ and idempotents. Analogous but weaker conditions are those of being *abundant* and *weakly abundant*, which are defined in the same way but with $\mathcal{R}$ and $\mathcal{L}$ replaced by $\mathcal{R}^*$ and $\mathcal{L}^*$, or $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$, respectively.

Our first main result, Theorem 3.2 is that for a biordered set E for which the basic products are trivial the semigroup $\mathrm{IG}(E)$ is abundant. Semilattices, indeed posets, and rectangular bands, provide examples of such biordered sets. We describe those bands which give rise to biordered sets with trivial basic products. Our second main result, Theorem 4.13, shows that for any band B, the semigroup $\mathrm{IG}(B)$ is weakly abundant and is such that $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ are, respectively, left and right congruences, a property called the *congruence condition*. We remark

[1] It is more usual to identify elements of E with those of $\overline{E}$, but it helps the clarity of our later arguments to make this distinction.

that regular, abundant and restriction semigroups always have the congruence condition. On the other hand, we give an example of a band B such that $\mathrm{IG}(B)$ is not abundant. In the positive direction we investigate a condition on a normal band B that guarantees abundancy of $\mathrm{IG}(B)$.

We proceed as follows. To make this article as self-contained as possible, in Section 2 we recall some basics of Green's relations and regular semigroups, and of generalised Green's relations and (weakly) abundant semigroups. We briefly describe how the presentation of any $\mathrm{IG}(E)$ naturally induces a reduction system. In Section 3 we begin our investigation of free idempotent generated semigroups by considering a biordered set E with trivial products. We show that every element of $\mathrm{IG}(E)$ has a unique normal form and consequently if E is finite then $\mathrm{IG}(E)$ has solvable word problem (a result that is known in the case where E is a poset). We then proceed to show that $\mathrm{IG}(E)$ is abundant. Finally in Section 3 we describe those bands having trivial basic products.

In Section 4 we proceed to look at $\mathrm{IG}(B)$ where B is an arbitrary band. In this case, we may lose uniqueness of normal forms in $\mathrm{IG}(B)$. To overcome this problem, we introduce the concept of *almost normal forms*. We prove that for any band B the semigroup $\mathrm{IG}(B)$ is weakly abundant with the congruence condition. We finish the section with an example of a four element non-normal band B such that $\mathrm{IG}(B)$ is not abundant. Section 5 considers a sufficient condition for a normal band to be abundant, and we give some examples where this is satisfied. One would naturally ask here whether $\mathrm{IG}(B)$ is abundant for an arbitrary normal band B. In Section 6 we construct a ten element normal band B with four $\mathcal{D}$-classes for which $\mathrm{IG}(B)$ is not abundant.

2. Preliminaries: (weakly) abundant semigroups and Reduction systems

We do not assume our readers have prior knowledge of all the various areas this article draws together. The aim of this section is to draw together the necessary technicalities. In addition, we recommend [13] for an excellent introduction to the requisite semigroup theory.

Throughout this paper, for $n \in \mathbb{N}$ we write $[1, n]$ to denote $\{1, \cdots, n\} \subseteq \mathbb{N}$. The free semigroup on a set A is denoted by A^+; the elements of A^+ are words in the letters of A and the binary operation is juxtaposition. The free monoid on A is denoted by A^*; notice that $A^* = A^+ \cup \{\varepsilon\}$ where ε is the empty word and the identity of A^*. The set of idempotents of a semigroup S is always denoted by $E(S)$ or more simply E.

We now recall an important tool for analysing ideals of a semigroup S and related notions of structure, called *Green's relations*. There are equivalence relations that characterise the elements of S in terms of the principal ideals they generate. The two most basic of Green's relations are $\mathcal{L}$ and $\mathcal{R}$, and are defined by

$$a \;\mathcal{L}\; b \Leftrightarrow S^1a = S^1b \text{ and } a \;\mathcal{R}\; b \Leftrightarrow aS^1 = bS^1,$$

where S^1 denotes S with an identity element adjoined (unless S already has one). Furthermore, we denote the intersection $\mathcal{L} \cap \mathcal{R}$ by $\mathcal{H}$ and the join $\mathcal{L} \vee \mathcal{R}$ by $\mathcal{D}$. One of the fundamental facts of semigroup theory tells us that $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, and hence $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

For our purposes we require two quasi-orders associated with $\mathcal{R}$ and $\mathcal{L}$, respectively, restricting ourselves to defining them on $E(S)$ where S is a semigroup. For $e, f \in E(S)$ we say that

$$e \leq_{\mathcal{R}} f \text{ if and only if } fe = e$$

and

$$e \leq_{\mathcal{L}} f \text{ if and only if } ef = e.$$

We leave the reader to check that $e \leq_{\mathcal{R}} f$ if and only if $eS^1 \subseteq fS^1$ and in this case, $fe \in E(S)$. Dual remarks apply to $\leq_{\mathcal{L}}$. It is then clear that $\mathcal{R}$ and $\mathcal{L}$ (more precisely, their restrictions to $E \times E$) are indeed the equivalence relations associated with $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$, respectively. We denote $\leq_{\mathcal{R}} \cap \leq_{\mathcal{L}}$ by $\leq$.

Much of this article is concerned with biordered sets that come from *bands*, where a band B is a semigroup such that $E(B) = B$. A commutative band is a *semilattice*. This terminology is used since, if Y is a semilattice, then the relation $\leq = \leq_{\mathcal{R}} = \leq_{\mathcal{L}}$ is a partial order and is such that the product of any pair of elements is their greatest lower bound. On the other hand, any partially ordered set P having this property may be made into a commutative band by setting $uv = u \wedge v$, for all $u, v \in P$. A band is *rectangular* if it satisfies the identity $x = xyx$. It is easy to see that in a rectangular band B we have $e \,\mathcal{R}\, ef \,\mathcal{L} f$ for any $e, f \in B$ so that $e \,\mathcal{D}\, f$: in fact, rectangular bands are precisely those bands that are a single $\mathcal{D}$-class. More generally, a band B is a *semilattice Y of rectangular bands B_α, $\alpha \in Y$* [13, Theorem 4.4.1]. This means that $B = \bigcup_{\alpha \in Y} B_\alpha$ where each B_α is a rectangular band, $B_\alpha \cap B_\beta = \emptyset$ for $\alpha \neq \beta$, and $B_\alpha B_\beta \subseteq B_{\alpha\beta}$, for all $\alpha, \beta \in Y$. One can check that the subsemigroups B_α are the $\mathcal{D}$-classes of B and B is a semilattice if and only if each B_α is trivial. At times we will use the foregoing notation without specific comment.

A band B is *normal* if it satisfies the identity $xyzx = xzyx$. Equivalently, B is a *strong semilattice Y of rectangular bands* $B_\alpha, \alpha \in Y$, that is,

$$B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$$

is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, such that for all $\alpha \geq \beta$ in Y there exists a morphism $\phi_{\alpha,\beta} : B_\alpha \to B_\beta$ such that

(B1) for all $\alpha \in Y$, $\phi_{\alpha,\alpha} = 1_{B_\alpha}$;

(B2) for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$, $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$,

and for all $\alpha, \beta \in Y$ and $x \in B_\alpha, y \in B_\beta$,

$$xy = (x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta}).$$

An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $a = axa$, that is, it is regular in the sense of von Neumann. The semigroup S is *regular* if it consists entirely of regular elements. It is well known that S is regular if and only if each $\mathcal{L}$-class (equivalently, each $\mathcal{R}$-class,) contains an idempotent. Regular semigroups are particularly amenable to analysis using Green's relations.

For the purpose of analysing a semigroup S that might not be regular, the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are defined on S by the rule that

$$a \;\mathcal{L}^*\; b \;\Leftrightarrow\; (\forall x, y \in S^1)\; (ax = ay \Leftrightarrow bx = by)$$

and

$$a \;\mathcal{R}^*\; b \;\Leftrightarrow\; (\forall x, y \in S^1)\; (xa = ya \Leftrightarrow xb = yb).$$

As commented in [9], it is easy to see that $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$, and if S is regular, then $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$. We denote by $\mathcal{H}^*$ the intersection $\mathcal{L}^* \cap \mathcal{R}^*$, and by $\mathcal{D}^*$ the join of $\mathcal{L}^* \vee \mathcal{R}^*$. Note that unlike Green's relations, generally $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$ (see [9, Example 1.11]).

A semigroup S is *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent. In the theory of abundant semigroups the relations $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{H}^*$ and $\mathcal{D}^*$ play a role which is analogous to that of Green's relations in the theory of regular semigroups.

As an easy but useful consequence of the definition of $\mathcal{L}^*$, we have the following lemma (a dual result holds for $\mathcal{R}^*$).

Lemma 2.1. *[9] Let S be a semigroup with $a \in S$ and $e \in E(S)$. Then the following statements are equivalent:*
(i) $a\ \mathcal{L}^*\ e$;
(ii) $ae = a$ and for any $x, y \in S^1$, $ax = ay$ implies $ex = ey$.

A third set of relations, extending the starred versions of Green's relations, and useful for semigroups that are not abundant, was introduced in [14]. The relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ on a semigroup S are defined by the rule

$$a\ \widetilde{\mathcal{L}}\ b \ \Leftrightarrow\ (\forall e \in E(S))\ (ae = a \Leftrightarrow be = b)$$

and

$$a\ \widetilde{\mathcal{R}}\ b \ \Leftrightarrow\ (\forall e \in E(S))\ (ea = a \Leftrightarrow eb = b)$$

for any $a, b \in S$.

Clearly $\mathcal{L}^* \subseteq \widetilde{\mathcal{L}}$ and $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$. If S is abundant, then $\mathcal{L}^* = \widetilde{\mathcal{L}}$ and $\mathcal{R}^* = \widetilde{\mathcal{R}}$ (see, for example, [14, Theorem 1.5]). Whereas $\mathcal{L}^*$ and $\mathcal{R}^*$ are always right and left congruences on S, respectively, the same is not necessarily true for $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ [14, Example 3.6]. A semigroup S is *weakly abundant* if each $\widetilde{\mathcal{L}}$-class and each $\widetilde{\mathcal{R}}$-class contains an idempotent. We say that a weakly abundant semigroup S satisfies the *congruence condition* if $\widetilde{\mathcal{L}}$ is a right congruence and $\widetilde{\mathcal{R}}$ is a left congruence. Clearly, an abundant semigroup is weakly abundant with the congruence condition.

The following lemma is an analogue of Lemma 2.1. Of course, a dual result holds for $\widetilde{\mathcal{R}}$.

Lemma 2.2. *[14] Let S be a semigroup with $a \in S$ and $e \in E(S)$. Then the following statements are equivalent:*
(i) $a\ \widetilde{\mathcal{L}}\ e$;
(ii) $ae = a$ and for any $f \in E(S)$, $af = a$ implies $ef = e$.

Easy observation yields the following useful lemmas.

Lemma 2.3. *Let S be a semigroup with $e, f \in E(S)$. Then $e\ \mathcal{L}\ f$ if and only if $e\ \widetilde{\mathcal{L}}\ f$ and $e\ \mathcal{R}\ f$ if and only if $e\ \widetilde{\mathcal{R}}\ f$.*

Lemma 2.4. *Let S be a semigroup, and let $a \in S$, $f \in E(S)$ be such that $a\ \widetilde{\mathcal{R}}\ f$ but a is not $\mathcal{R}^*$-related to f. Then a is not $\mathcal{R}^*$-related to* any *idempotent of S.*

Proof. Suppose that $a\ \mathcal{R}^*\ e$ for some idempotent $e \in E(S)$. Then $a\ \widetilde{\mathcal{R}}\ e$, as $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$, so that $e\ \widetilde{\mathcal{R}}\ f$ by assumption, and so $e\ \mathcal{R}\ f$ by Lemma 2.3. Hence $a\ \mathcal{R}^*\ f$ as $\mathcal{R} \subseteq \mathcal{R}^*$, a contradiction. ☐

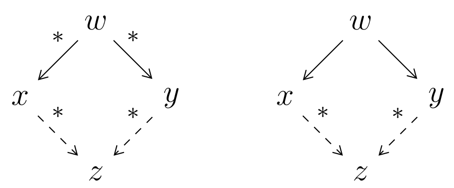
Lemma 2.5. *Let S be a semigroup with $a \in S$ and $e \in E(S)$ such that $a\ \widetilde{\mathcal{R}}\ e$. Then $a\ \mathcal{R}^*\ e$ if and only if for any $x, y \in S$, $xa = ya$ implies that $xe = ye$.*

Proof. Suppose that for all $x, y \in S$, if $xa = ya$ then $xe = ye$. By the dual of Lemma 2.1, we need only show that if $x \in S$ and $xa = a$, then $xe = e$. Suppose therefore that $x \in S$ and $xa = a$. As $a \, \widetilde{\mathcal{R}} \, e$, we have $xa = a = ea$, so that by assumption, $xe = ee = e$. □

We now recall the definition of reduction systems and their properties. As far as possible we follow standard notation and terminology, as may be found in [1].

Let A be a set and $\rightarrow$ a binary relation on A. We call the structure $(A, \rightarrow)$ a *reduction system* and the relation $\rightarrow$ a *reduction relation*. The reflexive, transitive closure of $\rightarrow$ is denoted by $\xrightarrow{*}$, while $\xleftrightarrow{*}$ denotes the smallest equivalence relation on A that contains $\rightarrow$. We denote the equivalence class of an element $x \in A$ by $[x]$. An element $x \in A$ is said to be *irreducible* if there is no $y \in A$ such that $x \rightarrow y$; otherwise, x is *reducible*. For any $x, y \in A$, if $x \xrightarrow{*} y$ and y is irreducible, then y is a *normal form* of x. A reduction system $(A, \rightarrow)$ is *noetherian* if there is no infinite sequence $x_0, x_1, \cdots \in A$ such that for all $i \geq 0$, $x_i \rightarrow x_{i+1}$.

We say that a reduction system $(A, \rightarrow)$ is *confluent* if whenever $w, x, y \in A$ are such that $w \xrightarrow{*} x$ and $w \xrightarrow{*} y$, then there is a $z \in A$ such that $x \xrightarrow{*} z$ and $y \xrightarrow{*} z$, as described by the figure below on the left, and $(A, \rightarrow)$ is *locally confluent* if whenever $w, x, y \in A$, are such that $w \rightarrow x$ and $w \rightarrow y$, then there is a $z \in A$ such that $x \xrightarrow{*} z$ and $y \xrightarrow{*} z$, as described by the figure below on the right.



Lemma 2.6. *[1] Let $(A, \rightarrow)$ be a reduction system.*

(i) If $(A, \rightarrow)$ is noetherian and confluent, then for each $x \in A$, $[x]$ contains a unique normal form.

(ii) If $(A, \rightarrow)$ is noetherian, then it is confluent if and only if it is locally confluent.

Let S be a semigroup with presentation $\langle X : u_i = v_i, i \in I \rangle$, where $u_i, v_i \in X^+$. We can form a reduction system $(X^+, \rightarrow)$ where

$$u \rightarrow v \Leftrightarrow (u = xu_iy, v = xv_iy \text{ for some } x, y \in X^*, i \in I).$$

It is clear that $\xleftrightarrow{*}$ is the congruence generated by $R = \{(u_i, v_i) : i \in I\}$. Thus if $\rightarrow$ is a confluent noetherian rewriting system then every element of S has a unique normal form as a word in X^+. Consequently, if X and I are finite, the word problem for S is decidable, that is, there is an effective procedure to determine when two elements of X^+ represent the same element of S.

Let E be a biordered set. Given that the reduction relations $\rightarrow$ corresponding to the presentation for $\mathrm{IG}(E)$ are clearly length reducing, we immediately deduce the next result.

Lemma 2.7. *Let E be a biordered set, and consider the presentation*

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef}, (e, f) \textit{ is a basic pair} \rangle.$$

Then $(\overline{E}^+, \rightarrow)$ forms a noetherian reduction system and $\mathrm{IG}(E)$ *is* $\overline{E}^+ / \xleftrightarrow{*}$.

