Reconsidering MacLane: the foundations of categorical coherence

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York – Maths Dept. – Oct. 2013

Previous versions

Talks is based on:

- Talk at AbramskyFest (Oxford, June 2013)
- Joint Maths / Computing Seminar (Oxford, March 2013)

Topic of talk:

Foundations of category theory & “MacLane’s Theorem”

Category Theory is simply

a calculus of mathematical[1] structures.

It studies:

- Mathematical structures.
- Structure-preserving mappings.
- Transformations between structures.

[1] or logical, or computational, or linguistic, or . . .

It arose from work by:

- **Samuel Eilenberg**,
- **Saunders MacLane**,

in *Algebraic Topology*.

Later applied (despite protests) in other subjects:

Theoretical Computing	...
Linguistics	...
Logic	...
Quantum Mechanics	...
Foundations of Mathematics	...

History & prehistory

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Precursors to category theory

John von Neumann (1925): Axiomatic theory of *classes*.

A formalism for working with **proper classes**:

All sets, all monoids, all lattices, &c.

Later became the von Neumann, Gödel, Bernay formalism

- **von Neumann** originated the theory. *(proto-cat. theory)*
- **Gödel** made it logically consistent.
- **Bernay** rewrote it to look like ZFC set theory ....

Applications of category theory in various fields

. . . a large range of texts.

The underlying theory of categories:

"Categories for the Working Mathematician"
— S. MacLane (1971)

... examples & applications taken from **algebraic topology**.

Category Theory textbooks

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The definition ...

A **category** $\mathcal{C}$ consists of

- A **class** of objects, $Ob(\mathcal{C})$.
- For all objects $A, B \in Ob(\mathcal{C})$, a **set** of arrows $\mathcal{C}(A, B)$.

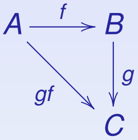
We will work diagrammatically:

An arrow $f \in \mathcal{C}(A, B)$ is drawn as

$$A \xrightarrow{f} B$$

The axioms ...

- Matching arrows can be composed



- Composition is associative

$$h(gf) = (hg)f$$

- There is an identity 1_A at each object A

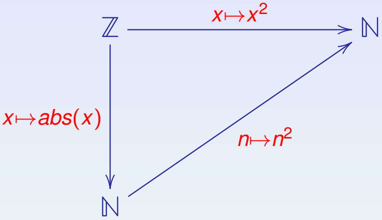
Examples of categories

- **Monoid**
  - (Objects:) *all monoids*.
  - (Arrows:) *homomorphisms*.
- **Set**
  - (Objects:) *all sets*.
  - (Arrows:) *functions*.
- **Poset**
  - (Objects:) *all partially ordered sets*.
  - (Arrows:) *order-preserving functions*.

Diagrams in categories

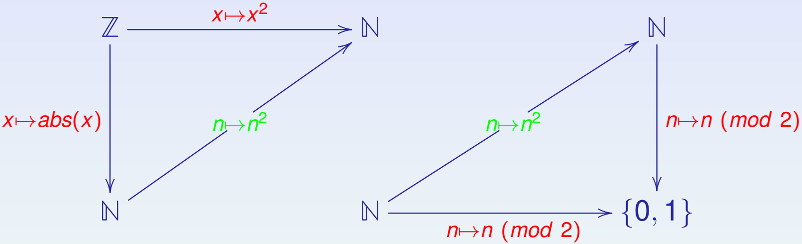
Identities and equations are usually expressed graphically.

A **diagram** in the category **Set**



A diagram **commutes** when all paths with the same source / target describe the same arrow.

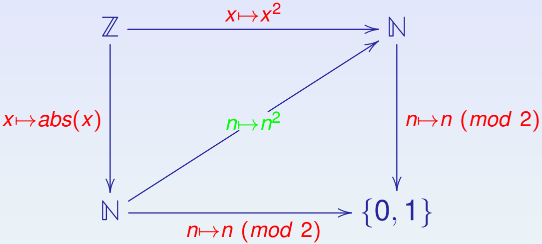
Commuting diagrams can be **pasted** *along a common edge.*



Both the above diagrams commute . . .

The art of diagram-chasing

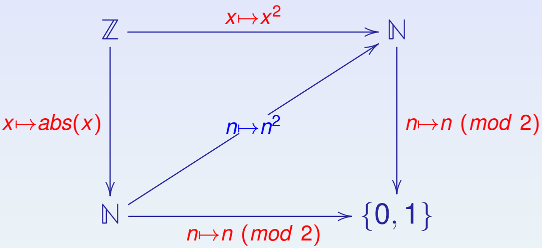
Commuting diagrams can be **pasted** *along a common edge.*



…this diagram also commutes!

The art of diagram-chasing

Edges can be **deleted** *in commuting diagrams.*



...

