Local embeddability into finite semigroups

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Basic notions

LEF (semi)group S

For every finite subset H of S there exists a finite (semi)group F_H and an injective function $f_H : H \to F_H$, such that for all $x, y \in H$ with $xy \in H$ we have $(xy)f_H = (xf_H)(yf_H)$.

Equivalent definitions

Let $\mathcal{F}$ be the class of finite semigroups.

- S is LEF;
- S is a model of $Th_\forall(\mathcal{F})$;
- S is embeddable into a model of $Th(\mathcal{F})$.

Example and non-example

Free semigroup $S_n = Sg\langle x_1, \ldots, x_n \rangle$

Any finite subset H of S_n embeds into semigroup

$$F_H = Sg\langle 0, x_1, \ldots, x_n | \mathcal{R} \rangle$$

where $\mathcal{R}$ is the set of relations stating $0u = u0 = 0$ for all u and $w = 0$ for $l(w) \geq m$ where $m = \max\{l(h) | h \in H\}$.

Bicyclic monoid $B = Mon\langle p, q | pq = 1 \rangle$

The finite subset $H = \{1, p, q, qp\}$ cannot embed into a finite semigroup as $ab = 1$, $ba \neq 1$ implies a and b generate a semigroup isomorphic to a bicyclic monoid.

History and study of the concept

- E. Gordon, A. Vershik, *Groups that are locally embeddable in the class of finite groups*, Algebra i Analiz **9**:1 (1997), 71–97;
- A. Ould Houcine, F. Point, *Alternatives for pseudofinite groups*, Journal of Group Theory , **16**:4 (2013), 461–495;
- V. Pestov, A. Kwiatkowska *An introduction to hyperlinear and sofic groups*, arXiv:0911.4266 (2012);
- O. Belegradek, *Local embeddability*, Algebra and Discrete Mathematics **14**:1 (2012), 14–28.

Group connection

Proposition

A group is an LEF semigroup if and only if it is an LEF group.

Universal groups

The semigroup $S = Sg\langle a, b | ab^2 = b^3 a\rangle$ is LEF.
Its universal group $G = Gp\langle a, b | ab^2 = b^3 a\rangle$ (one of the *Baumslag-Solitar groups*) is non-LEF.

Connection to residual finiteness, part 1

Residually finite (semi)group S

For every finite subset H of S there exists a finite (semi)group F_H and a homomorphism $\phi_H : S \to F_H$, such that ϕ_H is injective on H.

Lemma

Let S be a residually finite semigroup. Then S is LEF.

Proposition

Let S be a finitely presented LEF semigroup. Then S is residually finite.

Connection to residual finiteness, part 2

Consider a sequence $\{S_n\}_{n=1}^{\infty}$ of semigroups together with a system of homomorphisms $\phi_{n,m} : S_n \to S_m$ defined for all $m \geq n$, satisfying $\phi_{n,n} = \mathrm{id}_{S_n}$ and $\phi_{n,k} = \phi_{n,m}\phi_{m,k}$ for $n \leq m \leq k$.

Direct limit of $\{S_n\}_{n=1}^{\infty}$

The semigroup $\sqcup S_n/\sim$, where for $a_n \in S_n$ and $b_m \in S_m$ we have $a_n \sim b_m$ if and only if there exists an index $k \geq n, m$ such that $a_n\phi_{n,k} = b_m\phi_{m,k}$.

Theorem

A finitely generated semigroup is LEF if and only if it is isomorphic to a direct limit of residually finite semigroups.

Differences between LEF and residually finite semigroups

Let N be the semigroup $(\mathbb{N}, +)$.

Theorem (R. Gray, N. Ruškuc, 2012)

Let S be a semigroup. Then $N \times S$ is residually finite if and only if S is residually finite.

Proposition

Let S be a semigroup. Then $N \times S$ is LEF.

Linear semigroups

Proposition

Any finitely generated subsemigroup of the matrix semigroup $M_n(\mathbb{K})$ is residually finite.

Linear interpretation of LEF

Let S be a semigroup and $\mathcal{L} = \{\cdot\}$ be the language of semigroups. The following are equivalent.

- S is LEF;
- Every quantifier-free formula in $\mathcal{L}$ which is satisfiable in S can be satisfied in $M_n(\mathbb{K})$ for some $n \in \mathbb{N}$ and a field $\mathbb{K}$.

Inverse semigroups

Definition

A semigroup S is called *inverse* if for every $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

iLEF semigroup S

For every finite subset H of S there exists a finite inverse semigroup F_H and an injective function $f_H : H \to F_H$, such that for all $x, y \in H$ with $xy \in H$ we have $(xy)f_H = (xf_H)(yf_H)$.

Proposition

An inverse semigroup is an LEF semigroup if and only if it is an iLEF semigroup.

Structural properties, part 1

Proposition

Let S be a semigroup. The following are equivalent.

- S is LEF;
- S^0 (the semigroup S with adjoined zero) is LEF.
- S^1 (the semigroup S with adjoined identity) is LEF.

Proposition

Let S be an LEF semigroup and T be a subsemigroup of S. Then T is an LEF semigroup.

Structural properties, part 2

Proposition

A direct product of any given family of LEF semigroups is an LEF semigroup.

Wreath product of semigroups S and T

The semigroup, denoted $S \wr T$, with the underlying set $S^T \times T$ and the operation defined as follows: for $f, g \in S^T$ and $t, u \in T$ we have $(f, t)(g, u) = (f\,{}^t g, tu)$, where ${}^t g$ is an element of S^T defined by $(x)^t g = (xt)g$.

Proposition

Let S be an LEF semigroup and T be a locally finite semigroup. Then $S \wr T$ is an LEF semigroup.

Topological outlook

Consider the space Σ of all semigroups with generators $x_1, \dots, x_n$. For a semigroup $S \in \Sigma$ denote by ψ_S the natural homomorphism from the free semigroup $Sg\langle x_1, \dots, x_n \rangle$ to S.

Topology on Σ

$\mathcal{T}$ defined by base
$B_k(S_0) = \{S | \forall u, v : l(u), l(v) \leq k \text{ implies } u(\ker \psi_{S_0})v \iff u(\ker \psi_S)v\}$,
$k \in \mathbb{N}$

Proposition

Let S be a semigroup generated by $x_1, \dots, x_n$. The following are equivalent.

- S is LEF;
- There exists a sequence of finite semigroups which converges to S in the space Σ with topology $\mathcal{T}$.

A "converse" property

LWF (semi)group S

For every finite subset H of S there exists a finite (semi)group D_H and a function $d_H : D_H \to S$, such that $H \subset D_H d_H$ and for all $x', y' \in D_H$ with $x' d_H, y' d_H \in H$ it holds that $(x'y')d_H = (x'd_H)(y'd_H)$.

Proposition

An LEF (semi)group is LWF.

Theorem (E. Gordon, A. Vershik, 1997)

A group is LEF if and only if it is LWF.

Conjecture

A semigroup is LEF if and only if it is LWF.

Classifying non-LEF semigroups

For a given non-LEF semigroup S we can view the non-embedding finite set H as a partial multiplication table which cannot be completed to a multiplication table of a finite semigroup.

Example

The semigroup $S = \langle a, b | aab = a \rangle$ is non-LEF. One of its non-embedding sets $\{a, b, ab, aba\}$ gives rise to the following table.

	a	b	ab	aba
a	$*$	ab	a	$*$
b	$*$	$*$	$*$	$*$
ab	aba	$*$	$*$	$*$
aba	$*$	$*$	aba	$*$

Question

Is there an infinite number of independent non-completing tables?

Thank you!