3. Free idempotent generated semigroups over a biordered set with trivial basic products

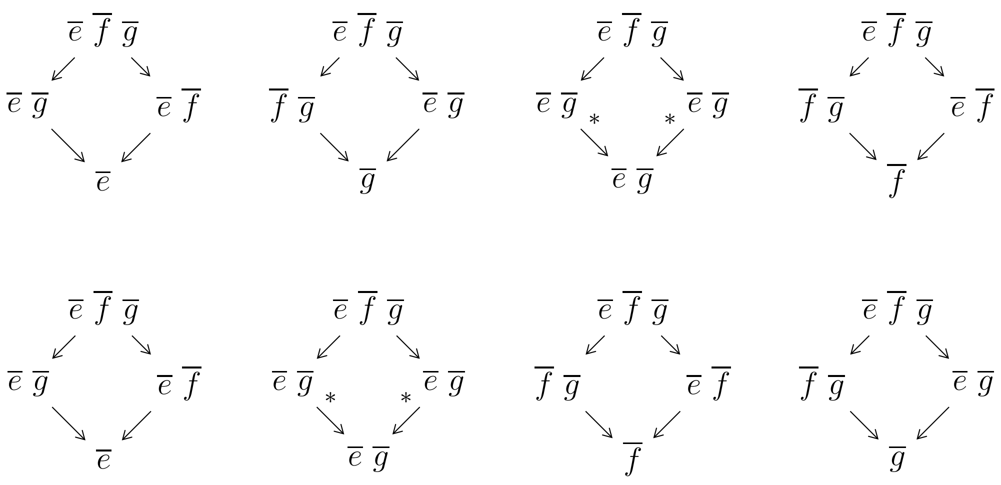
We start our investigation of free idempotent generated semigroups $\mathrm{IG}(E)$ over a biordered set E in the case that the basic products are *trivial*, that is, if (e, f) is a basic pair, then $\{ef, fe\} \subseteq \{e, f\}$. It is easy to see that in this case a pair (e, f) is basic if and only if $e \leq f$, $e \geq f$, $e \,\mathcal{L}\, f$ or $e \,\mathcal{R}\, f$. Clearly, semilattices and rectangular bands provide us with examples of biordered sets with trivial basic products; at the end of this section we consider exactly which bands give biordered sets with trivial basic products. Further, any poset may be regarded as a biordered set in which the quasi-orders coincide (see [15, Page 8]) and as such clearly has trival basic products. We prove below that if E is a biordered set with trivial basic products, then $\mathrm{IG}(E)$ is an abundant semigroup; however, it need not be regular.

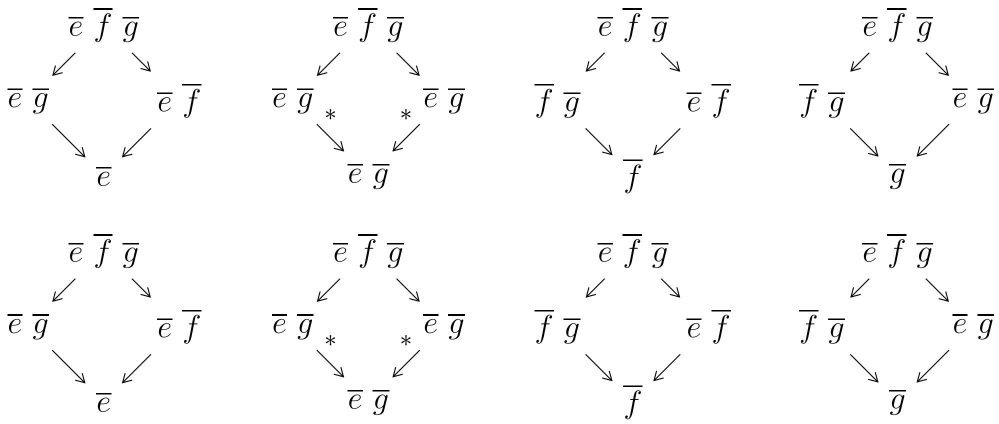
This article is not concerned with maximal subgroups of $\mathrm{IG}(E)$; however, it is easy to see that if E has trivial basic products then it has no non-trivial singular squares and hence from, for example, [16, Theorem 3], [2, Theorems 3.6 and 4.2], the maximal subgroups of $\mathrm{IG}(E)$ are all free groups. This result is known in some cases, for example, where E is the biordered set of a rectangular band [18].

The next result is well-known in the case that E is a poset (regarded as a biordered set in which the quasi-orders coincide with the partial order). Note that an element $\overline{x_1} \cdots \overline{x_n} \in \mathrm{IG}(E)$ is in normal form if and only if (x_i, x_{i+1}) is not basic, for all $i \in [1, n-1]$.

Lemma 3.1. *Let E be a biordered set with trivial basic products. Then every element in $\mathrm{IG}(E)$ has a unique normal form and consequently, if E is finite, then $\mathrm{IG}(E)$ has solvable word problem.*

Proof. In view of Lemmas 2.6 and 2.7, to show the required result we only need to argue that $(\overline{E}^+, \rightarrow)$ is locally confluent. For this purpose, it is sufficient to consider an arbitrary word of length 3, say $\overline{e}\,\overline{f}\,\overline{g} \in \overline{E}^+$, where (e, f) and (f, g) are basic. There are sixteen cases, namely, $e \leq f \leq g$, $e \geq f \geq g$, $e \leq f \geq g$, $e \geq f \leq g$, $e \,\mathcal{L}\, f \,\mathcal{L}\, g$, $e \,\mathcal{L}\, f \,\mathcal{R}\, g$, $e \,\mathcal{R}\, f \,\mathcal{L}\, g$, $e \,\mathcal{R}\, f \,\mathcal{R}\, g$, $e \,\mathcal{L}\, f \leq g$, $e \,\mathcal{L}\, f \geq g$, $e \,\mathcal{R}\, f \leq g$, $e \,\mathcal{R}\, f \geq g$, $e \leq f \,\mathcal{L}\, g$, $e \leq f \,\mathcal{R}\, g$, $e \geq f \,\mathcal{L}\, g$ and $e \geq f \,\mathcal{R}\, g$, for which we have the following 16 diagrams:





Thus $(\overline{E}^+, \to)$ is locally confluent, so that every element in $\mathrm{IG}(E)$ has a unique normal form. $\square$

For the result below it is convenient to use the relations $<$ and $>$ on a biordered set where $e < f$ if $e \leq f$ and $e \neq f$; similarly for $>$.

Theorem 3.2. *Let E be a biordered set with trivial basic products. The free idempotent generated semigroup* $\mathrm{IG}(E)$ *is abundant.*

Proof. We show that $\overline{e}\ \mathcal{R}^*\ \overline{e}\ \overline{f}$ for any $\overline{e}\ \overline{f} \in \mathrm{IG}(E)$ in normal form. Induction and duality yield $\overline{e_1}\ \mathcal{R}^*\ \overline{e_1}\cdots\overline{e_n}\ \mathcal{L}^*\ \overline{e_n}$, for any $\overline{e_1}\cdots\overline{e_n} \in \mathrm{IG}(E)$ in normal form; hence $\mathrm{IG}(E)$ is abundant. Clearly $\overline{e}\,\overline{e}\ \overline{f} = \overline{e}\ \overline{f}$. In view of the dual of Lemma 2.1, we must show that for any $X, Z \in IG(E)^1$, the equality $X\overline{e}\,\overline{f} = Z\overline{e}\,\overline{f}$ implies the equality $X\overline{e} = Z\overline{e}$.

Let $X = \overline{x_1}\cdots\ \overline{x_n}$ and $Y = \overline{e}\ \overline{f}$ be elements of $\mathrm{IG}(E)$ in normal form. We begin by considering the product XY. If (x_n, e) is not a basic pair then clearly $XY = \overline{x_1}\cdots\ \overline{x_n}\ \overline{e}\ \overline{f}$ is in normal form. Otherwise, (x_n, e) is basic and there are four cases to consider: $x_n > e$, $x_n < e$, $x_n\ \mathcal{L}\ e$ or $x_n\ \mathcal{R}\ e$.

Suppose that $x_n > e$ and hence $XY = \overline{x_1}\cdots\ \overline{x_{n-1}}\ \overline{e}\ \overline{f}$. Either $\overline{x_1}\cdots\ \overline{x_{n-1}}\ \overline{e}\ \overline{f}$ is in normal form, or (x_{n-1}, e) is basic (in which case $n \geq 2$). Notice that we cannot have $x_{n-1} < e$, $x_{n-1}\ \mathcal{L}\ e$ or $x_{n-1}\ \mathcal{R}\ e$, else (x_{n-1}, x_n), would be basic, contradicting the irreducibility of $\overline{x_1}\cdots\ \overline{x_n}$. Thus $x_{n-1} > e$ and continuing we obtain that XY has normal form $\overline{x_1}\cdots\ \overline{x_{t-1}}\ \overline{e}\ \overline{f}$, where $1 \leq t \leq n$, $x_n, \cdots, x_t > e$, and either $t = 1$ (in which case $\overline{x_1}\ \cdots\ \overline{x_{t-1}}$ is the empty product) or (x_{t-1}, e) is not basic. In this case we say that XY *reduces by* $>$. Similarly, if $x_n < e$, then XY has normal form $\overline{x_1}\cdots\ \overline{x_n}\ \overline{f}$ where (x_n, f) is not basic; or $\overline{x_1}\cdots\ \overline{x_n}$ where $x_n < e, f$. In this case we say that XY *reduces by* $<$.

If $x_n\ \mathcal{L}\ e$ then XY reduces to $\overline{x_1}\cdots\ \overline{x_n}\ \overline{f}$. The latter expression is in normal form if (x_n, f) is not basic; in the case it is in normal form we say that XY *reduces by* $\mathcal{L}$. On the other hand, if (x_n, f) is basic, then as above we can rule out three cases and deduce that $x_n\ \mathcal{R}\ f$ and so XY has normal form $\overline{x_1}\cdots\ \overline{x_{n-1}}\ \overline{f}$ with $e\ \mathcal{L}\ x_n\ \mathcal{R}\ f$ and (x_{n-1}, f) not basic, or $\overline{x_1}\cdots\ \overline{x_{n-1}}$ with $e\ \mathcal{L}\ x_n\ \mathcal{R}\ f\ \mathcal{L}\ x_{n-1}$. In the first case we say that XY *reduces by* $\mathcal{LR}$ and in the second that XY reduces by $\mathcal{LL}$.

Similarly, if $x_n\ \mathcal{R}\ e$, then the normal form of XY is $\overline{x_1}\cdots\ \overline{x_{n-1}}\ \overline{e}\ \overline{f}$ with (x_{n-1}, e) not basic; or $\overline{x_1}\cdots\ \overline{x_{n-1}}\ \overline{f}$ with $x_n\ \mathcal{R}\ e\ \mathcal{L}\ x_{n-1}$ and (x_{n-1}, f) not basic; or $\overline{x_1}\cdots\ \overline{x_{n-2}}\ \overline{f}$ with $x_n\ \mathcal{R}\ e\ \mathcal{L}\ x_{n-1}\ \mathcal{R}\ f$ and (x_{n-2}, f) not basic; or $\overline{x_1}\cdots\ \overline{x_{n-2}}$ with $x_n\ \mathcal{R}\ e\ \mathcal{L}\ x_{n-1}\ \mathcal{R}\ f\ \mathcal{L}\ x_{n-2}$, We say that XY *reduces by* $\mathcal{R}$, $\mathcal{RL}$, $\mathcal{RR}$ and $\mathcal{RL}$, respectively.

Suppose now that $X = \overline{x_1} \cdots \overline{x_n}, Z = \overline{z_1} \cdots \overline{z_k}$ and $Y = \overline{e}\,\overline{f} \in \mathrm{IG}(E)^1$ are in normal form such that

$$XY = ZY$$

in $\mathrm{IG}(E)$. Here we assume $n \geq 1, k \geq 0$. We proceed to prove that $X\overline{e} = Z\overline{e}$ in $\mathrm{IG}(E)$. To our end, we consider a number of cases, using symmetry to reduce the number of cases we mention explicitly.

Case $k = 0$: In this case $XY = Y$ and by uniqueness of normal forms we must have that (x_n, e) is basic. If XY reduces by $>$, then it has normal form

$$XY = \overline{x_1} \cdots \overline{x_{t-1}}\,\overline{e}\,\overline{f}$$

where $t = 1$ or (x_{t-1}, e) is not basic and $x_t, \cdots, x_n > e$. To avoid contradiction, we must have that $t = 1$ and then $X\overline{e} = \overline{e}$. On the other hand, XY cannot reduce by $<$, for if it did, it would have normal form

$$XY = \overline{x_1} \cdots \overline{x_n}\,\overline{f} \text{ or } XY = \overline{x_1} \cdots \overline{x_n}$$

where $x_n < e$ and (x_n, f) is not basic, or $x_n < e, f$. Since $XY = Y$, we must have $x_n = e$ or f, a contradiction.

If XY reduces by $\mathcal{R}$ to $\overline{x_1} \cdots \overline{x_{n-1}}\,\overline{e}\,\overline{f}$, then we must have that $n = 1$ and $x_1 \,\mathcal{R}\, e$. On the other hand, if XY reduces by a single step $\mathcal{L}$ to $\overline{x_1} \cdots \overline{x_n}\,\overline{f}$, we again have $n = 1$ and $x_1 = e$. In each case, $X\overline{e} = \overline{x_1}\,\overline{e} = \overline{e}$.

No other reduction is possible, for each case yields the contradiction that (x_n, x_{n-1}) or (x_{n-2}, x_{n-1}) is basic.

From this point, we assume that $k \geq 1$. By referring below to Case (A, B) the convention is that XY reduces as per procedure A and ZY as per procedure B, or if A or B is N then we mean that the original expression for XY or ZY is in normal form. Where it is easily seen to be possible we call upon duality to reduce the number of cases under consideration. For clarity we separate procedures $\mathcal{K}\mathcal{K}'$ where $\mathcal{K}, \mathcal{K}'$ are $\mathcal{R}$ or $\mathcal{L}$ from procedure $\mathcal{K}$ although in some cases some awkward conflation is possible.

Case (N, N): Here it is clear that $X = Z$ and so $X\overline{e} = Z\overline{e}$.

Case $(N, <)$: Here $XY = ZY$ in normal form (that is, each side of the equation is expressed in normal form) is

$$\overline{x_1} \cdots \overline{x_n}\,\overline{e}\,\overline{f} = \overline{z_1} \cdots \overline{z_k}\,\overline{f} \text{ or } \overline{x_1} \cdots \overline{x_n}\,\overline{e}\,\overline{f} = \overline{z_1} \cdots \overline{z_k}$$

where $z_k < e$ and (z_k, f) is not basic, or $z_k < e, f$. We deduce that $z_k = e$ or $z_k = f$, a contradiction.

Case $(N, >)$: Here $XY = ZY$ in normal form is

$$\overline{x_1} \cdots \overline{x_n}\,\overline{e}\,\overline{f} = \overline{z_1} \cdots \overline{z_{t-1}}\,\overline{e}\,\overline{f}$$

where $1 \leq t \leq k, z_k, \ldots, z_t > e$ and $t = 1$ or (z_{t-1}, e) is not basic. In this case $\overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_{t-1}}$ and so

$$X\overline{e} = \overline{x_1} \cdots \overline{x_n}\,\overline{e} = \overline{z_1} \cdots \overline{z_{t-1}}\,\overline{e} = \overline{z_1} \cdots \overline{z_{t-1}}\,\overline{z_t} \cdots \overline{z_k}\,\overline{e} = Z\overline{e}$$

as required.

Case $(>, <)$: If this case held, we would have $XZ = YZ$ expressed in normal form as

$$\overline{x_1} \cdots \overline{x_{s-1}}\,\overline{e}\,\overline{f} = \overline{z_1} \cdots \overline{z_k}\,\overline{f} \text{ or } \overline{x_1} \cdots \overline{x_{s-1}}\,\overline{e}\,\overline{f} = \overline{z_1} \cdots \overline{z_k}$$

where $1 \le s \le n, x_n, \dots, x_s > e, s = 1$ or (x_{s-1}, e) is not basic, and $z_k < e, (z_k, f)$ is not basic, or $z_k < e, f$. This would give $z_k = e$ or $z_k = f$, a contradiction.