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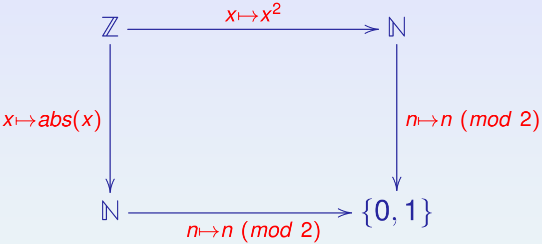

…this is still a commuting diagram.

Maps between categories

A mapping between categories $\mathcal{C}$ and $\mathcal{D}$ is a **functor** $\Gamma : \mathcal{C} \to \mathcal{D}$.

- **Objects** of $\mathcal{C}$ are mapped to **objects** of $\mathcal{D}$.
- **Arrows** of $\mathcal{C}$ are mapped to **arrows** of D.

$$A \xrightarrow{\ f\ } B \qquad\qquad \text{Category } \mathcal{C}$$

$$\Big\Downarrow \Gamma$$

$$\Gamma(A) \xrightarrow[\Gamma(f)]{} \Gamma(B) \qquad\qquad \text{Category } \mathcal{D}$$

Functors

Functors must preserve **composition** and **identities.**

$$\Gamma(1_X) = 1_{\Gamma(X)} \quad , \quad \Gamma(gf) = \Gamma(g)\Gamma(f)$$

Functors preserve **commutativity of diagrams.**

commutes in $\mathcal{C}$

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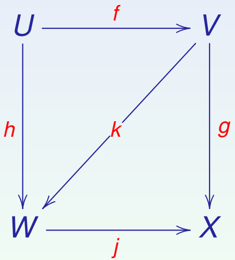

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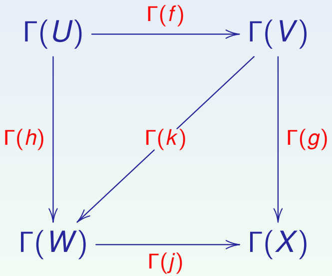

commutes in $\mathcal{D}$

Examples of functors (I)

A functor from **Set** to **Monoid**.

- Take a set X.
- Form the *free monoid* X^* *(All finite words over X).*

Every **function** $f : X \to Y$

induces a **homomorphism**

$map(f) : X^* \to Y^*$

This is a functor $Free$: **Set** $\to$ **Monoid**.

Examples of functors (II)

A functor from **Top$_*$** to **Group**.

- Take a pointed topological space T
- Form its *fundamental group* $\pi_1(T)$

> Every **continuous map**
> $c : S \to T$
>
> induces a **homomorphism**
>
> $\pi(f) : \pi_1(S) \to \pi_1(T)$

This is a functor π : **Top$_*$** $\to$ **Group**.

A sweeping generalisation

In general:

- finding **invariants** (e.g. fundamental group, K_0 group, &c.)
- using **constructors** (e.g. monoid semi-ring construction)
- type **re-assignments** (e.g. **Int** $\rightarrow$ **Real**)
- forming **algebraic models**
  (e.g. Brouwer-Heyting-Kolmogorov interpretation)
- . . .

are all examples of functors.

Monoidal Categories

and

MacLane's Theorem

Monoidal Categories $\equiv$ Categories with Tensors.

A **tensor** $\_ \otimes \_$ on a category is:

a way of combining two objects / arrows
to make a new object / arrow of the same category.

- **Objects:** Given X, Y, we can form $X \otimes Y$.
- **Arrows:** Given f, g, we can form $f \otimes g$.

Properties of tensors:

A **tensor** is a functor:

$$\_ \otimes \_ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

Functoriality implies:

1/ Types match:

$$A \xrightarrow{f} B$$

$$X \xrightarrow{h} Y$$

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Properties of tensors:

A **tensor** is a functor:

$$\_ \otimes \_ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

Functoriality implies:

2/ Composition is preserved:

$$A \otimes X \xrightarrow{(g \otimes k)(f \otimes h) = gf \otimes kh} C \otimes Z$$

Familiar examples

- **Tensor product** of Hilbert spaces / bounded linear maps
- **Cartesian product** (pairing) of Sets / functions
- **Direct sum** of Vector spaces / matrices
- **Disjoint union** of Sets / functions
- **Combining** Binary trees
- ...

The final conditions

We also require:

- **Associativity**

$$f \otimes (g \otimes h) = (f \otimes g) \otimes h$$

- **A unit object** $I \in Ob(\mathcal{C})$

$$X \otimes I = X = I \otimes X \quad \text{for all objects } X \in Ob(\mathcal{C})$$

Trivial objects

Monoidal categories usually[2] have a **unit object** $I \in Ob(\mathcal{C})$

$$A \otimes I \;=\; A \;=\; I \otimes A \qquad \text{for all objects } A \in Ob(\mathcal{C})$$

These are *trivial* objects within a category:

- The **single-element set.**
- The **trivial monoid.**
- The **empty space.**
- The **underlying scalar field.**
- The **trivially true** proposition.

---

[2] Part of the original definition. Later shown not to be essential (Saavedra72 / Kock08 / PH13).

A problem, and MacLane's solution

The problem ...

In real-world examples, the condition

$$f \otimes (g \otimes h) \; = \; (f \otimes g) \otimes h$$

is almost never satisfied.

... and its solution.

MacLane's theorem lets us *pretend* that

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with no harmful side-effects.

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Failure of associativity - an example

Associativity often fails, *in a trivial way*!

The disjoint union of sets

Given sets A, B,

$$A \uplus B \ = \ \{(a, 0)\} \ \cup \ \{(b, 1)\}$$

This is not associative . . . for ridiculous reasons.

Non-associativity of disjoint union

- $A \uplus (B \uplus C) =$

$$\{(a,0)\} \ \cup \ \{(b,01)\} \ \cup \ \{(c,11)\}$$

- $(A \uplus B) \uplus C =$

$$\{(a,00)\} \ \cup \ \{(b,10)\} \ \cup \ \{(c,1)\}$$

These are not the same set – for annoying syntactical reasons.

There is an obvious isomorphism between them ...

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Replacing equality by isomorphism:

- **Strict** associativity:

$$A \otimes (B \otimes C) \xrightarrow{\quad = \quad} (A \otimes B) \otimes C$$

- Associativity up to **isomorphism**

$$A \otimes (B \otimes C) \underset{\tau_{ABC}^{-1}}{\overset{\tau_{ABC}}{\rightleftarrows}} (A \otimes B) \otimes C$$

How to ignore isomorphisms

Provided the *associativity isomorphisms* satisfy:

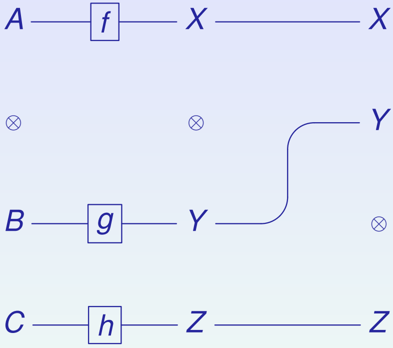
1. **naturality**
2. A **coherence condition**

we can *ignore them completely*.

Natural examples generally satisfy these conditions!

Naturality

We can 'push arrows through associativity isomorphisms'



$$\tau(f \otimes (g \otimes h)) = ((f \otimes g) \otimes h)\tau$$

Naturality

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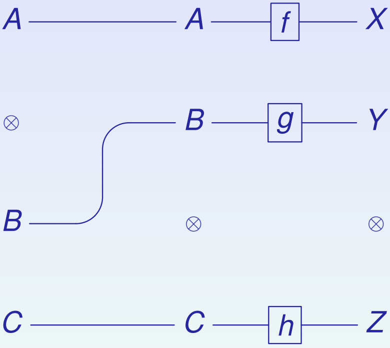

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Coherence

MacLane's coherence condition

The two ways of re-arranging

$$A \otimes (B \otimes (C \otimes D))$$

into

$$((A \otimes B) \otimes C) \otimes D$$

must be *identical*.

Also called **MacLane's Pentagon condition**

$$\tau \, \tau \;=\; (\tau \otimes 1) \, \tau \, (1 \otimes \tau)$$

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$$\tau \, \tau \; = \; (\tau \otimes 1) \, \tau \, (1 \otimes \tau)$$

Rebracketing four symbols

$$A \otimes (B \otimes (C \otimes D))$$

$$A \otimes ((B \otimes C) \otimes D)$$

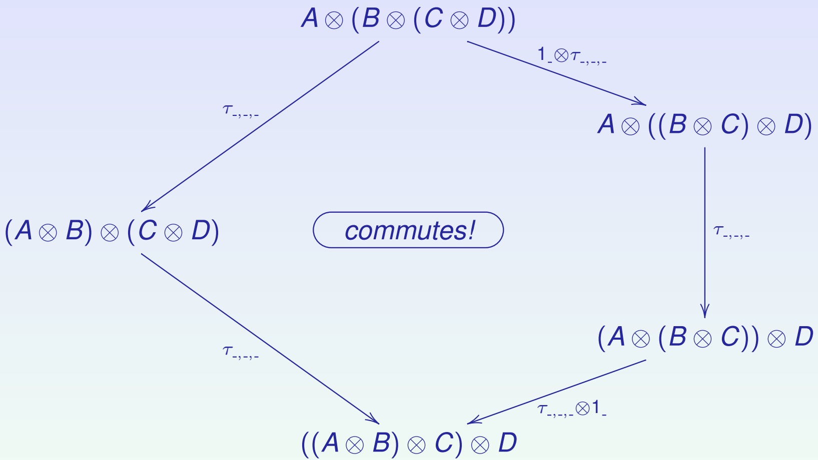
$$(A \otimes B) \otimes (C \otimes D)$$

$$(A \otimes (B \otimes C)) \otimes D$$

$$((A \otimes B) \otimes C) \otimes D$$

Yes, there are two paths you can go by, but ...