Case $(>, >)$: Here we have $XZ = YZ$ expressed in normal form as

$$\overline{x_1} \cdots \overline{x_{s-1}}\, \overline{e}\, \overline{f} = \overline{z_1} \cdots \overline{z_{t-1}}\, \overline{e}\, \overline{f}$$

where $1 \le s \le n, x_n, \dots, x_s > e, s = 1$ or (x_{s-1}, e) is not basic, and $1 \le t \le k, z_k, \dots, z_t > e$ and $t = 1$ or (z_{t-1}, e) is not basic. We deduce that $t = s$ and $x_i = z_i$, for $1 \le i \le t-1$. Then

$$X\overline{e} = \overline{x_1} \cdots \overline{x_{s-1}}\, \overline{x_s} \cdots \overline{x_n}\, \overline{e} = \overline{x_1} \cdots \overline{x_{s-1}}\, \overline{e} = \overline{z_1} \cdots \overline{z_{t-1}}\, \overline{e} = \overline{z_1} \cdots \overline{z_{t-1}}\, \overline{z_t} \cdots \overline{z_k}\, \overline{e} = Z\overline{e}.$$

Case $(<, <)$: In this case $XY = ZY$ has normal form

$$\overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_k} \text{ or } \overline{x_1} \cdots \overline{x_n}\, \overline{f} = \overline{z_1} \cdots \overline{z_k}\, \overline{f} \text{ or } \overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_k}\, \overline{f} \text{ or } \overline{x_1} \cdots \overline{x_n}\, \overline{f} = \overline{z_1} \cdots \overline{z_k}$$

where the respective constraints are: $x_n < e, f$ and $z_k < e, f$, or $x_n < e, (x_n, f)$ is not basic and $z_k < e, (z_k, f)$ is not basic, or $x_n < e, f$ and $z_k < e, (z_k, f)$ is not basic, or $x_n < e, (x_n, f)$ is not basic and $z_k < e, f$. In the first two cases we must have $X = Z$, and so $X\overline{e} = Z\overline{e}$. In the last two cases we have x_n or $z_k = f < e$, a contradiction.

Case $(N, \mathcal{R})$: We obtain $XY = ZY$ in normal form as

$$\overline{x_1} \cdots \overline{x_n}\, \overline{e}\, \overline{f} = \overline{z_1} \cdots \overline{z_{k-1}}\, \overline{e}\, \overline{f}$$

where $z_k \,\mathcal{R}\, e$. By uniqueness of normal forms we have $\overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_{k-1}}$ so that using the fact that $z_k \,\mathcal{R}\, e$,

$$X\overline{e} = \overline{x_1} \cdots \overline{x_n}\, \overline{e} = \overline{z_1} \cdots \overline{z_{k-1}}\, \overline{e} = \overline{z_1} \cdots \overline{z_{k-1}}\, \overline{z_k}\, \overline{e} = Z\overline{e}.$$

Case $(N, \mathcal{L})$: We obtain $XY = ZY$ in normal form as

$$\overline{x_1} \cdots \overline{x_n}\, \overline{e}\, \overline{f} = \overline{z_1} \cdots \overline{z_k}\, \overline{f}$$

where $e \,\mathcal{L}\, z_k$. We obtain that $z_k = e$ and $\overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_{k-1}}$, whence $X\overline{e} = Z\overline{e}$ as required.

Case $(N, \mathcal{LL})$: We obtain $XY = ZY$ in normal form as

$$\overline{x_1} \cdots \overline{x_n}\, \overline{e}\, \overline{f} = \overline{z_1} \cdots \overline{z_{k-1}}$$

where $e \,\mathcal{L}\, z_k \,\mathcal{R}\, f \,\mathcal{L}\, z_{k-1}$. But then $z_{k-1} = f \;\mathcal{R}\; z_k$, a contradiction.

Cases $(N, \mathcal{LR})$, $(N, \mathcal{RL})$, $(N, \mathcal{RR})$: These all lead to a contradiction, in a similar fashion to Case $(N, \mathcal{LL})$.

We remark that, in view of symmetry, we have now dealt with all cases where A or B is N.

Case $(<, \mathcal{L})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1} \cdots \overline{x_n}\, \overline{f} = \overline{z_1} \cdots \overline{z_k}\, \overline{f} \text{ or } \overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_k}\, \overline{f}$$

where $z_k \;\mathcal{L}\; e$ and $x_n < e$. In the first case we have $X = Z$, and so $X\overline{e} = Z\overline{e}$. In the second case, we have $x_n = f < e$, a contradiction.

Case $(<, \mathcal{R})$: Here we would have $XY = ZY$ in normal form as

$$\overline{x_1} \cdots \overline{x_n}\, \overline{f} = \overline{z_1} \cdots \overline{z_{k-1}}\, \overline{e}\, \overline{f} \text{ or } \overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_{k-1}}\, \overline{e}\, \overline{f}$$

where $z_k \;\mathcal{R}\; e$ and $x_n < e$ with (x_n, f) not basic, or $x_n < e, f$. But our conditions force $x_n = e$ in the first case or $x_n = f$ in the second, yielding a contradiction.

Case $(<, \mathcal{LL})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1} \cdots \overline{x_n}\, \overline{f} = \overline{z_1} \cdots \overline{z_{k-1}} \text{ or } \overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_{k-1}}$$

where $e\,\mathcal{L}\,z_k\,\mathcal{R}\,f\,\mathcal{L}\,z_{k-1}$ and $x_n < e$ with (x_n, f) not basic, or $x_n < e, f$. In the first case $z_{k-1} = f\ \mathcal{R}\ z_k$, a contradiction. In the second we have the contradiction $x_n = z_{k-1}\ \mathcal{L}\ f$.

Cases $(<, \mathcal{LR})$, $(<, \mathcal{RR})$, $(<, \mathcal{RL})$ similarly do not occur.

We remark that, in view of symmetry, we have now dealt with all cases where A or B is N or $<$.

Case $(>, \mathcal{L})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1}\cdots\overline{x_{s-1}}\,\overline{e}\ \overline{f} = \overline{z_1}\cdots\overline{z_k}\,\overline{f}$$

where $1 \leq s \leq n, x_n, \ldots, x_s > e, s = 1$ or (x_{s-1}, e) is not basic, and $z_k\,\mathcal{L}\,e$. Clearly we must have that $z_k = e$ and $\overline{x_1}\cdots\overline{x_{s-1}} = \overline{z_1}\cdots\overline{z_{k-1}}$, whence familiar arguments give that $X\overline{e} = Z\overline{e}$.

Case $(>, \mathcal{R})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1}\cdots\overline{x_{s-1}}\,\overline{e}\ \overline{f} = \overline{z_1}\cdots\overline{z_{k-1}}\,\overline{e}\ \overline{f}$$

where $1 \leq s \leq n, x_n, \ldots, x_s > e, s = 1$ or (x_{s-1}, e) is not basic, and $z_k\,\mathcal{R}\,e$. Now $\overline{x_1}\cdots\overline{x_{s-1}} = \overline{z_1}\cdots\overline{z_{k-1}}$. Making use of the fact that $\overline{z_k}\,\overline{e} = \overline{e}$, we deduce that $X\overline{e} = Z\overline{e}$.

Case $(>, \mathcal{LL})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1}\cdots\overline{x_{s-1}}\,\overline{e}\ \overline{f} = \overline{z_1}\cdots\overline{z_{k-1}}$$

where $1 \leq s \leq n, x_n, \ldots, x_s > e, s = 1$ or (x_{s-1}, e) is not basic, and $e\,\mathcal{L}\,z_k\,\mathcal{R}\,f\,\mathcal{L}\,z_{k-1}$. But this gives that $z_{k-1} = f\ \mathcal{R}\ z_k$, a contradiction.

Cases $(>, \mathcal{LR})$, $(>, \mathcal{RR})$, $(>, \mathcal{RL})$ similarly do not occur.

We remark that, in view of symmetry, we have now dealt with all cases where A or B is $N, <$ or $>$.

Case $(\mathcal{R}, \mathcal{R})$: Here the normal form $XY = ZY$ is

$$\overline{x_1}\cdots\overline{x_{n-1}}\ \overline{e}\ \overline{f} = \overline{z_1}\cdots\overline{z_{k-1}}\ \overline{e}\ \overline{f},$$

where $x_n\,\mathcal{R}\,e\,\mathcal{R}\,z_k$. By the latter, and uniqueness of normal forms, we have

$$X\overline{e} = \overline{x_1}\cdots\overline{x_n}\ \overline{e} = \overline{x_1}\cdots\overline{x_{n-1}}\ \overline{e} = \overline{z_1}\cdots\overline{z_{k-1}}\ \overline{e} = \overline{z_1}\cdots\overline{z_k}\ \overline{e} = Z\overline{e}.$$

Case $(\mathcal{R}, \mathcal{L})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1}\cdots\overline{x_{n-1}}\ \overline{e}\ \overline{f} = \overline{z_1}\cdots\overline{z_k}\ \overline{f}$$

where $x_n\,\mathcal{R}\,e\,\mathcal{L}\,z_k$. We obtain $k = n$ and $\overline{x_1}\cdots\overline{x_{n-1}}\ \overline{e} = \overline{z_1}\cdots\overline{z_k}$ and then

$$X\overline{e} = \overline{x_1}\cdots\overline{x_n}\ \overline{e} = \overline{x_1}\cdots\overline{x_{n-1}}\ \overline{e} = \overline{z_1}\cdots\overline{z_k} = \overline{z_1}\cdots\overline{z_k}\ \overline{e} = Z\overline{e}.$$

Case $(\mathcal{L}, \mathcal{L})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1}\cdots\overline{x_n}\ \overline{f} = \overline{z_1}\cdots\overline{z_k}\ \overline{f}$$

where $x_n\,\mathcal{L}\,e\,\mathcal{L}\,z_k$. Immediately we obtain $X = Z$ and $X\overline{e} = Z\overline{e}$.

We remark that, in view of symmetry, we have now dealt with all cases where both A and B are $\mathcal{L}$ or $\mathcal{R}$.

Case $(\mathcal{R}, \mathcal{RR})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1}\cdots\overline{x_{n-1}}\ \overline{e}\ \overline{f} = \overline{z_1}\cdots\ \overline{z_{k-2}}\ \overline{f}$$

where $z_k\ \mathcal{R}\ e\ \mathcal{L}\ z_{k-1}\ \mathcal{R}\ f$ and (z_{k-2}, f) is not basic. But this gives that $z_{k-2} = e\ \mathcal{L}\ z_{k-1}$, a contradiction.

Cases $(\mathcal{R},\mathcal{LR})$, $(\mathcal{R},\mathcal{RL})$, $(\mathcal{R},\mathcal{LL})$ similarly do not occur.

We remark that, in view of symmetry, we have now dealt with all cases where one of A or B is $N,<,>$ or $\mathcal{R}$.

Case $(\mathcal{L},\mathcal{LR})$: Here we have $XY=ZY$ in normal form as

$$\overline{x_1}\cdots\overline{x_n}\,\overline{f}=\overline{z_1}\cdots\ \overline{z_{k-1}}\,\overline{f}$$

where $e\,\mathcal{L}\,x_n$ and $e\ \mathcal{L}\ z_k\ \mathcal{R}\ f$ and (z_{k-1},f) is not basic. But this gives $z_{k-1}=x_n\ \mathcal{L}\ e\ \mathcal{L}\ z_k$, a contradiction.

Case $(\mathcal{L},\mathcal{LL})$: In normal form $XY=ZY$ is

$$\overline{x_1}\cdots\overline{x_n}\,\overline{f}=\overline{z_1}\cdots\overline{z_{k-1}}$$

where $e\,\mathcal{L}\,x_n$ and $e\,\mathcal{L}\,z_k\,\mathcal{R}\,f\,\mathcal{L}\,z_{k-1}$. We must have $z_{k-1}=f\ \mathcal{R}\ z_k$, a contradiction.

Case $(\mathcal{L},\mathcal{RL})$: Here we have $XY=ZY$ in normal form as

$$\overline{x_1}\cdots\overline{x_n}\,\overline{f}=\overline{z_1}\cdots\ \overline{z_{k-2}}\text{ or }\overline{x_1}\cdots\overline{x_n}\,\overline{f}=\overline{z_1}\cdots\ \overline{z_{k-1}}\,\overline{f}$$

where $e\,\mathcal{L}\,x_n$, and $z_k\ \mathcal{R}\ e\ \mathcal{L}\ z_{k-1}\ \mathcal{R}\ f\ \mathcal{L}\ z_{k-2}$ or $z_k\ \mathcal{R}\ e\ \mathcal{L}\ z_{k-1}$. The first case gives us $z_{k-2}=f\ \mathcal{R}\ z_{k-1}$, a contradiction. The second case leads to $\overline{x_1}\cdots\overline{x_n}=\overline{z_1}\cdots\overline{z_{k-1}}$, and so as $z_k\ \mathcal{R}\ e$ we have

$$X\overline{e}=\overline{x_1}\cdots\overline{x_n}\,\overline{e}=\overline{z_1}\cdots\overline{z_{k-1}}\,\overline{e}=\overline{z_1}\cdots\overline{z_k}\,\overline{e}=Z\overline{e}.$$

Case $(\mathcal{L},\mathcal{RR})$: Here we have $XY=ZY$ in normal form as

$$\overline{x_1}\cdots\overline{x_n}\,\overline{f}=\overline{z_1}\cdots\ \overline{z_{k-2}}\,\overline{f}$$

where $e\ \mathcal{L}\ x_n$ and $z_k\ \mathcal{R}\ e\ \mathcal{L}\ z_{k-1}\ \mathcal{R}\ f$ with (z_{k-2},f) not basic. But this gives $z_{k-2}=x_n\ \mathcal{L}\ e\ \mathcal{L}\ z_{k-1}$, a contradiction.

We remark that, in view of symmetry, we have now dealt with all cases where A or B is $N,<,>,\mathcal{R}$ or $\mathcal{L}$.

Case $(\mathcal{RR},\mathcal{RR})$: Here we have $XY=ZY$ in normal form as

$$\overline{x_1}\cdots\ \overline{x_{n-2}}\,\overline{f}=\overline{z_1}\cdots\ \overline{z_{k-2}}\,\overline{f}$$

where $x_n\ \mathcal{R}\ e\ \mathcal{L}\ x_{n-1}\ \mathcal{R}\ f$ and $z_k\ \mathcal{R}\ e\ \mathcal{L}\ z_{k-1}\ \mathcal{R}\ f$ with (x_{n-2},f) and (z_{k-2},f) not basic. Since $E=E(S)$ for a semigroup S, and idempotents of (group) $\mathcal{H}$-classes are unique, $x_{n-1}\ \mathcal{L}\ e\ \mathcal{L}\ z_{k-1},x_{n-1}\ \mathcal{R}\ f\ \mathcal{R}\ z_{k-1}$ gives us $x_{n-1}=z_{k-1}$. Uniqueness of normal forms gives $\overline{x_1}\cdots\ \overline{x_{n-2}}=\overline{z_1}\cdots\ \overline{z_{k-2}}$, so $\overline{x_1}\cdots\overline{x_{n-1}}=\overline{z_1}\cdots\overline{z_{k-1}}$, and

$$X\overline{e}=\overline{x_1}\cdots\overline{x_{n-1}}\,\overline{e}=\overline{z_1}\cdots\overline{z_{k-1}}\,\overline{e}=Z\overline{e}.$$

Case $(\mathcal{RR},\mathcal{LR})$: Here we have $XY=ZY$ in normal form as

$$\overline{x_1}\cdots\ \overline{x_{n-2}}\,\overline{f}=\overline{z_1}\cdots\ \overline{z_{k-1}}\,\overline{f}$$

where $x_n\,\mathcal{R}\,e\,\mathcal{L}\,x_{n-1}\,\mathcal{R}\,f$ and $e\ \mathcal{L}\ z_k\ \mathcal{R}\ f$ with (z_{k-1},f) not basic. Similarly to Case $(\mathcal{RR},\mathcal{RR})$ we deduce that $x_{n-1}=z_k$. Also, note that $\overline{x_1}\cdots\ \overline{x_{n-2}}=\overline{z_1}\cdots\ \overline{z_{k-1}}$, so $\overline{x_1}\cdots\ \overline{x_{n-1}}=\overline{z_1}\cdots\ \overline{z_k}$, giving $X\overline{e}=\overline{x_1}\cdots\overline{x_{n-1}}\,\overline{e}=\overline{z_1}\cdots\ \overline{z_k}\,\overline{e}=Z\overline{e}$.