MacLane's pentagon



MacLane's coherence theorem:

When we have

1. Naturality
2. Coherence

every **canonical diagram** – built up using

$$\tau_{\_,\_,\_} \ , \ \_ \otimes \_ \ \text{ and } \ 1_{\_}$$

is *guaranteed* to commute.

A consequence:

Given a tensor that is *associative up to isomorphism*,

$$A \otimes (B \otimes C) \underset{\tau^{-1}_{ABC}}{\overset{\tau_{ABC}}{\rightleftarrows}} A \otimes (B \otimes C)$$

We can 'pretend it is *strictly associative*'

$$A \otimes (B \otimes C) \overset{=}{\text{———}} A \otimes (B \otimes C)$$

with no "harmful side-effects".

The conclusion

The theory of coherence has written itself out of existence!

By appealing to MacLane's theorem ...

We can completely ignore questions of coherence, naturality, pentagons, canonical diagrams, &c.

Two common descriptions of MacLane's theorem:

1. Every canonical diagram commutes.

2. We can treat

$$A \otimes (B \otimes C) \underset{\tau^{-1}_{A,B,C}}{\overset{\tau_{A,B,C}}{\rightleftarrows}} (A \otimes B) \otimes C$$

as a strict identity

$$A \otimes B \otimes C \overset{=}{\text{———}} A \otimes B \otimes C$$

with no 'harmful side-effects'.

Two inaccurate descriptions of MacLane's theorem:

1. ~~Every canonical diagram commutes.~~

2. ~~We can treat~~

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~~as a strict identity~~

$$A \otimes B \otimes C \overset{=}{\longrightarrow} A \otimes B \otimes C$$

~~with no 'harmful side-effects'.~~

Two contrary claims:

- Not every canonical diagram commutes.

  (Claim 1)

- Treating associativity **isomorphisms** as **strict identities** can have major consequences.[3]

  (Claim 2)

[3] *everything collapses to a triviality ...*

A simple example:

The **Cantor monoid** $\mathcal{U}$ (single-object category).

- Single object $\mathbb{N}$.
- Arrows: all bijections on $\mathbb{N}$.

The tensor

We have a tensor $(\_ \star \_) : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$.

$$(f \star g)(n) = \begin{cases} 2.f\left(\frac{n}{2}\right) & n \text{ even,} \\ 2.g\left(\frac{n-1}{2}\right) + 1 & n \text{ odd.} \end{cases}$$

Properties of the Cantor monoid (I)

The Cantor monoid has only one object —

$$\mathbb{N} \star (\mathbb{N} \star \mathbb{N}) = \mathbb{N} = (\mathbb{N} \star \mathbb{N}) \star \mathbb{N}$$

$(\_ \star \_) : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is associative *up to a natural isomorphism*

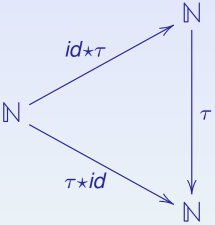
$$\tau(n) = \begin{cases} 2n & n \ (mod\ 2) = 0, \\ n+1 & n \ (mod\ 4) = 1, \\ \frac{n-1}{2} & n \ (mod\ 4) = 3. \end{cases}$$

that satisfies MacLane's pentagon condition.

This is not the identity map!

Properties of the Cantor monoid (II)

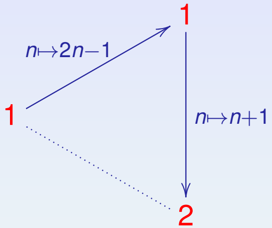
Not all canonical diagrams commute:



This diagram does *not* commute.

Properties of the Cantor monoid (II)

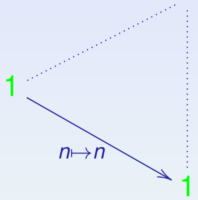
Using an actual number:



On the upper path, $1 \mapsto 2$.

Properties of the Cantor monoid (II)

Taking the right hand path:



1 $\neq$ 2, so this diagram does *not* commute.

What does MacLane's thm. actually say?

A recent (May 2013) report

*"Hines uses MacLane's theorem – **the fact that all canonical diagrams commute** – to construct a large class of examples where . . . "*

— Anonymous Referee

(Category Theory / Theoretical Computing journal).

If in doubt ...

. . . ask the experts:

http://en.wikipedia.org/wiki/Monoidal_category

WIKIPEDIA
The Free Encyclopedia

*"It follows that **any diagram** whose morphisms are built using [canonical isomorphisms], identities and tensor product commutes."*

Tinker, Tailor, Soldier, Sarcasm

Untangling The Web – *N.S.A. guide to internet use*

- *Do not as a rule rely on Wikipedia as your sole source of information.*
- *The best thing about Wikipedia are the external links from entries.*

MacLane, on MacLane's theorem

Categories for the working mathematician (1^{st} ed.)

- *Moreover all diagrams involving [canonical iso.s] must commute.* **(p. 158)**
- *These three [coherence] diagrams imply that "all" such diagrams commute.* **(p. 159)**
- *We can only prove that every "formal" diagram commutes.* **(p. 161)**

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What does his theorem say?

MacLane's coherence theorem for associativity

All diagrams *within the image of a certain functor* are guaranteed to commute.

This **usually** means all canonical diagrams.

In some circumstances, this is **not** the case.

Dissecting MacLane's theorem

— a closer look

A technicality:

In common with MacLane, we study *monogenic categories*.

Objects are generated by:

- Some object S,
- The tensor $(\_ \otimes \_)$.

This is not a restriction — S is thought of as a 'variable symbol'.

Dissecting MacLane's theorem

— a closer look

A technicality:

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Objects are generated by:

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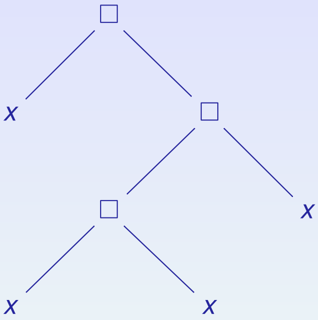
The source of the functor

This is based on (non-empty) *binary trees*.



- **Leaves** labelled by x,
- **Branchings** labelled by $\Box$.

The **rank** of a tree is the number of leaves.

A posetal category of trees

MacLane's category $\mathcal{W}$.

- **(Objects)** All non-empty binary trees.
- **(Arrows)** A **unique** arrow between any two trees *of the same rank*.

— write this as $(v \leftarrow u) \in \mathcal{W}(u, v)$.

Key points:

1. $(\_\square\_)$ is a tensor on $\mathcal{W}$.
2. $\mathcal{W}$ is **posetal** — all diagrams over $\mathcal{W}$ commute.

MacLane's theorem relies on a monoidal (i.e. tensor-preserving) functor

$$\mathcal{W}Sub : (\mathcal{W}, \Box) \to (\mathcal{C}, \otimes)$$

This is based on a notion of *substitution*.

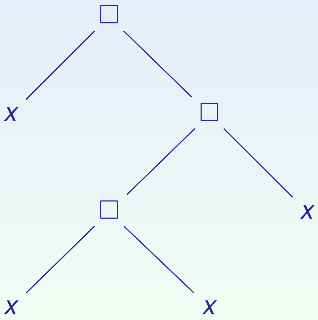
i.e. mapping *formal* symbols to *concrete* objects & arrows.

The functor itself

On objects:

- $\mathcal{W}Sub(x) = S$,
- $\mathcal{W}Sub(u\Box v) = \mathcal{W}Sub(u) \otimes \mathcal{W}Sub(v)$.

An object of $\mathcal{W}$:



An inductively defined functor (I)

On objects:

- $\mathcal{W}Sub(x) = S$,
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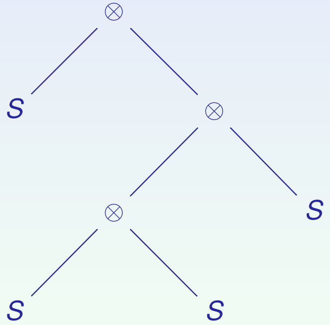
An object of $\mathcal{C}$:



On arrows:

- $\mathcal{W}Sub(u \leftarrow u) = 1_{\_}$.
- $\mathcal{W}Sub(a\Box v \leftarrow a\Box u) = 1_{\_} \otimes \mathcal{W}Sub(v \leftarrow u)$.
- $\mathcal{W}Sub(v\Box b \leftarrow u\Box b) = \mathcal{W}Sub(v \leftarrow u) \otimes 1_{\_}$.
- $\mathcal{W}Sub((a\Box b)\Box c \leftarrow a\Box(b\Box c)) = \tau_{\_,\_,\_}$.

The role of the Pentagon

The Pentagon condition $\implies$ $\mathcal{W}Sub$ is a monoidal functor.

An inductively defined functor (II)

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We have a functor $\mathcal{W}Sub : (\mathcal{W}, \Box) \to (\mathcal{C}, \otimes)$.

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As a corollary:

The image of **every diagram** in $(\mathcal{W}, \Box)$ **commutes** in $(\mathcal{C}, \otimes)$.

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“A beautiful (useful) theory slain by an ugly counterexample”?

A full theory of coherence for associativity is:

- *more mathematically elegant,*
- *much more practically useful!*

$\mathcal{W}Sub : (\mathcal{W}, \Box) \rightarrow (\mathcal{C}, \otimes)$ can **never** be an embedding when $\mathcal{C}$ has a **finite** set of objects.

The *Cantor monoid* has precisely one object

Where did this come from?

Hilbert's Hotel

A children's story about infinity.