Case $(\mathcal{RR},\mathcal{LL})$: Here we have $XY=ZY$ in normal form as

$$\overline{x_1}\cdots\ \overline{x_{n-2}}\,\overline{f}=\overline{z_1}\cdots\overline{z_{k-1}}$$

where $x_n\ \mathcal{R}\ e\ \mathcal{L}\ x_{n-1}\ \mathcal{R}\ f$ and $e\,\mathcal{L}\,z_k\,\mathcal{R}\,f\,\mathcal{L}\,z_{k-1}$. But this implies that $z_{k-1}=f\ \mathcal{R}\ z_k$, a contradiction.

Cases $(\mathcal{KR}, \mathcal{K}'\mathcal{L})$ *for* $\mathcal{K}, \mathcal{K}' = \mathcal{L}$ *or* $\mathcal{R}$: These cases are entirely similar to Case $(\mathcal{RR}, \mathcal{LL})$.

We remark that, in view of the duality, we have now dealt with all cases where A or B is $N, <, >, \mathcal{R}, \mathcal{L}$ or $\mathcal{RR}$.

Case $(\mathcal{LR}, \mathcal{LR})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1} \cdots \overline{x_{n-1}}\ \overline{f} = \overline{z_1} \cdots \overline{z_{k-1}}\ \overline{f}$$

where $e \mathrel{\mathcal{L}} x_n \mathrel{\mathcal{R}} f$ and $e \mathrel{\mathcal{L}} z_k \mathrel{\mathcal{R}} f$. Again, this implies $x_n = z_k$. As $\overline{x_1} \cdots \overline{x_{n-1}} = \overline{z_1} \cdots \overline{z_{k-1}}$ by uniqueness, we have $\overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_k}$, and so $X\overline{e} = Z\overline{e}$.

We remark that, in view of symmetry and earlier remarks, we have now dealt with all cases where A or B is $N, <, >, \mathcal{R}, \mathcal{L}, \mathcal{RR}$ or $\mathcal{LR}$.

Case $(\mathcal{LL}, \mathcal{LL})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1} \cdots \overline{x_{n-1}} = \overline{z_1} \cdots \overline{z_{k-1}}$$

where $e \mathrel{\mathcal{L}} x_n \mathrel{\mathcal{R}} f \mathrel{\mathcal{L}} x_{n-1}$ and $e \mathrel{\mathcal{L}} z_k \mathrel{\mathcal{R}} f \mathrel{\mathcal{L}} z_{k-1}$. This implies $x_n = z_k$, so $X = Z$ and the case is done.

Case $(\mathcal{LL}, \mathcal{RL})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1} \cdots \overline{x_{n-1}} = \overline{z_1} \cdots \overline{z_{k-1}}\ \overline{f} \text{ or } \overline{x_1} \cdots \overline{x_{n-1}} = \overline{z_1} \cdots \overline{z_{k-2}}$$

where $e \mathrel{\mathcal{L}} x_n \mathrel{\mathcal{R}} f \mathrel{\mathcal{L}} x_{n-1}$, and $z_k \mathrel{\mathcal{R}} e \mathrel{\mathcal{L}} z_{k-1}$ with (z_{k-1}, f) not basic, or $z_k \mathrel{\mathcal{R}} e \mathrel{\mathcal{L}} z_{k-1} \mathrel{\mathcal{R}} f \mathrel{\mathcal{L}} z_{k-2}$. In the first case, we have $x_{n-1} = f \mathrel{\mathcal{R}} x_n$, a contradiction. In the second case $x_n = z_{k-1}$, so $\overline{x_1} \cdots \overline{x_{n-1}}\ \overline{x_n} = \overline{z_1} \cdots \overline{z_{k-2}}\ \overline{z_{k-1}}$ and

$$X\overline{e} = \overline{z_1} \cdots \overline{z_{k-2}}\ \overline{z_{k-1}}\ \overline{e} = Z\overline{e}.$$

We are left with one case remaining.

Case $(\mathcal{RL}, \mathcal{RL})$: Here we have $XY = ZY$ in normal form as

$$\overline{x_1} \cdots \overline{x_{n-2}} = \overline{z_1} \cdots \overline{z_{k-2}} \text{ or } \overline{x_1} \cdots \overline{x_{n-1}}\ \overline{f} = \overline{z_1} \cdots \overline{z_{k-1}}\ \overline{f}$$

or

$$\overline{x_1} \cdots \overline{x_{n-2}} = \overline{z_1} \cdots \overline{z_{k-1}}\ \overline{f} \text{ or } \overline{x_1} \cdots \overline{x_{n-1}}\ \overline{f} = \overline{z_1} \cdots \overline{z_{k-2}}$$

where $x_n \mathrel{\mathcal{R}} e \mathrel{\mathcal{L}} x_{n-1} \mathrel{\mathcal{R}} f \mathrel{\mathcal{L}} x_{n-2}$ and $z_k \mathrel{\mathcal{R}} e \mathrel{\mathcal{L}} z_{k-1} \mathrel{\mathcal{R}} f \mathrel{\mathcal{L}} z_{k-2}$, or $x_n \mathrel{\mathcal{R}} e \mathrel{\mathcal{L}} x_{n-1}$ and $z_k \mathrel{\mathcal{R}} e \mathrel{\mathcal{L}} z_{k-1}$, or $x_n \mathrel{\mathcal{R}} e \mathrel{\mathcal{L}} x_{n-1} \mathrel{\mathcal{R}} f \mathrel{\mathcal{L}} x_{n-2}$ and $z_k \mathrel{\mathcal{R}} e \mathrel{\mathcal{L}} z_{k-1}$, or $x_n \mathrel{\mathcal{R}} e \mathrel{\mathcal{L}} x_{n-1}$ and $z_k \mathrel{\mathcal{R}} e \mathrel{\mathcal{L}} z_{k-1} \mathrel{\mathcal{R}} f \mathrel{\mathcal{L}} z_{k-2}$. Familiar arguments give that in the first two cases $\overline{x_1} \cdots \overline{x_{n-1}} = \overline{z_1} \cdots \overline{z_{k-1}}$ and then $X\overline{e} = Z\overline{e}$, and the second two cases lead to contradictions. □

We remark that if E is a biordered set with trivial basic products, then Theorem 3.2 immediately tells us what are the $\mathcal{R}^*$- and $\mathcal{L}^*$-classes in $\mathrm{IG}(E)$. For $e \in E$ the $\mathcal{R}^*$-class of $\overline{e}$ is

$$\{\overline{e}\ \overline{e_1} \cdots \overline{e_m} : m \geq 0, e_i \in E, 1 \leq i \leq m, \overline{e}\ \overline{e_1} \cdots \overline{e_m} \text{ is in normal form}\}$$

with $\mathcal{L}^*$-class of $\overline{e}$ being defined dually.

Example 3.3. *[2, Example 2]* Let $Y = \{e, f, g\}$ be the semilattice with $e, f \geq g$ and e, f incomparable. Then $\mathrm{IG}(Y)$ is not regular.

Proof. First, we observe that

$$\mathrm{IG}(Y) = \{\overline{e}, \overline{f}, \overline{g}, (\overline{e}\ \overline{f})^n, (\overline{f}\ \overline{e})^n, (\overline{e}\ \overline{f})^n\ \overline{e}, (\overline{f}\ \overline{e})^n\ \overline{f} :\ n \in \mathbb{N}\}.$$

It is easy to check that for any $n \in \mathbb{N}$, $(\overline{e}\ \overline{f})^n \in \mathrm{IG}(Y)$ is not regular, as for any $w \in \mathrm{IG}(Y)$, $(\overline{e}\ \overline{f})^n w(\overline{e}\ \overline{f})^n = \overline{g}$ if w contains $\overline{g}$ as a letter; otherwise $(\overline{e}\ \overline{f})^n w(\overline{e}\ \overline{f})^n = (\overline{e}\ \overline{f})^m$ for some $m \geq 2n \in \mathbb{N}$. Therefore, $\mathrm{IG}(Y)$ is not a regular semigroup. □

On the other hand, by Theorem 3.2 we have that $\mathrm{IG}(Y)$ is an abundant semigroup. Furthermore, the $\mathcal{R}^*$-classes are

$$\{\overline{e}, (\overline{e}\ \overline{f})^n, (\overline{e}\ \overline{f})^n\ \overline{e} :\ n \in \mathbb{N}\}, \{\overline{f}, (\overline{f}\ \overline{e})^n, (\overline{f}\ \overline{e})^n\ \overline{f} :\ n \in \mathbb{N}\}, \{\overline{g}\}$$

and the $\mathcal{L}^*$-classes are

$$\{\overline{e}, (\overline{f}\ \overline{e})^n, (\overline{e}\ \overline{f})^n\ \overline{e} :\ n \in \mathbb{N}\}, \{\overline{f}, (\overline{e}\ \overline{f})^n, (\overline{f}\ \overline{e})^n\ \overline{f} :\ n \in \mathbb{N}\}, \{\overline{g}\}.$$

Note that we have

$$\mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$$

in $\mathrm{IG}(Y)$, and there are two $\mathcal{D}^*$-classes of $\mathrm{IG}(Y)$, namely, $\{\overline{g}\}$ and $\mathrm{IG}(Y) \setminus \{\overline{g}\}$, the latter of which can be depicted by the following $*$-analogue of a traditional egg-box picture:

$\overline{e}, (\overline{e}\ \overline{f})^n\overline{e}$	$(\overline{e}\ \overline{f})^n$
$(\overline{f}\ \overline{e})^n$	$\overline{f}, (\overline{f}\ \overline{e})^n\overline{f}$

We commented earlier that if B is a rectangular band then it has trivial basic products, and so $\mathrm{IG}(B)$ is abundant by Theorem 3.2. In fact it is well known [18, Theorem 6.4] that $\mathrm{IG}(B)$ is what is known as a completely simple semigroup, which immediately tells us it is regular and further, that for any element $\overline{e_1}\,\overline{e_2} \cdots \overline{e_n} \in \mathrm{IG}(B)$ we have that $\overline{e_1}\,\mathcal{R}\,\overline{e_1}\,\overline{e_2} \cdots \overline{e_n}\,\mathcal{L}\,\overline{e_n}$. In addition, Pastijn shows that the maximal subgroups of $\mathrm{IG}(B)$ in this case are free groups and determines their rank.

Corollary 3.4. *Let B be a semilattice Y of rectangular bands B_α, $\alpha \in Y$. Then for any $x_1, \cdots, x_n \in B_\alpha$, we have $\overline{x_1}\,\mathcal{R}\,\overline{x_1} \cdots \overline{x_n}\,\mathcal{L}\,\overline{x_n}$ in $\mathrm{IG}(B)$. Consequently, $\overline{x_1} \cdots \overline{x_n}$ is a regular element of $\mathrm{IG}(B)$.*

Proof. It is clear from the presentations of $\mathrm{IG}(B_\alpha)$ and $\mathrm{IG}(B)$ that there is a well defined morphism

$$\overline{\psi} : \mathrm{IG}(B_\alpha) \to \mathrm{IG}(B), \text{ such that } \overline{e}\ \overline{\psi} = \overline{e}$$

for each $e \in B_\alpha$. It suffices to recall that $\overline{\psi}$ preserves Green's relations and use the remark preceding the corollary. □

We now characterise those bands such that the corresponding biordered sets have trivial basic products.

Proposition 3.5. *The following conditions are equivalent for a band $B = \bigcup_{\alpha \in Y} B_\alpha$:*

1. *the biordered set B has trivial basic products;*
2. *for all $\alpha, \beta \in Y$ with $\beta > \alpha$, $u \in B_\alpha$ and $v \in B_\beta$, we have $uv = vu = u$;*
3. *for all $\alpha \in Y$ and $e \in B_\alpha$, the subsemigroup $eBe = \{e\} \cup \bigcup_{\beta < \alpha} B_\beta$.*

Proof. Suppose that (1) holds. Let $u \in B_\alpha$ and $v \in B_\beta$ where $\beta > \alpha$. It is clear that $u \,\mathcal{L}\, vu \leq_{\mathcal{R}} v$ so that (vu, v) is a basic pair, but vu is not $\mathcal{D}$-related to v. From a comment at the beginning of the section, we deduce that $vu < v$, so that $vuv = vu$. Dually we obtain $vuv = uv \,\mathcal{R}\, u$, so that $u = uv = vu$ and (2) holds.

The implication from (2) to (1) is clear from the structure of B as a semilattice of rectangular bands. The equivalence of (2) and (3) is immediate. □

In view of Proposition 3.5, a zero-direct union of bands with trivial basic products has the same property. Subsemigroups of the form eBe in Proposition 3.5 are known as *local* subsemigroups; thus if B has trivial basic products then the local subsemigroups are as large as possible. Such bands form a rather restricted class, since if $\alpha = \beta\gamma$ where $\beta \neq \alpha \neq \gamma$, then we must have that B_α is trivial. To see this, choose any $e \in B_\beta, f \in B_\gamma$ and $g \in B_\alpha$. Then $efg = g = gef$ so that $g = ef$. If B is normal, then to have trivial basic products is equivalent to B_α being trivial for all non-maximal $\alpha \in Y$. We also note that any band B with trivial basic products lies in the variety of *regular bands*, that is, it satisfies the identity $xyxzx = xyzx$. To see this, let $x \in B_\alpha, y \in B_\beta$ and $z \in B_\gamma$. If $\alpha = \alpha\beta\gamma$, clearly $xyxzx = x = xyzx$. Otherwise, $\alpha > \alpha\beta\gamma$ and

$$xyxzx = x(yxz)(yxz)x = xyx(zyx)zx = xy(zyx)zx = (xyzy)xzx = xyzyzx = xyzx.$$

4. Free idempotent generated semigroups over bands

Our aim here is to investigate the general structure of $\mathrm{IG}(B)$ for an arbitrary band B. As is our convention, we may assume without comment that $B = \bigcup_{\alpha \in Y} B_\alpha$ is a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$. We prove that for any such B, the semigroup $\mathrm{IG}(B)$ is weakly abundant with the congruence condition. However, we demonstrate a band B for which $\mathrm{IG}(B)$ is not abundant.

Lemma 4.1. *Let S and T be semigroups with biordered sets of idempotents $U = E(S)$ and $V = E(T)$, respectively, and let $\theta : S \to T$ be a morphism. Then the map from $\overline{U}$ to $\overline{V}$ defined by $\overline{e} \mapsto \overline{e\theta}$, for all $e \in U$, lifts to a well defined morphism $\overline{\theta} : \mathrm{IG}(U) \to \mathrm{IG}(V)$.*

Proof. Since θ is a morphism by assumption, we have that (e, f) is a basic pair in U implies $(e\theta, f\theta)$ is a basic pair in V, so that there exists a morphism $\overline{\theta} : \mathrm{IG}(U) \to \mathrm{IG}(V)$ defined by $\overline{e}\,\overline{\theta} = \overline{e\theta}$, for all $e \in U$. □

Let B be a band. The mapping θ defined by

$$\theta : B \to Y, x \mapsto \alpha$$

where $x \in B_\alpha$, is a morphism with kernel $\mathcal{D}$. Applying Lemma 4.1 to this θ, we have the following corollary.

Corollary 4.2. *The map $\overline{\theta} : \mathrm{IG}(B) \to \mathrm{IG}(Y)$ defined by*

$$(\overline{x_1} \cdots \overline{x_n})\,\overline{\theta} = \overline{\alpha_1} \cdots \overline{\alpha_n}$$

is a morphism, where $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$.

To proceed further we need the following definition of *left to right significant indices* of elements in $\mathrm{IG}(B)$.