Hilbert's "Grand Hotel"

An infinite corridor, with rooms numbered $0, 1, 2, 3, \ldots$

$\mathbb{N} \hookrightarrow \mathbb{N}$	the successor function.
$\mathbb{N} \cong \mathbb{N} \uplus \mathbb{N}$	the Cantor pairing.
$\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$	an exercise!
$[\mathbb{N} \to \{0, 1\}]$	is not isomorphic to $\mathbb{N}$

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Self-similarity

The categorical identity $S \cong S \otimes S$

Exhibited by two canonical isomorphisms:

- **(Code)** $\lhd : S \otimes S \to S$
- **(Decode)** $\rhd : S \to S \otimes S$

These are *unique* (up to *unique isomorphism*).

Examples

- *The natural numbers $\mathbb{N}$, Separable Hilbert spaces, Infinite matrices, Cantor set & other fractals, &c.*
- *C-monoids, and other untyped (single-object) categories with tensors*
- *Any unit object I of a monoidal category . . .*

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- *Any unit object I of a monoidal category . . .*

A tensor on a single object

At a self-similar object S, we may define a tensor by

$$\begin{array}{ccc} S \otimes S & \xrightarrow{t \otimes u} & S \otimes S \\ \uparrow \rhd & & \downarrow \lhd \\ S & \xrightarrow[t \star u]{} & S \end{array}$$

$(\_ \star \_)$ makes $C(S, S)$ a single-object monoidal category!

Associativity at a single object

The tensor $(\_ \star \_)$ is associative *up to isomorphism*.

$$\begin{array}{ccccc} S & \xrightarrow{\ \rhd\ } & S \otimes S & \xrightarrow{\ 1_S \otimes \rhd\ } & S \otimes (S \otimes S) \\ \big\downarrow{\scriptstyle \tau} & & & & \big\downarrow{\scriptstyle \tau_{S,S,S}} \\ S & \xleftarrow[\ \lhd\ ]{} & S \otimes S & \xleftarrow[\ \lhd \otimes 1_S\ ]{} & (S \otimes S) \otimes S \end{array}$$

Claim: This is the identity arrow

precisely when

the object S is trivial.

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constructing

categories where all

canonical diagrams commute

How to Rectify the Anomaly

Given a **badly-behaved** category $(\mathcal{C}, \otimes)$, we can *build a **well-behaved** (non-strict) version*.

Think of this as the **Platonic Ideal** of $(\mathcal{C}, \otimes)$.

We (still) assume $\mathcal{C}$ is *monogenic*, with objects generated by $\{S, \_ \otimes \_\}$

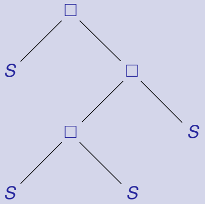
Building the 'Platonic Ideal'

We will construct $Plat_{\mathcal{C}}$

A version of $\mathcal{C}$ for which $\mathcal{W}Sub$ is an *embedding*.

Constructing $Plat_{\mathcal{C}}$

Objects are free binary trees



Leaves labelled by $S \in Ob(\mathcal{C})$,

Branchings labelled by $\Box$.

There is an **instantiation map** $Inst : Ob(Plat_{\mathcal{C}}) \to Ob(\mathcal{C})$

$$S\Box((S\Box S)\Box S) \;\mapsto\; S \otimes ((S \otimes S) \otimes S)$$

This is not just a matter of syntax!

Constructing $Plat_{\mathcal{C}}$

What about arrows?

Homsets are copies of homsets of $\mathcal{C}$

Given trees T_1, T_2,

$$Plat_{\mathcal{C}}(T_1, T_2) \ = \ \mathcal{C}(Inst(T_1), Inst(T_2))$$

Composition is inherited from $\mathcal{C}$ in the obvious way.

The tensor $(\_\Box\_) : Plat_{\mathcal{C}} \times Plat_{\mathcal{C}} \to Plat_{\mathcal{C}}$

$$\left.\begin{array}{l} A \xrightarrow{\ f\ } X \\ \\ B \xrightarrow{\ g\ } Y \end{array}\right\} \qquad A\Box X \xrightarrow{\ f\Box g\ } B\Box Y$$

The tensor of $Plat_{\mathcal{C}}$ is

- **(Objects)** A free formal pairing, $A\Box B$,
- **(Arrows)** Inherited from $(\mathcal{C}, \otimes)$, so $f\Box g \stackrel{def.}{=} f \otimes g$.

Some properties of the platonic ideal ...

1. The functor

$$\mathcal{W}Sub : (\mathcal{W}, \Box) \to (Plat_{\mathcal{C}}, \Box)$$

is always **monic**.

2. As a corollary:

All canonical diagrams of $(Plat_{\mathcal{C}}, \Box)$ commute.

3. Instantiation defines an **epic** monoidal functor

$$Inst : (Plat_{\mathcal{C}}, \Box) \to (\mathcal{C}, \otimes)$$

through which McL'.s substitution functor always factors.