Let $\overline{x_1} \cdots \overline{x_n} \in \overline{B}^+$ with $x_i \in B_{\alpha_i}$, for all $1 \leq i \leq n$. Then a set of numbers

$$\{i_1, \cdots, i_r\} \subseteq [1, n] \text{ with } i_1 < \cdots < i_r$$

is called the *left to right significant indices* of $\overline{x_1} \cdots \overline{x_n}$, if these numbers are picked out in the following manner:

i_1 : the largest number such that $\alpha_1, \cdots, \alpha_{i_1} \geq \alpha_{i_1}$;
k_1 : the largest number such that $\alpha_{i_1} \leq \alpha_{i_1}, \alpha_{i_1+1}, \cdots, \alpha_{k_1}$.

We pause here to remark that $\alpha_{i_1}, \alpha_{k_1+1}$ are incomparable. This is because, if $\alpha_{i_1} \leq \alpha_{k_1+1}$, then we add 1 to k_1, contradicting the choice of k_1; and if $\alpha_{i_1} > \alpha_{k_1+1}$, then $\alpha_1, \cdots, \alpha_{i_1}, \cdots, \alpha_{k_1} \geq \alpha_{k_1+1}$, contradicting the choice of i_1. Now we continue our process:

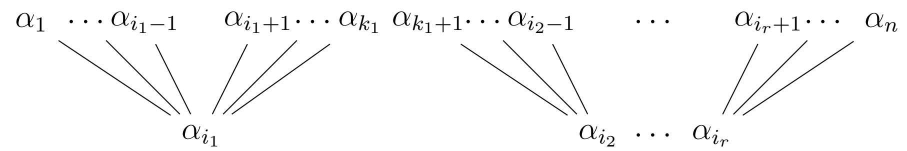
i_2 : the largest number such that $\alpha_{k_1+1}, \cdots, \alpha_{i_2} \geq \alpha_{i_2}$;
k_2 : the largest number such that $\alpha_{i_2} \leq \alpha_{i_2}, \alpha_{i_2+1}, \cdots, \alpha_{k_2}$.
$\vdots$
i_r : the largest number such that $\alpha_{k_{r-1}+1}, \cdots, \alpha_{i_r} \geq \alpha_{i_r}$;
$k_r = n$: here we have $\alpha_{i_r} \leq \alpha_{i_r}, \alpha_{i_r+1}, \cdots, \alpha_n$. Of course, we may have $i_r = k_r = n$.

Corresponding to the so called left to right significant indices $i_1, \cdots, i_r$, we have

$$\alpha_{i_1}, \cdots, \alpha_{i_r} \in Y.$$

We claim that for all $1 \leq s \leq r-1$, α_{i_s} and $\alpha_{i_{s+1}}$ are incomparable. If not, suppose that there exists some $1 \leq s \leq r-1$ such that α_{i_s} and $\alpha_{i_{s+1}}$ are comparable. If $\alpha_{i_s} \leq \alpha_{i_{s+1}}$ then $\alpha_{i_s} \leq \alpha_{k_s+1}$ as $\alpha_{i_{s+1}} \leq \alpha_{k_s+1}$, a contradiction; if $\alpha_{i_s} \geq \alpha_{i_{s+1}}$, then $\alpha_{i_{s+1}} \leq \alpha_{i_{s+1}}, \alpha_{i_{s+1}-1}, \cdots, \alpha_{k_{s-1}+1}$ with $k_0 = 0$, contradicting our choice of i_s. Therefore, we deduce that $\overline{\alpha_{i_1}} \cdots \overline{\alpha_{i_r}}$ is the unique normal form of $\overline{\alpha_1} \cdots \overline{\alpha_n}$ in IG(Y).

We can use the following *Hasse diagram* to depict the relationship among $\alpha_1, \cdots, \alpha_{i_r}$:



Dually, we can define the *right to left significant indices* $\{l_1, \cdots, l_s\} \subseteq [1, n]$ of the element $\overline{x_1} \cdots \overline{x_n} \in \overline{B}^+$, where $l_1 < \cdots < l_s$. Note that as $\overline{\alpha_{i_1}} \cdots \overline{\alpha_{i_r}}$ must equal $\overline{\alpha_{l_1}} \cdots \overline{\alpha_{l_s}}$ in $\overline{B}^+$, we have $r = s$.

Lemma 4.3. *Let $\overline{x_1} \cdots \overline{x_n} \in \overline{B}^+$ with $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$, and left to right significant indices $i_1, \cdots, i_r$. Suppose also that $\overline{y_1} \cdots \overline{y_m} \in \overline{B}^+$ with $y_i \in \beta_i$, for all $i \in [1, m]$, and left to right significant indices $l_1, \cdots, l_s$. Then*

$$\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}$$

in IG(B) *implies $s = r$ and $\alpha_{i_1} = \beta_{l_1}, \cdots, \alpha_{i_r} = \beta_{l_r}$.*

Proof. It follows from Corollary 4.2 and the discussion above that

$$\overline{\alpha_{i_1}} \cdots \overline{\alpha_{i_r}} = \overline{\alpha_1} \cdots \overline{\alpha_n} = \overline{\beta_1} \cdots \overline{\beta_m} = \overline{\beta_{l_1}} \cdots \overline{\beta_{l_s}}$$

in IG(Y). By uniqueness of normal forms, we have that $s = r$ and $\alpha_{i_1} = \beta_{l_1}, \cdots, \alpha_{i_r} = \beta_{l_r}$. $\square$

In view of the above observations, we introduce the following notions.

Let $w = \overline{x_1}\ \cdots \overline{x_n}$ be a word in $\overline{B}^+$ with $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$. Suppose that w has left to right significant indices $i_1, \cdots, i_r$. Then we call the natural number r the *Y-length*, and $\alpha_{i_1}, \cdots, \alpha_{i_r}$ the *ordered Y-components* of the equivalence class of w in $\mathrm{IG}(B)$.

In what follows whenever we write $w \sim w'$ for $w, w' \in \overline{B}^+$, we mean that the word w' can be obtained from the word w from a single step $\rightarrow$ or its reverse $\leftarrow$ as in Lemma 2.7. We interpret ε and $\overline{\varepsilon}$ as added identities.

Lemma 4.4. *Let $\overline{x_1}\ \cdots\ \overline{x_n} \in \overline{B}^+$ with left to right significant indices $i_1, \cdots, i_r$, where $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$. Let $\overline{y_1} \cdots\ \overline{y_m} \in \overline{B}^+$ be such that $\overline{y_1} \cdots\ \overline{y_m} \sim \overline{x_1}\ \cdots\ \overline{x_n}$, and suppose that the left to right significant indices of $\overline{y_1} \cdots\ \overline{y_m}$ are $j_1, \cdots, j_r$. Then for all $l \in [1, r]$, we have*

$$\overline{y_1}\ \cdots\ \overline{y_{j_l}} = \overline{x_1}\ \cdots\ \overline{x_{i_l}}\ \overline{u}$$

in $\mathrm{IG}(B)$ where $y_{j_l} = u' x_{i_l} u$ and one of the following holds: (i) $u' = u = \varepsilon$; (ii) $u = \varepsilon$ and $u' \in B_\sigma$ for some $\sigma \geq \alpha_{i_l}$; (iii) $u' = \varepsilon$ and $u \in B_\delta$ for some $\delta > \alpha_{i_l}$; or (iv) $u' = \varepsilon$, $u = y_{j_l}$ and there exists $v \in B_\theta$ for some $\theta > \alpha_{i_l}$ such that $vu = u$ and $uv = x_{i_l}$.

Proof. Suppose that we split $x_k = ef$ for some $k \in [1, n]$, where ef is a basic product with $e \in B_\mu$ and $f \in B_\tau$, so that $\alpha_k = \mu\tau$. Then

$$\overline{x_1}\ \cdots\ \overline{x_n} \sim \overline{x_1}\ \cdots\ \overline{x_{k-1}}\ \overline{e}\ \overline{f}\ \overline{x_{k+1}}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}.$$

If $k < i_l$, then clearly $y_{j_l} = x_{i_l}$ and

$$\overline{y_1}\ \cdots\ \overline{y_{j_l}} = \overline{x_1}\ \cdots\ \overline{x_{k-1}}\ \overline{e}\ \overline{f}\ \overline{x_{k+1}} \cdots\ \overline{x_{i_l}} = \overline{x_1}\ \cdots\ \overline{x_{i_l}},$$

so we may take $u = u' = \varepsilon$.

Suppose that $k = i_l$ and so $\mu\tau = \alpha_{i_l}$. If $\mu \geq \tau$, then $y_{j_l} = f$ and again

$$\overline{y_1}\ \cdots\ \overline{y_{j_l}} = \overline{x_1}\ \cdots \overline{x_{i_l-1}}\ \overline{e}\ \overline{f} = \overline{x_1}\ \cdots\ \overline{x_{i_l}}.$$

As $x_{i_l} = ef\ \mathcal{L}\ f$, we have $y_{j_l} = f = f x_{i_l}$. Put $u' = y_{j_l}$ and $u = \varepsilon$. Note also that $x_{i_l} = ef = e y_{j_l}$.

On the other hand, if $\mu < \tau$, then $y_{j_l} = e$. As ef is a basic product, $ef = e = x_{i_l}$ or $fe = e$. We first consider the case where $ef = e = x_{i_l}$. Here

$$\overline{y_1} \cdots \overline{y_{j_l}} = \overline{x_1} \cdots \overline{x_{i_l-1}}\ \overline{e} = \overline{x_1} \cdots \overline{x_{i_l}},$$

and $y_{j_l} = e = x_{i_l}$ so that we may take $u = u' = \varepsilon$. The other situation is where $fe = e$. Here, as $x_k = ef\ \mathcal{R}\ e$ and $e = efe$,

$$\overline{y_1} \cdots \overline{y_{j_l}} = \overline{x_1} \cdots\ \overline{x_{i_l-1}}\ \overline{e} = \overline{x_1} \cdots\ \overline{x_{k-1}}\ \overline{ef}\ \overline{e} = \overline{x_1} \cdots\ \overline{x_{i_l}}\ \overline{e}$$

and $y_{j_l} = x_{i_l} e$ where $fe = e$; we put $u' = \varepsilon, u = e$ and $v = f$ to satisfy the conditions of the Lemma. We also note that

$$\overline{x_1} \cdots\ \overline{x_{i_l}} = \overline{x_1} \cdots\ \overline{x_{k-1}}\ \overline{ef} = \overline{x_1}\ \cdots\ \overline{x_{i_l}}\ \overline{e}\ \overline{f} = \overline{y_1}\ \cdots\ \overline{y_{j_l}}\ \overline{f}$$

and $x_{i_l} = y_{j_l} f$.

Finally, suppose that $k > i_l$. Then it is obvious that $j_l = i_l$, $x_{i_l} = y_{j_l}$ and

$$\overline{y_1}\ \cdots\ \overline{y_{j_l}} = \overline{x_1}\ \cdots\ \overline{x_{i_l}}$$

so again we take $u = u' = \varepsilon$.

We now remark that at each stage of the argument we have shown that, not only is $\overline{x_1}\ \cdots\ \overline{x_{i_l}}$ a left factor of $\overline{y_1}\ \cdots\ \overline{y_{j_l}}$, but also the dual conditions hold for $\overline{y_1}\ \cdots\ \overline{y_{j_l}}$ to be a left factor of $\overline{x_1}\ \cdots\ \overline{x_{i_l}}$. Thus the lemma is proven. □

It follows immediately from Lemma 4.4 that

Corollary 4.5. *Suppose that $\overline{y_1}\cdots\ \overline{y_m} = \overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ with left to right significant indices $j_1, \cdots, j_r$ and $i_1, \cdots, i_r$, respectively, and suppose $x_i \in B_{\alpha_i}$ for all $i \in [1, n]$. Then for all $l \in [1, r]$, we have*

$$\overline{y_1}\ \cdots\ \overline{y_{j_l}} = \overline{x_1}\ \cdots\ \overline{x_{i_l}}\ \overline{u_1}\ \overline{u_2}\cdots\ \overline{u_{s_l}}$$

in IG(B) *where $y_{j_l} = u'_{s_l}\cdots u'_1 x_{i_l} u_1 \cdots u_{s_l}$, and for all $t \in [1, s_l]$, one of the following holds: (i) $u'_t = u_t = \varepsilon$; (ii) $u_t = \varepsilon$ and $u'_t \in B_{\sigma_t}$ for some $\sigma_t \geq \alpha_{i_l}$; (iii) $u'_t = \varepsilon$ and $u_t \in B_\delta$ for some $\delta > \alpha_{i_l}$; or (iv) $u'_t = \varepsilon$, $u_t \in B_{\alpha_{i_l}}$ and there exists $v_t \in B_{\theta_t}$ for some $\theta_t > \alpha_{i_l}$ such that $v_t u_t = u_t$.*

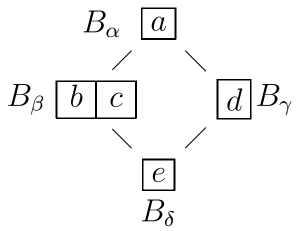
Consequently, $\overline{y_1}\ \cdots\ \overline{y_{j_l}}\ \mathcal{R}\ \overline{x_1}\ \cdots\ \overline{x_{i_l}}$ in IG(B) *and hence $y_1\ \cdots\ y_{j_l}\ \mathcal{R}\ x_1\ \cdots\ x_{i_l}$ in B.*

Proof. The proof follows from Lemma 4.4 by finite induction. □

Note that the duals of Lemma 4.4 and Corollary 4.5 hold for right to left significant indices.

From Lemma 3.1, we know that if B is a semilattice or a rectangular band, then every element in IG(B) has a unique normal form. However, it may not be true for an arbitrary band B, even if B is normal.

Example 4.6. Let $B = \mathcal{B}(Y; B_\mu, \phi_{\mu,\kappa})$ be a strong semilattice $Y = \{\alpha, \beta, \gamma, \delta\}$ of rectangular bands B_μ, $\mu \in Y$ (see the figure below), such that $\phi_{\alpha,\beta}$ is defined by $a\phi_{\alpha,\beta} = b$, the remaining morphisms being defined in the obvious unique manner. It is easy to see that a pair (u, v) is basic if and only if $u \in B_\eta, v \in B_\nu$ where η, ν are comparable in Y.



By an easy calculation, we have $ca = c(a\phi_{\alpha\beta}) = cb = b$ and so

$$\overline{c}\ \overline{d} = \overline{c}\ \overline{ad} = \overline{c}\ \overline{a}\ \overline{d} = \overline{ca}\ \overline{d} = \overline{b}\ \overline{d}$$

in IG(B), showing that not every element in IG(B) has a unique normal form.

Lemma 4.7. *Let $\overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ with $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$, and let $y \in B_\beta$ with $\beta \leq \alpha_i$, for all $i \in [1, n]$. Then in* IG(B) *we have*

$$\overline{x_1}\ \cdots\ \overline{x_n}\ \overline{y} = \overline{x_1 \cdots x_n y x_n \cdots x_1}\ \cdots\ \overline{x_{n-1} x_n y x_n x_{n-1}}\ \overline{x_n y x_n}\ \overline{y}$$

and

$$\overline{y}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{y}\ \overline{x_1 y x_1}\ \overline{x_2 x_1 y x_1 x_2}\ \cdots\ \overline{x_n \cdots x_1 y x_1 \cdots x_n}.$$

Proof. First, we notice that for any $x \in B_\alpha, y \in B_\beta$ such that $\alpha \geq \beta$, we have $yx\ \mathcal{R}\ y$, so that (y, yx) is a basic pair and $(yx)y = y$. On the other hand, as $(yx)x = yx$, we have that (x, yx) is a basic pair, so that

$$\overline{x}\ \overline{y} = \overline{x}\ \overline{(yx)y} = \overline{x}\ \overline{yx}\ \overline{y} = \overline{xyx}\ \overline{y}.$$

The first required equality follows from the above observation by finite induction. The second is dual. $\square$

Corollary 4.8. *Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band and let $\overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ be such that $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$. Let $y \in B_\beta$ with $\beta \leq \alpha_i$, for all $i \in [1, n]$. Then in* IG(B) *we have*

$$\overline{x_1}\ \cdots\ \overline{x_n}\ \overline{y} = \overline{x_1\phi_{\alpha_1,\beta}}\ \cdots\ \overline{x_n\phi_{\alpha_n,\beta}}\ \overline{y}$$

and

$$\overline{y}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{y}\ \overline{x_1\phi_{\alpha_1,\beta}}\ \cdots\ \overline{x_n\phi_{\alpha_n,\beta}}.$$

Corollary 4.9. *Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a chain Y of rectangular bands B_α, $\alpha \in Y$. Then* IG(B) *is a regular semigroup, and hence a chain of completely simple semigroups.*

Proof. Let $\overline{u_1}\ \cdots\ \overline{u_n}$ be an element in IG(B). From Lemma 4.7 it follows that $\overline{u_1}\ \cdots\ \overline{u_n}$ can be written as an element of IG(B) in which all letters come from B_γ, where γ is the minimum of the ordered Y-components $\{\alpha_1, \cdots, \alpha_n\}$, so that $\overline{u_1}\ \cdots\ \overline{u_n}$ is regular by Lemma 3.4. $\square$

Notice from [3] that if B is left or right seminormal the subgroups of IG(B) in Corollary 4.9 are certainly free (see Section 6).