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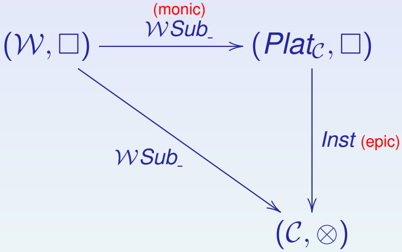
A monic / epic decomposition

MacLane's substitution functor always factors through the platonic ideal:



This gives a monic / epic decomposition of his functor.

A highly relevant question ...

What does the Platonic Ideal of a **single-object** category actually look like?

The simplest possible case:

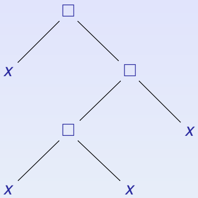
The trivial monoidal category $(\mathcal{I}, \otimes)$.

- **Objects:** $Ob(\mathcal{I}) = \{x\}$.
- **Arrows:** $\mathcal{I}(x, x) = \{1_x\}$.
- **Tensor:**

$$x \otimes x = x \quad , \quad 1_x \otimes 1_x = 1_x$$

What is the platonic ideal of $\mathcal{I}$?

(Objects) All non-empty binary trees:



(Arrows) For all trees T_1, T_2,

$$Plat_{\mathcal{I}}(T_1, T_2) \text{ is a single-element set.}$$

There is a unique arrow between any two trees!

A la recherche du tensors perdu

(PhD Thesis) The prototypical self-similar category $(\mathcal{X}, \Box)$

- **Objects:** *All non-empty binary trees.*
- **Arrows:** *A unique arrow between any two objects.*

This monoidal category:

1. was introduced to study **self-similarity** $S \cong S \otimes S$,
2. contains MacLane's $(\mathcal{W}, \Box)$ as a subcategory.

Coherence for Self-Similarity

(a special case of a much more general theory)

A straightforward coherence theorem

We base this on the category $(\mathcal{X}, \Box)$

- **Objects** All non-empty binary trees.
- **Arrows** A unique arrow between any two trees.

This category is posetal — all diagrams over $\mathcal{X}$ commute.

We will define a monoidal substitution functor:

$$\mathcal{X}Sub : (\mathcal{X}, \Box) \rightarrow (\mathcal{C}, \otimes)$$

The self-similarity substitution functor

An inductive definition of $\mathcal{X}Sub : (\mathcal{X}, \Box) \rightarrow (\mathcal{C}, \otimes)$

On objects:

$$\begin{array}{rcl} x & \mapsto & S \\ u\Box v & \mapsto & \mathcal{X}Sub(u) \otimes \mathcal{X}Sub(v) \end{array}$$

On arrows:

$$\begin{array}{rcl} (x \leftarrow x) & \mapsto & 1_S \in \mathcal{C}(S,S) \\ & & \\ (x \leftarrow x\Box x) & \mapsto & \lhd \in \mathcal{C}(S\otimes S, S) \\ (x\Box x \leftarrow x) & \mapsto & \rhd \in \mathcal{C}(S, S\otimes S) \\ & & \\ (b\Box v \leftarrow a\Box u) & \mapsto & \mathcal{X}Sub(b \leftarrow a) \otimes \mathcal{X}Sub(v \leftarrow u) \end{array}$$

Interesting properties:

1. $\mathcal{X}Sub : (\mathcal{X}, \Box) \to (\mathcal{C}, \otimes)$ is always functorial.
2. Every arrow built up from

$$\{\lhd \ , \ \rhd \ , \ 1_S \ , \ \_ \otimes \_\}$$

is the image of an arrow in $\mathcal{X}$.

3. The image of every diagram in $\mathcal{X}$ commutes.

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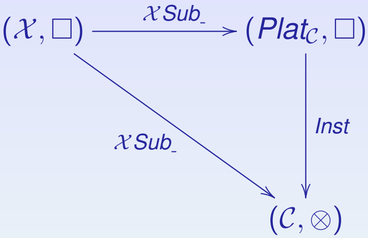
$\mathcal{X}$ *Sub* factors through the Platonic ideal

There is a monic-epic decomposition of $\mathcal{X}Sub$.



Every canonical (for self-similarity) diagram in $(Plat_{\mathcal{C}}, \Box)$ commutes.

Relating associativity and self-similarity

A tale of two functors

Comparing the *associativity* and *self-similarity* categories.

MacLane's $(\mathcal{W}, \Box)$

Objects: Binary trees.

Arrows: Unique arrow between two trees *of the same rank*.

The category $(\mathcal{X}, \Box)$

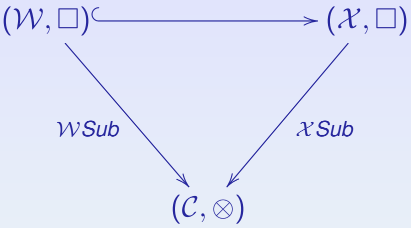
Objects: Binary trees.

Arrows: Unique arrow between *any two trees*.