Given the above observations, we now introduce the idea of almost normal form for elements in IG(B).

Definition 4.10. *An element $\overline{x_1}\ \cdots\ \overline{x_n} \in \overline{B}^+$ is said to be in* almost normal form *if there exists a sequence*

$$1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n$$

with

$$\{x_1, \cdots, x_{i_1}\} \subseteq B_{\alpha_1}, \{x_{i_1+1}, \cdots, x_{i_2}\} \in B_{\alpha_2}, \cdots, \{x_{i_{r-1}+1}, \cdots x_n\} \subseteq B_{\alpha_r}$$

where α_i, α_{i+1} are incomparable for all $i \in [1, r-1]$.

The reader should note that a word being in almost normal form does not imply that it is in normal form, for we do not insist in the above expression that the pair (x_j, x_{j+1}) is not basic for $x_j, x_{j+1} \in B_{\alpha_i}, 1 \leq i \leq r$. On the other hand, an element in normal form need not be in almost normal form. For example, if $x \in B_\alpha, y \in B_\beta$ with $\alpha > \beta$ and (x, y) not a basic pair, then $\overline{x}\,\overline{y}$ is in normal form but not almost normal form.

It is obvious that the element $\overline{x_1}\ \cdots\ \overline{x_n} \in \overline{B}^+$ above has Y-length r, ordered Y-components $\alpha_1, \cdots, \alpha_r$, left to right significant indices $i_1, i_2, \cdots, i_{r-1}, i_r = n$ and right to left significant indices $1, i_1 + 1, \cdots, i_{r-2} + 1, i_{r-1} + 1$. Note that, in general, the almost normal forms of elements of IG(B) are not unique. Further, if $\overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}$ are in almost normal form, then they have the same Y-length and ordered Y-components, but the significant indices of the expressions on each side can differ.

The next result is immediate from the definition of significant indices and Lemma 4.7.

Lemma 4.11. *Let B be a band. Then every element of* IG(B) *can be written in almost normal form.*

We have the following lemma regarding the almost normal form of the product of two almost normal forms.

Lemma 4.12. *Let* $\overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ *be in almost normal form with* Y*-length* r*, left to right significant indices* $i_1, \cdots, i_r = n$ *and ordered* Y*-components* $\alpha_1, \cdots, \alpha_r$*, let* $\overline{y_1}\ \cdots\ \overline{y_m} \in \mathrm{IG}(B)$ *be in almost normal form with* Y*-length* s*, left to right significant indices* $l_1, \cdots, l_s = m$ *and ordered* Y*-components* $\beta_1, \cdots, \beta_s$*. Then (with* $i_0 = 0$*)*

(i) α_r *and* β_1 *incomparable implies that* $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$ *is in almost normal form;*

(ii) $\alpha_r \geq \beta_1$ *implies*

$$\overline{x_1}\cdots\overline{x_{i_t}}\ \overline{x_{i_t+1}\cdots x_{i_r}y_1x_{i_r}\cdots x_{i_t+1}}\ \cdots\ \overline{x_{i_r}y_1x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$$

is an almost normal form of the product $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$*, for some* $t \in [0, r-1]$ *such that* $\alpha_r, \cdots, \alpha_{t+1} \geq \beta_1$ *and* $t = 0$ *or* α_t, β_1 *are incomparable;*

(iii) $\alpha_r \leq \beta_1$ *implies*

$$\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1x_{i_r}y_1}\ \cdots\ \overline{y_{l_v}\cdots y_1x_{i_r}y_1\cdots y_{l_v}}\ \overline{y_{l_v+1}}\ \cdots\ \overline{y_{l_s}}$$

is an almost normal form of the product $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$ *for some* $v \in [1, s]$ *such that* $\alpha_r \leq \beta_1, \cdots, \beta_v$ *and* $v = s$ *or* β_{v+1}, α_r *are incomparable.*

Proof. Clearly, the statement (i) is true. We now aim to show (ii). Since $\alpha_r \geq \beta_1$, we have

$$\overline{x_{i_{r-1}+1}}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1} = \overline{x_{i_{r-1}+1}\cdots x_{i_r}y_1x_{i_r}\cdots x_{i_{r-1}+1}}\ \cdots\ \overline{x_{i_r}y_1x_{i_r}}\ \overline{y_1}$$

by Corollary 4.7. Consider α_{r-1} and β_1, then we either have $\alpha_{r-1} \geq \beta_1$ or they are incomparable, as $\alpha_{r-1} < \beta_1$ would imply $\alpha_r > \alpha_{r-1}$, which contradicts the almost normal form of $\overline{x_1}\ \cdots\ \overline{x_{i_r}}$. By finite induction we have that

$$\overline{x_1}\cdots\overline{x_{i_t}}\ \overline{x_{i_t+1}\cdots x_{i_r}y_1x_{i_r}\cdots x_{i_t+1}}\ \cdots\ \overline{x_{i_r}y_1x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$$

is an almost normal form of the product $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$, for some $t \in [0, r-1]$, such that $\alpha_r, \cdots, \alpha_{t+1} \geq \beta_1$ and $t = 0$ or α_t, β_1 are incomparable. Similarly, we can show (iii). ☐

Theorem 4.13. *Let* B *be band. Then* $\mathrm{IG}(B)$ *is a weakly abundant semigroup with the congruence condition.*

Proof. As usual, we let B be a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$. Let $\overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, and ordered Y-components $\alpha_1, \cdots, \alpha_r$. Clearly $\overline{x_1}\,\overline{x_1}\ \cdots\ \overline{x_n} = \overline{x_1}\ \cdots\ \overline{x_n}$. Let $e \in B_\delta$ be such that $\overline{e}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{x_1}\ \cdots\ \overline{x_n}$. By Corollary 4.2, applying $\overline{\theta}$, we have that $\overline{\delta}\ \overline{\alpha_1}\ \cdots\ \overline{\alpha_r} = \overline{\alpha_1}\ \cdots\ \overline{\alpha_r}$. It follows from Lemma 3.1 that $\delta \geq \alpha_1$, so that by Corollary 4.5 we have

$$ex_1\cdots x_{i_1}\ \mathcal{R}\ x_1\cdots x_{i_1}.$$

On the other hand, $x_1\cdots x_{i_1}\ \mathcal{R}\ x_1$ so that $ex_1\ \mathcal{R}\ x_1$, and we have $x_1 \leq_{\mathcal{R}} e$. Thus $\overline{e}\,\overline{x_1} = \overline{ex_1} = \overline{x_1}$. Therefore $\overline{x_1}\ \cdots\ \overline{x_n}\ \widetilde{\mathcal{R}}\ \overline{x_1}$. Dually, $\overline{x_1}\ \cdots\ \overline{x_n}\ \widetilde{\mathcal{L}}\ \overline{x_n}$, so that $\mathrm{IG}(B)$ is a weakly abundant semigroup as required.

Next we show that $\mathrm{IG}(B)$ satisfies the congruence condition.

Let $\overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ be defined as above and let $\overline{y_1}\ \cdots\ \overline{y_m} \in \mathrm{IG}(B)$ be in almost normal form with Y-length u, left to right significant indices $l_1, \cdots, l_u = m$ and ordered Y-components $\beta_1, \cdots, \beta_u$. From above and Lemma 2.3, we have $\overline{x_1}\ \cdots\ \overline{x_n}\ \widetilde{\mathcal{R}}\ \overline{y_1}\ \cdots\ \overline{y_m}$ if and only if $\overline{x_1}\ \mathcal{R}\ \overline{y_1}$. Since the biorders in $\mathrm{IG}(B)$ and B are isomorphic, the latter is equivalent to $x_1\,\mathcal{R}\,y_1$ in B.

Suppose now that $x_1 \mathcal{R} y_1$, so that $\alpha_1 = \beta_1$. Let $\overline{z_1} \cdots \overline{z_s} \in \mathrm{IG}(B)$, where, without loss of generality, we can assume it is in almost normal form with Y-length t, left to right significant indices $j_1, \cdots, j_t = s$, and ordered Y-components $\gamma_1, \cdots, \gamma_t$. We aim to show that

$$\overline{z_1} \cdots \overline{z_s}\, \overline{x_1} \cdots \overline{x_n}\, \widetilde{\mathcal{R}}\, \overline{z_1} \cdots \overline{z_s}\, \overline{y_1} \cdots \overline{y_m}.$$

We consider the following three cases.

(i) If $\alpha_1 = \beta_1, \gamma_t$ are incomparable, then it is clear that

$$\overline{z_1} \cdots \overline{z_s}\, \overline{x_1} \cdots \overline{x_n} \text{ and } \overline{z_1} \cdots \overline{z_s}\, \overline{y_1} \cdots \overline{y_m}$$

are in almost normal form, so clearly we have

$$\overline{z_1} \cdots \overline{z_s}\, \overline{x_1} \cdots \overline{x_n}\, \widetilde{\mathcal{R}}\, \overline{z_1}\, \widetilde{\mathcal{R}}\, \overline{z_1} \cdots \overline{z_s}\, \overline{y_1} \cdots \overline{y_m}.$$

(ii) Let $\beta_1 = \alpha_1 \leq \gamma_1$. By Lemma 4.12

$$\overline{z_1} \cdots \overline{z_s}\, \overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_{j_v}}\, \overline{z_{j_v+1} \cdots z_s x_1 z_s \cdots z_{j_v+1}} \cdots \overline{z_s x_1 z_s}\, \overline{x_1} \cdots \overline{x_n}$$

and

$$\overline{z_1} \cdots \overline{z_s}\, \overline{y_1} \cdots \overline{y_m} = \overline{z_1} \cdots \overline{z_{j_v}}\, \overline{z_{j_v+1} \cdots z_s y_1 z_s \cdots z_{j_v+1}} \cdots \overline{z_s y_1 z_s}\, \overline{y_1} \cdots \overline{y_m}$$

where $v \in [0, t-1]$, $\gamma_{v+1}, \cdots, \gamma_t \geq \alpha_1 = \beta_1$ and γ_v, β_1 are incomparable or $v = 0$. Note that the right hand sides are in almost normal form.

If $v \geq 1$, then clearly the required result is true, as the above two almost normal forms begin with the same idempotent. If $v = 0$, then we need to show that

$$z_1 \cdots z_s x_1 z_s \cdots z_1 \mathcal{R} z_1 \cdots z_s y_1 z_s \cdots z_1.$$

Since $x_1 \mathcal{R} y_1$, it follows from the structure of B that

$$z_1 \cdots z_s x_1 z_s \cdots z_1 \mathcal{R} z_1 \cdots z_s x_1 \mathcal{R} z_1 \cdots z_s y_1 \mathcal{R} z_1 \cdots z_s y_1 z_s \cdots z_1$$

as required.

(iii) Let $\beta = \alpha_1 \geq \gamma_1$. By Lemma 4.12

$$\overline{z_1} \cdots \overline{z_s}\, \overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_s}\, \overline{x_1 z_s x_1} \cdots \overline{x_{i_k} \cdots x_1 z_s x_1 \cdots x_{i_k}}\, \overline{x_{i_k+1}} \cdots \overline{x_n}$$

and

$$\overline{z_1} \cdots \overline{z_s}\, \overline{y_1} \cdots \overline{y_m} = \overline{z_1} \cdots \overline{z_s}\, \overline{y_1 z_s y_1} \cdots \overline{y_{l_p} \cdots y_1 z_s y_1 \cdots y_{l_p}}\, \overline{y_{l_p+1}} \cdots \overline{y_m},$$

where $k \in [1, r]$, $\alpha_1, \cdots, \alpha_k \geq \gamma_1$, and α_{k+1}, γ_1 are incomparable or $k = r$, and $p \in [1, u]$, $\beta_1, \cdots, \beta_p \geq \gamma_1$, and β_{p+1}, γ_1 are incomparable or $p = u$. Clearly, the right hand sides are in almost normal form, so that

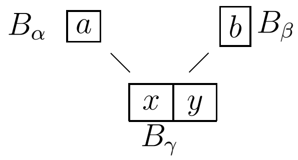
$$\overline{z_1} \cdots \overline{z_s}\, \overline{x_1} \cdots \overline{x_n}\, \widetilde{\mathcal{R}}\, \overline{z_1}\, \widetilde{\mathcal{R}}\, \overline{z_1} \cdots \overline{z_s}\, \overline{y_1} \cdots \overline{y_m}.$$

Similarly, we can show that $\widetilde{\mathcal{L}}$ is a right congruence, so that $\mathrm{IG}(B)$ is a weakly abundant semigroup satisfying the congruence condition. This completes the proof. $\square$

We finish this section by constructing a band B for which $\mathrm{IG}(B)$ is not an abundant semigroup.

Example 4.14. Let $B = B_\alpha \cup B_\beta \cup B_\gamma$ be a band with semilattice decomposition structure and multiplication table defined by

	a	b	x	y
a	a	y	x	y
b	y	b	x	y
x	x	y	x	y
y	y	y	x	y



First, it is easy to check that B is indeed a semigroup. We now show that $\mathrm{IG}(B)$ is not abundant by arguing that the element $\overline{a}\,\overline{b} \in \mathrm{IG}(B)$ is not $\mathcal{R}^*$-related to any idempotent of $\mathrm{IG}(B)$. It follows from the proof of Theorem 4.13 that $\overline{a}\,\overline{b}\,\widetilde{\mathcal{R}}\,\overline{a}$. However, $\overline{a}\,\overline{b}$ is not $\mathcal{R}^*$-related to $\overline{a}$, because

$$\overline{x}\,\overline{a}\,\overline{b} = \overline{y} = \overline{y}\,\overline{a}\,\overline{b} \text{ but } \overline{x}\,\overline{a} = \overline{x} \neq \overline{y} = \overline{y}\,\overline{a}.$$

From Lemma 2.4, $\overline{a}\,\overline{b}$ is not $\mathcal{R}^*$-related to any idempotent of $\overline{B}$, and hence $\mathrm{IG}(B)$ is not an abundant semigroup.

5. Condition (P)

We have shown that for any band B, the semigroup $\mathrm{IG}(B)$ is always weakly abundant with the congruence condition, but not necessarily abundant. We know from Theorem 3.2 that if B has trivial basic products, then $\mathrm{IG}(B)$ is abundant. This section is devoted to finding some further special kinds of bands B for which $\mathrm{IG}(B)$ is abundant. As a means to this end we introduce a technical condition.

Definition 5.1. *We say that the semigroup* $\mathrm{IG}(B)$ *satisfies Condition* (P) *if for any two almost normal forms* $\overline{u_1}\ \cdots\ \overline{u_n} = \overline{v_1}\ \cdots\ \overline{v_m} \in \mathrm{IG}(B)$ *with* Y*-length* r*, left to right significant indices* $i_1, \cdots, i_r = n$ *and* $l_1, \cdots, l_r = m$*, respectively, the following statements (with* $i_0 = l_0 = 0$*) hold:*
(i) $u_{i_s}\ \mathcal{L}\ v_{l_s}$ *implies* $\overline{u_1}\ \cdots \overline{u_{i_s}} = \overline{v_1}\ \cdots\ \overline{v_{l_s}}$*, for all* $s \in [1, r]$.
(ii) $u_{i_t+1}\ \mathcal{R}\ v_{l_t+1}$ *implies* $\overline{u_{i_t+1}}\ \cdots \overline{u_n} = \overline{v_{l_t+1}}\ \cdots\ \overline{v_m}$*, for all* $t \in [0, r-1]$.

We immediately show one case where Condition (P) is guaranteed to hold.

Lemma 5.2. *Let* B *be a band with trivial basic products, and let* $\overline{x_1}\ \cdots\ \overline{x_n}$*,* $\overline{y_1}\ \cdots\ \overline{y_m} \in \mathrm{IG}(B)$ *have left to right significant indices* $i_1, \cdots, i_r$ *and* $j_1, \cdots, j_r$*, respectively. If* $\overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}$*, then for any* $l \in [1, r]$*,* $\overline{x_1}\ \cdots\ \overline{x_{i_l}} = \overline{y_1}\ \cdots\ \overline{y_{j_l}}$*. Of course,* B *satisfies Condition (P).*

Proof. It follows from Proposition 3.5 that a band with trivial basic products is just a band $B = \bigcup_{\alpha \in Y} B_\alpha$ such that for all $\alpha, \beta \in Y$ with $\beta > \alpha$, $u \in B_\alpha$ and $v \in B_\beta$, we have $uv = vu = u$. Suppose that $x_i \in B_{\alpha_i}$ for all $i \in [1, r]$. It is enough to consider a single step, so suppose that

$$\overline{x_1}\ \cdots\ \overline{x_n} \sim \overline{y_1}\ \cdots\ \overline{y_m}.$$

By Lemma 4.4, for any $l \in [1, r]$, we have

$$\overline{y_1}\ \cdots\ \overline{y_{j_l}} = \overline{x_1}\ \cdots\ \overline{x_{i_l}}\ \overline{u}$$

and $y_{j_l} = u'x_{i_l}u$, where u and u' are defined by various cases as exhibited in Lemma 4.4. In each case $\overline{x_{i_l}}\ \overline{u} = \overline{x_{i_l}}$, either trivially, if $u = \varepsilon$, or because $x_{i_l}u = x_{i_l}$ is a basic product in B, and so $\overline{y_1}\ \cdots\ \overline{y_{j_l}} = \overline{x_1}\ \cdots\ \overline{x_{i_l}}$. □

More examples of bands satisfying Condition (P) will be given later in this section. However, it is a consequence of our results and Example 6.5 that not every band has Condition (P), in particular, not every normal band has Condition (P).

Proposition 5.3. *Let B be a normal band for which* $\mathrm{IG}(B)$ *satisfies Condition* (P)*. Then* $\mathrm{IG}(B)$ *is an abundant semigroup.*

Proof. Let $\overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, and ordered Y-components $\alpha_1, \cdots, \alpha_r$. By Theorem 4.13, $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \widetilde{\mathcal{R}}\ \overline{x_1}$. We aim to show that $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \mathcal{R}^*\ \overline{x_1}$. From Lemma 2.5, we only need to show that for any two almost normal forms $\overline{y_1}\ \cdots\ \overline{y_m}, \overline{z_1}\ \cdots\ \overline{z_h} \in \mathrm{IG}(B)$ we have

$$\overline{z_1}\ \cdots\ \overline{z_h}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}\ \overline{x_1}\ \cdots\ \overline{x_n} \Rightarrow \overline{z_1}\ \cdots\ \overline{z_h}\ \overline{x_1} = \overline{y_1}\ \cdots\ \overline{y_m}\ \overline{x_1}.$$

Suppose that $\overline{y_1}\ \cdots\ \overline{y_m}$ has Y-length m, left to right significant indices $l_1, \cdots, l_s = m$, and ordered Y-components $\beta_1, \cdots, \beta_s$, and $\overline{z_1}\ \cdots\ \overline{z_h} \in \mathrm{IG}(B)$ has Y-length t, left to right significant indices $j_1, \cdots, j_t = h$, and ordered Y-components $\gamma_1, \cdots, \gamma_t$.

Assume now that

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$$

(it will be convenient to use the indices i_r, l_s, j_t). We consider the following cases, the remainder following by considerations of duality.

(i) If γ_t, α_1 and β_s, α_1 are incomparable, then both sides of the above equality are in almost normal form, so that by Condition (P)

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}.$$

Since $\overline{x_1} \cdots\ \overline{x_{i_1}}\ \mathcal{R}\ \overline{x_1}$ by Corollary 3.4, we have $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}$.

(ii) Suppose now that $\gamma_t \leq \alpha_1$ and β_s, α_1 are incomparable. By Lemma 4.12, $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$ has an almost normal form

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}\ \overline{x_{i_v+1}}\ \cdots\ \overline{x_{i_r}},$$

for some $v \in [1, r]$, where $\gamma_t \leq \alpha_1, \cdots, \alpha_v$ and $v = r$ or γ_t, α_{v+1} are incomparable. Hence we have

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}\ \overline{x_{i_v+1}}\ \cdots\ \overline{x_{i_r}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}.$$

Note that both sides of the above equality are in almost normal form. It follows from Corollary 4.2 that

$$(\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}\ \overline{x_{i_v+1}}\ \cdots\ \overline{x_{i_r}})\ \overline{\theta} = (\overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}})\ \overline{\theta}$$

and so

$$\overline{\gamma_1}\ \cdots\ \overline{\gamma_t}\ \overline{\alpha_{v+1}}\ \cdots\ \overline{\alpha_r} = \overline{\beta_1}\ \cdots\ \overline{\beta_s}\ \overline{\alpha_1}\ \cdots\ \overline{\alpha_r}.$$

Since $v \geq 1$, we have $\gamma_t = \alpha_v$. To avoid contradiction, $v = 1$, and hence by Condition (P)

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_1} \cdots x_1 z_{j_t} x_1 \cdots x_{i_1}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}.$$

As $\gamma_t = \alpha_v$,

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

so that $\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} = \overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1}$.

(iii) Suppose that $\gamma_t \leq \alpha_1$ and $\beta_s \leq \alpha_1$. By Lemma 4.12 we have the following two almost normal forms for $\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} \cdots \overline{x_{i_r}}$ and $\overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_r}}$, namely,

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}\, \overline{x_{i_v+1}} \cdots \overline{x_{i_r}}$$

where $v \in [1, r]$ such that $\gamma_t \leq \alpha_1, \cdots, \alpha_v$ and $v = r$ or γ_t, α_{v+1} are incomparable, and

$$\overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1 y_{l_s} x_1} \cdots \overline{x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}}\, \overline{x_{i_u+1}} \cdots \overline{x_{i_r}}$$

where $u \in [1, r]$ with $\beta_s \leq \alpha_1, \cdots, \alpha_u$ and $u = r$ or β_s, α_{u+1} are incomparable. Hence by Corollary 4.2,

$$\overline{\gamma_1} \cdots \overline{\gamma_t}\, \overline{\alpha_{v+1}} \cdots \overline{\alpha_r} = \overline{\beta_1} \cdots \overline{\beta_s}\, \overline{\alpha_{u+1}} \cdots \overline{\alpha_r}$$

If $v > u$, then $\gamma_t = \alpha_v$, so to avoid contradiction $v = 1$. But then $u < 1$, again contradiction. Similarly, $v < u$ is impossible. We deduce that $v = u$, and so $t = s$ and $\beta_s = \gamma_t$.

We have

$$x_1 z_{j_t} x_1 = x_1 \phi_{\alpha_1, \gamma_t} = x_1 \phi_{\alpha_1, \beta_s} = x_1 y_{l_s} x_1$$

$$\vdots$$

$$x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v} = x_{i_v} \phi_{\alpha_v, \gamma_t} = x_{i_u} \phi_{\alpha_u, \beta_s} = x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}.$$

By Condition (P) we deduce

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}} = \overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1 y_{l_s} x_1} \cdots \overline{x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}}.$$

On the other hand, we have

$$\overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}} = \overline{x_1 y_{l_s} x_1} \cdots \overline{x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}}$$

which by Lemma 3.4 is $\mathcal{R}$-related to $\overline{x_1 z_{j_t} x_1}$ in $\mathrm{IG}(B_{\gamma_t})$ and hence in $\mathrm{IG}(B)$. It follows that

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1 z_{j_t} x_1} = \overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1 y_{l_s} x_1},$$

and so

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} = \overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1}.$$

(iv) Suppose that $\gamma_t \leq \alpha_1$ and $\beta_s \geq \alpha_1$. By Lemma 4.12 we have the following two almost normal forms for $\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} \cdots \overline{x_{i_r}}$ and $\overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_r}}$, namely,

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}\, \overline{x_{i_v+1}} \cdots \overline{x_{i_r}}$$

for some $v \in [1, r]$ with $\gamma_t \leq \alpha_1, \cdots, \alpha_v$ and $v = r$ or γ_t, α_{v+1} are incomparable, and

$$\overline{y_1} \cdots \overline{y_{l_u}}\, \overline{y_{l_u+1} \cdots y_{l_s} x_1 y_{l_s} \cdots y_{l_u+1}} \cdots \overline{y_{l_s} x_1 y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_r}}$$

for some $u \in [0, s-1]$ with $\beta_{u+1}, \cdots, \beta_s \geq \alpha_1$ and β_u, α_1 are incomparable or $u = 0$. It follows from Corollary 4.2 that

$$\overline{\gamma_1} \cdots \overline{\gamma_t}\, \overline{\alpha_{v+1}} \cdots \overline{\alpha_r} = \overline{\beta_1} \cdots \overline{\beta_u}\, \overline{\alpha_1} \cdots \overline{\alpha_r}.$$

Note that both sides of the above equality are normal forms of $\mathrm{IG}(Y)$. As $v \geq 1$, we have $\gamma_t = \alpha_v$, so that to avoid contradiction we have $v = 1$ and then $x_{i_1} \cdots x_1 z_{j_t} x_1 \cdots x_{i_1} = x_{i_1}$. Hence by Condition (P)

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_1} \cdots x_1 z_{j_t} x_1 \cdots x_{i_1}}$$

$$= \overline{y_1} \cdots \overline{y_{l_u}}\, \overline{y_{l_u+1} \cdots y_{l_s} x_1 y_{l_s} \cdots y_{l_u+1}} \cdots \overline{y_{l_s} x_1 y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_1}}$$

and so

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\cdots\overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}},$$

which implies $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1}\cdots\overline{y_{l_s}}\ \overline{x_1}$.

(v) Suppose that $\gamma_t \geq \alpha_1$ and $\beta_s \geq \alpha_1$. By Lemma 4.12 we have the following two almost normal forms for $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$ and $\overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$, namely,

$$\overline{z_1}\cdots\overline{z_{j_v}}\ \overline{z_{j_v+1}\cdots z_{j_t}x_1z_{j_t}\cdots z_{j_v+1}}\ \cdots\ \overline{z_{j_t}x_1z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\ \cdots\ \overline{x_{i_r}}$$

for some $v \in [0, t-1]$ such that $\gamma_{v+1}, \cdots, \gamma_t \geq \alpha_1$ and γ_v, α_1 are incomparable or $v = 0$, and

$$\overline{y_1}\cdots\overline{y_{l_u}}\ \overline{y_{l_u+1}\cdots y_{l_s}x_1y_{l_s}\cdots y_{l_u+1}}\ \cdots\ \overline{y_{l_s}x_1y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\ \cdots\ \overline{x_{i_r}}$$

for some $u \in [0, s-1]$ such that $\beta_{u+1}, \cdots, \beta_s \geq \alpha_1$ and β_u, α_1 are incomparable or $u = 0$. Hence by Condition (P),

$$\begin{aligned}&\overline{z_1}\cdots\overline{z_{j_v}}\ \overline{z_{j_v+1}\cdots z_{j_t}x_1z_{j_t}\cdots z_{j_v+1}}\ \cdots\ \overline{z_{j_t}x_1z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\\ &= \overline{y_1}\cdots\overline{y_{l_u}}\ \overline{y_{l_u+1}\cdots y_{l_s}x_1y_{l_s}\cdots y_{l_u+1}}\ \cdots\ \overline{y_{l_s}x_1y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}},\end{aligned}$$

so that

$$\overline{z_1}\cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

and hence $\overline{z_1}\cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1}\cdots\ \overline{y_{l_s}}\ \overline{x_1}$.

(vi) Suppose that $\gamma_t \geq \alpha_1$ and β_s, α_1 are incomparable. By Lemma 4.12

$$\overline{z_1}\cdots\overline{z_{j_v}}\ \overline{z_{j_v+1}\cdots z_{j_t}x_1z_{j_t}\cdots z_{j_v+1}}\ \cdots\ \overline{z_{j_t}x_1z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\ \cdots\ \overline{x_{i_r}} = \overline{y_1}\cdots\overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\ \cdots\ \overline{x_{i_r}}$$

for some $v \in [0, t-1]$ with $\gamma_{v+1}, \cdots, \gamma_t \geq \alpha_1$ and γ_v, α_1 are incomparable or $v = 0$. Note that both sides of the above equality are in almost normal form. Again by Condition (P)

$$\overline{z_1}\cdots\overline{z_{j_v}}\ \overline{z_{j_v+1}\cdots z_{j_t}x_1z_{j_t}\cdots z_{j_v+1}}\ \cdots\ \overline{z_{j_t}x_1z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\cdots\overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

so that

$$\overline{z_1}\cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

and hence $\overline{z_1}\cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1}\cdots\ \overline{y_{l_s}}\ \overline{x_1}$.

From the above case-by-case analysis, we deduce that $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \mathcal{R}^*\ \overline{x_1}$, and similarly we can show that $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \mathcal{L}^*\ \overline{x_{i_r}}$, so that IG$(B)$ is an abundant semigroup. □

We now aim to find examples of normal bands B for which IG(B) satisfies Condition (P), so that by Proposition 5.3, IG(B) is abundant.

A band $B = \bigcup_{\alpha\in Y} B_\alpha$ is called *Y-basic* if it is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, where B_α is either a left zero band or a right zero band. Any left or right regular band (that is, where *every* B_α is left zero, or *every* B_α is right zero) is Y-basic, but the class of Y-basic bands is easily seen to be larger. We now justify the terminology.

Lemma 5.4. *Let $B = \bigcup_{\alpha\in Y} B_\alpha$ be a band. Then B is Y-basic if and only if it has the property that for any $e \in B_\alpha$ and $f \in B_\beta$ the pair (e, f) being basic in B is equivalent to the pair (α, β) being basic in Y, that is, to α and β being comparable.*

Proof. Suppose that B has the given property on basic pairs. For any $\alpha \in Y$ fix $e \in B_\alpha$; since (e, f) must be basic in B for any $f \in B_\alpha$, clearly B_α is a left or a right zero semigroup.

Conversely, suppose that B is Y-basic. Let $e \in B_\alpha$ and $f \in B_\beta$. If (e, f) is basic, certainly so is (α, β). For the converse, without loss of generality, suppose that $\alpha \leq \beta$. Then $ef, fe \in B_\alpha$. As B is a Y-basic band, we have B_α is either a left zero band or a right zero band. If B_α is a left zero band, then $e(ef) = e$, i.e. $ef = e$, so (e, f) is a basic pair. If B_α is a right zero band, then $(fe)e = e$, i.e. $fe = e$, which again implies that (e, f) is a basic pair. □

It follows from Lemma 5.4 that for a Y-basic band B, every element of $\mathrm{IG}(B)$ has a normal form, say, $\overline{x_1} \cdots \overline{x_n}$ with $x_i \in B_{\alpha_i}$ and α_i and α_{i+1} incomparable, for all $i \in [1, n-1]$. Notice that in this case any normal form must also be an almost normal form. Of course, as Example 4.6 demonstrates, the normal forms need not be unique.

Lemma 5.5. *Let B be a Y-basic band. Then* $\mathrm{IG}(B)$ *satisfies Condition* (*P*).