There is an obvious inclusion $(\mathcal{W}, \Box) \hookrightarrow (\mathcal{X}, \Box)$

Is *associativity* a restriction of *self-similarity*?

Does the following diagram commute?



Does the **associativity** functor

factor through

the **self-similarity** functor?

Proof by contradiction:

Let's assume this is the case.

Special arrows of $(\mathcal{X}, \Box)$

For arbitrary trees u, e, v,

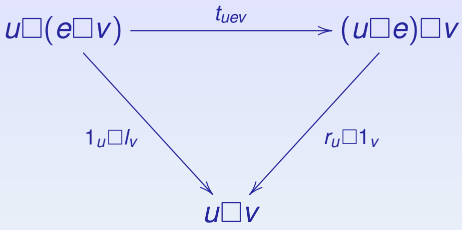
$$t_{uev} = ((u \Box e) \Box v \leftarrow u \Box (e \Box v))$$

$$l_v = (v \leftarrow e \Box v)$$

$$r_u = (u \leftarrow u \Box e)$$

Since all diagrams over X commute:

The following diagram over $(\mathcal{X}, \Box)$ commutes:



$$\begin{array}{ccc} u\Box(e\Box v) & \xrightarrow{t_{uev}} & (u\Box e)\Box v \\ & {\scriptstyle 1_u\Box l_v}\searrow \quad \swarrow{\scriptstyle r_u\Box 1_v} & \\ & u\Box v & \end{array}$$

Let's apply $\mathcal{X}Sub$ to this diagram.

By Assumption: $t_{uev} \mapsto \tau_{U,E,V}$ (assoc. iso.)

Notation: $u \mapsto U$, $v \mapsto V$, $e \mapsto E$, $l_v \mapsto \lambda_V$, $r_u \mapsto \rho_U$

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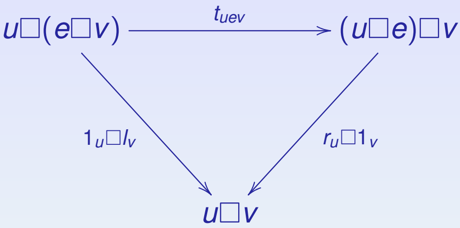

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The following diagram over $(\mathcal{C}, \otimes)$ commutes:

$$\begin{array}{ccc} U \otimes (E \otimes V) & \xrightarrow{\tau_{UEV}} & (U \otimes E) \otimes V \\ & {\scriptstyle 1_U \otimes \lambda_U} \searrow \quad \swarrow {\scriptstyle \rho_U \otimes 1_V} & \\ & U \otimes V & \end{array}$$

This is MacLane's **units triangle**
— the defining equation for a unit (trivial) object.

The choice of e was *arbitrary* — every object is trivial!

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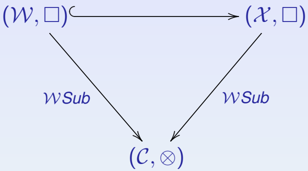
A general result

The following diagram commutes



exactly when $(\mathcal{C}, \otimes)$ is **degenerate** —

i.e. all objects are trivial.

An important special case:

What is **strict self-similarity**?

Can the code / decode maps

$$\lhd : S \otimes S \to S \quad , \quad \rhd : S \to S \otimes S$$

be **strict identities**?

In **single-object** monoidal categories:

We only have one object, so $S \otimes S = S$.

Take the identity as both the **code** and **decode** arrows.

Untyped $\equiv$ Strictly Self-Similar.

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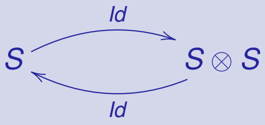

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Untyped $\equiv$ Strictly Self-Similar.

Generalising Isbell's argument

1. **Strict associativity:** $A \otimes (B \otimes C) = (A \otimes B) \otimes C$
   All arrows of $(\mathcal{W}, \Box)$ *are mapped to identities of* $(\mathcal{C}, \otimes)$

2. **Strict self-similarity:** $S \otimes S = S$.
   All arrows of $(\mathcal{X}, \Box)$ *are mapped to the identity of* $(\mathcal{C}, \otimes)$.

$\mathcal{W}Sub$ trivially factors through $\mathcal{X}Sub$.

The conclusion

Strictly associative untyped monoidal categories are **degenerate**.

This is seen in various fields ...

We see special cases of this in many areas:

- **(Monoid Theory)**

  Congruence-freeness (e.g. the polycyclic monoids).

- **(Group Theory)**

  No normal subgroups (e.g. Thompson's group $\mathcal{F}$).

- **(λ calculus / Logic)**

  Hilbert-Post completeness / Girard's dynamical algebra.

- **(Linguistics)**

  Recently (re)discovered ... not yet named!

Another perspective ...

Another way of looking at things:

The 'No Simultaneous Strictness' Theorem

One cannot have both

(I) Associativity $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$

(II) Self-Similarity $S \cong S \otimes S$

as strict equalities.