Proof. Let $\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, $j_1, \cdots, j_r = m$, respectively, and ordered Y-components $\alpha_1, \cdots, \alpha_r$. It then follows from Corollary 4.5 that for any $s \in [1, r]$, either

$$\overline{y_1} \cdots \overline{y_{j_s}} = \overline{x_1} \cdots \overline{x_{i_s}}$$

and we are done, or

$$\overline{y_1} \cdots \overline{y_{j_s}} = \overline{x_1} \cdots \overline{x_{i_s}}\, \overline{e_1} \cdots \overline{e_m}$$

where for all $k \in [1, m]$, $e_k \in B_{\delta_k}$ with $\delta_k \geq \alpha_s$. In this case by Lemma 5.4, we have

$$\overline{x_{i_s}}\, \overline{e_1} \cdots \overline{e_m} = \overline{x_{i_s} e_1 \cdots e_m},$$

so that if we assume $x_{i_s} \,\mathcal{L}\, y_{j_s}$, then

$$\overline{y_1} \cdots \overline{y_{j_s}} = \overline{y_1} \cdots \overline{y_{j_s}}\, \overline{x_{i_s}} = \overline{x_1} \cdots \overline{x_{i_s} e_1 \cdots e_m}\, \overline{x_{i_s}} = \overline{x_1} \cdots \overline{x_{i_s} e_1 \cdots e_m x_{i_s}} = \overline{x_1} \cdots \overline{x_{i_s}}.$$

Together with the dual, we have shown that $\mathrm{IG}(B)$ satisfies Condition (P). □

Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band. Clearly B is locally small in the sense that the local submonoids eBe are as small as they can be, that is, for $e \in B_\alpha$, we have $eBe = \{e\} \cup \{e\phi_{\alpha,\beta} : \alpha > \beta\} = \{e\phi_{\alpha,\beta} : \alpha \geq \beta\}$. We say that B is *pliant* if for every $\alpha \in Y$, there exists an $a_\alpha \in B_\alpha$ such that for all $\beta > \alpha$ and $u \in B_\beta$, we have $u\phi_{\beta,\alpha} = a_\alpha$.

Lemma 5.6. *Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a pliant normal band. Then* $\mathrm{IG}(B)$ *satisfies Condition* (*P*).

Proof. First note that since B is a pliant normal band, there exists $a_\alpha \in B_\alpha$ such that for any $\beta > \alpha$ and $u \in B_\beta$, $u\phi_{\beta,\alpha} = a_\alpha$.

Let $\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, $j_1, \cdots, j_r = m$, respectively, and ordered Y-components $\alpha_1, \cdots, \alpha_r$. Without loss of generality (excluding the trivial empty case), we may assume from Corollary 4.5 that

$$\overline{y_1} \cdots \overline{y_{j_l}} = \overline{x_1} \cdots \overline{x_{i_l}}\, \overline{u_1} \cdots \overline{u_s}$$

such that for all $k \in [1, s]$ we have $u_k \in B_{\delta_k}$ with $\delta_k > \alpha_l$, so that $u_k\phi_{\delta_k,\alpha_l} = a_{\alpha_l}$; or $u_k \in B_{\alpha_l}$ with $v_k u_k = u_k$ for some $v_k \in B_{\eta_k}$ such that $\eta_k > \alpha_l$, and in this case we have $a_{\alpha_l} u_k = u_k$, so that $a_{\alpha_l} \,\mathcal{R}\, u_k$. Thus the idempotents $u_1\phi_{\delta_1,\alpha_l}, \cdots, u_s\phi_{\delta_s,\alpha_l}$ are all $\mathcal{R}$-related, and so calling upon Corollary 4.8 we have

$$\overline{x_{i_l}}\, \overline{u_1} \cdots \overline{u_s} = \overline{x_{i_l}}\, \overline{u_1\phi_{\delta_1,\alpha_l}} \cdots \overline{u_s\phi_{\delta_s,\alpha_l}} = \overline{x_{i_l}}\, \overline{u_1\phi_{\delta_1,\alpha_l} \cdots u_s\phi_{\delta_s,\alpha_l}} = \overline{x_{i_l}}\, \overline{u_s\phi_{\delta_s,\alpha_l}} = \overline{x_{i_l}}\, \overline{u_s}.$$

On the other hand, again using Corollary 4.5 we have $y_{j_l} = wx_{i_l}u_1 \cdots u_s$, for some w, where $w = \varepsilon$ or $w \in B_{\alpha_l}$. Hence, for the purposes of verifying Condition (P), if we assume that $x_{i_l} \mathcal{L} y_{j_l}$, then $x_{i_l} = x_{i_l}u_s$, so that

$$\overline{x_{i_l}}\ \overline{u_s} = \overline{x_{i_l}u_s} = \overline{x_{i_l}}.$$

Hence $\overline{y_1} \cdots \overline{y_{j_l}} = \overline{x_1} \cdots \overline{x_{i_l}}$ as required. □

As an immediate consequence of Proposition 5.3 and Lemmas 5.5 and 5.6 we have the following result.

Theorem 5.7. *Let B be a normal band that is Y-basic or pliant. Then* IG(B) *is abundant.*

6. A normal band B for which IG(B) is not abundant

From Section 5, we know that the free idempotent idempotent generated semigroup IG(B) over a normal band B satisfying Condition (P) is an abundant semigroup. Therefore, one would like to ask whether IG(B) is abundant for any normal band B. In this section we answer the question in the negative by constructing a 10-element normal band B such that IG(B) is not abundant.

Throughout this section, we will use $\mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ as standard notation for a normal band.

Lemma 6.1. *Let B be a normal band, and let $x \in B_\beta, y \in B_\gamma$ with $\beta, \gamma \geq \alpha$. Then (x, y) is a basic pair implies $(x\phi_{\beta,\alpha}, y\phi_{\gamma,\alpha})$ is a basic pair and*

$$(x\phi_{\beta,\alpha})(y\phi_{\gamma,\alpha}) = (xy)\phi_{\delta,\alpha},$$

where δ is the minimum of β and γ.

Proof. Let (x, y) be a basic pair with $x \in B_\beta, y \in B_\gamma$. Then β, γ are comparable. If $\beta \geq \gamma$, then we either have $xy = y$ or $yx = y$. If $xy = y$, then $(x\phi_{\beta,\gamma})y = y$, so

$$y\phi_{\gamma,\alpha} = ((x\phi_{\beta,\gamma})y)\phi_{\gamma,\alpha} = (x\phi_{\beta,\alpha})(y\phi_{\gamma,\alpha}),$$

so $(x\phi_{\beta,\alpha}, y\phi_{\gamma,\alpha})$ is a basic pair. If $yx = y$, then $y(x\phi_{\beta,\gamma}) = y$, so

$$y\phi_{\gamma,\alpha} = (y(x\phi_{\beta,\gamma}))\phi_{\gamma,\alpha} = (y\phi_{\gamma,\alpha})(x\phi_{\beta,\alpha}),$$

so that $(x\phi_{\beta,\alpha}, y\phi_{\gamma,\alpha})$ is a basic pair.

A similar argument holds if $\gamma \geq \beta$. The final part of the lemma is clear. □

Lemma 6.2. *Let B be a normal band and let $\overline{u_1} \cdots \overline{u_n} \in$ IG(B) with $u_i \in B_{\alpha_i}$ and $\alpha_i \geq \alpha$ for all $i \in [1, n]$. Suppose that $\overline{v_1} \cdots \overline{v_m} \in$ IG(B) with $v_i \in B_{\beta_i}$ for all $i \in [1, m]$ and $\overline{u_1} \cdots \overline{u_n} \sim \overline{v_1} \cdots \overline{v_m}$. Note that $\beta_i \geq \alpha$, for all $i \in [1, m]$. Then in* IG(B_α) *we have*

$$\overline{u_1\phi_{\alpha_1,\alpha}} \cdots \overline{u_n\phi_{\alpha_n,\alpha}} = \overline{v_1\phi_{\beta_1,\alpha}} \cdots \overline{v_m\phi_{\beta_m,\alpha}}.$$

Proof. Suppose that $u_i = xy$ is a basic product with $x \in B_\delta, y \in B_\eta$, for some $i \in [1, n]$. Note that the minimum of δ and η is α_i. Then

$$\overline{u_1} \cdots \overline{u_n} \sim \overline{u_1} \cdots \overline{u_{i-1}}\ \overline{x}\ \overline{y}\ \overline{u_{i+1}} \cdots \overline{u_n}.$$

If follows from Lemma 6.1 that in IG(B_α)

$$\begin{aligned}\overline{u_1\phi_{\alpha_1,\alpha}} \cdots \overline{u_n\phi_{\alpha_n,\alpha}} &= \overline{u_1\phi_{\alpha_1,\alpha}} \cdots \overline{u_{i-1}\phi_{\alpha_{i-1},\alpha}}\ \overline{u_i\phi_{\alpha_i,\alpha}}\ \overline{u_{i+1}\phi_{\alpha_{i+1},\alpha}} \cdots \overline{u_n\phi_{\alpha_n,\alpha}} \\ &= \overline{u_1\phi_{\alpha_1,\alpha}} \cdots \overline{u_{i-1}\phi_{\alpha_{i-1},\alpha}}\ \overline{x\phi_{\delta,\alpha}y\phi_{\eta,\alpha}}\ \overline{u_{i+1}\phi_{\alpha_{i+1},\alpha}} \cdots \overline{u_n\phi_{\alpha_n,\alpha}} \\ &= \overline{u_1\phi_{\alpha_1,\alpha}} \cdots \overline{u_{i-1}\phi_{\alpha_{i-1},\alpha}}\ \overline{x\phi_{\delta,\alpha}}\ \overline{y\phi_{\eta,\alpha}}\ \overline{u_{i+1}\phi_{\alpha_{i+1},\alpha}} \cdots \overline{u_n\phi_{\alpha_n,\alpha}}\end{aligned}$$

as required. Note that the case where the elementary transition is a contraction is similar by symmetry of the underlying conditions. ☐

The final part of the next corollary is well known [18, Theorem 6.5] and was extended to left (right) seminormal bands by Dolinka in [3, Theorem 1].

Corollary 6.3. *Let B be a normal band and let $x_1, \cdots, x_n, y_1, \cdots, y_m \in B_\alpha$. Then $\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}$ in $\mathrm{IG}(B_\alpha)$ if and only if the equality holds in $\mathrm{IG}(B)$. Consequently, every maximal subgroup of $\mathrm{IG}(B)$ is free.*

Proof. The necessity is obvious, as any basic pair in B_α must also be basic in B. Suppose now that we have

$$\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}$$

in $\mathrm{IG}(B)$. Then there exists a sequence

$$\overline{x_1} \cdots \overline{x_n} \sim \overline{u_1} \cdots \overline{u_s} \sim \overline{v_1} \cdots \overline{v_t} \sim \cdots \sim \overline{w_1} \cdots \overline{w_l} \sim \overline{y_1} \cdots \overline{y_m}.$$

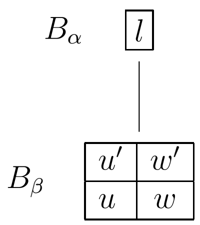
Note that all idempotents involved in the above sequence lie in components B_β where $\beta \geq \alpha$, so that successive applications of Lemma 6.2 give $\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}$ in $\mathrm{IG}(B_\alpha)$.

To see that the maximal subgroups of $\mathrm{IG}(B)$ are free, we recall from [12, Lemma 1] that every element in the $\mathcal{D}$-class of $\overline{e}$ is a product of idempotents that are $\mathcal{D}$-related to $\overline{e}$ in $\mathrm{IG}(B)$ and hence in B. Thus, if $e \in B_\alpha$, then every element of the $\mathcal{D}$-class of $\overline{e}$ is a product $\overline{e_1} \cdots \overline{e_n}$ where $e_i \in B_\alpha, 1 \leq i \leq n$. But two elements of this form are equal in $\mathrm{IG}(B)$ if and only if they are equal in $\mathrm{IG}(B_\alpha)$. Since the latter is known to have maximal subgroups that are free [18], it follows that the maximal subgroups of $\mathrm{IG}(B)$ are also free. ☐

We remark here that for an arbitrary band B, Corollary 6.3 need not be true.

Example 6.4. Let $B = B_\alpha \cup B_\beta$ be a band with semilattice structure and multiplication table defined by

	l	u	w	u'	w'
l	l	u'	w'	u'	w'
u	u	u	w	u	w
w	w	u	w	u	w
u'	u'	u'	w'	u'	w'
w'	w'	u'	w'	u'	w'



It is easy to check that B forms a band. By the uniqueness of normal forms in $\mathrm{IG}(B_\beta)$, we have $\overline{u'}\ \overline{w} \neq \overline{w'}$ in $\mathrm{IG}(B_\beta)$. However in $\mathrm{IG}(B)$ we have

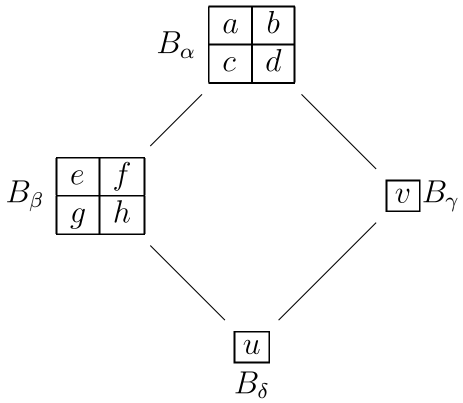
$$\begin{aligned} \overline{u'}\ \overline{w} &= \overline{u'l}\ \overline{w} \\ &= \overline{u'}\ \overline{l}\ \overline{w} \quad \text{(as } (u', l) \text{ is a basic pair)} \\ &= \overline{u'}\ \overline{lw} \quad \text{(as } (l, w) \text{ is a basic pair)} \\ &= \overline{u'}\ \overline{w'} \\ &= \overline{w'} \end{aligned}$$

With the above preparations, we now construct a 10-element normal band B for which $\mathrm{IG}(B)$ is not abundant.

Example 6.5. Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice $Y = \{\alpha, \beta, \gamma, \delta\}$ of rectangular bands (see the figure below), where $\phi_{\alpha,\beta} : B_\alpha \to B_\beta$ is defined by

$$a\phi_{\alpha,\beta} = e, b\phi_{\alpha,\beta} = f, c\phi_{\alpha,\beta} = g, d\phi_{\alpha,\beta} = h$$

the remaining morphisms being defined in the obvious unique manner.



Considering the element $\overline{e}\,\overline{v} \in \mathrm{IG}(B)$, we have

$$\begin{aligned}
\overline{e}\,\overline{v} &= \overline{e}\,\overline{dv} \\
&= \overline{e}\,\overline{d}\,\overline{v} \quad \text{(as } (d, v) \text{ is a basic pair)} \\
&= \overline{e}\,\overline{h}\,\overline{v} \quad \text{(as } \overline{e}\,\overline{d} = \overline{e}\,\overline{d\phi_{\alpha,\beta}} = \overline{e}\,\overline{h} \text{ by Corollary 4.8)} \\
&= \overline{e}\,\overline{h}\,\overline{av} \\
&= \overline{e}\,\overline{h}\,\overline{a}\,\overline{v} \quad \text{(as } (a, v) \text{ is a basic pair)} \\
&= \overline{e}\,\overline{h}\,\overline{e}\,\overline{v} \quad \text{(as } \overline{h}\,\overline{a} = \overline{h}\,\overline{a\phi_{\alpha,\beta}} = \overline{h}\,\overline{e} \text{ by Corollary 4.8)}
\end{aligned}$$

However, $\overline{e}\,\overline{h}\,\overline{e} \neq \overline{e}$ in $\mathrm{IG}(B_\beta)$ by the uniqueness of normal forms, so by Corollary 6.3, we have $\overline{e}\,\overline{h}\,\overline{e} \neq \overline{e}$ in $\mathrm{IG}(B)$, which implies $\overline{e}\,\overline{v}$ is not $\mathcal{R}^*$-related to $\overline{e}$. On the other hand, we have known from Theorem 4.13 that $\overline{e}\,\overline{v}\;\widetilde{\mathcal{R}}\;\overline{e}$, so that by Lemma 2.4 that $\overline{e}\,\overline{v}$ is not $\mathcal{R}^*$-related any idempotent of B, so that $\mathrm{IG}(B)$ is not an abundant semigroup.

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