Idempotent tropical matrices

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arXiv:1106.4525v1 [math.RA]

29th February 2012

Research supported by EPSRC Grant EP/H000801/1, Israel Science Foundation grant number 448/09 and the Mathematisches Forschungsinstut Oberwolfach.

The tropical semifield

Let $\mathbb{FT} = (\mathbb{R}, \oplus, \otimes)$ where $\oplus$ and $\otimes$ denote two binary operations defined by

$$a \oplus b := \max(a, b), \quad a \otimes b := a + b,$$

- $(\mathbb{FT}, \oplus)$ is a commutative semigroup;
- $(\mathbb{FT}, \otimes)$ is a commutative group with identity element 0;
- $\otimes$ distributes over $\oplus$;
- For all $a \in \mathbb{FT}$ we have $a \oplus a = a$.

We say that $\mathbb{FT}$ is an idempotent semifield.
It is often referred to as the max-plus or **tropical semifield**.

Motivation

The tropical semifield has applications in areas such as...

- analysis of discrete event systems
- combinatorial optimisation and scheduling problems
- formal languages and automata
- statistical inference
- algebraic geometry
- computing eigenvalues of matrix polynomials...

Problems in application areas typically involve finding solutions to a system of linear equations over the tropical semifield.

We are therefore interested in properties of matrices with entries in the tropical semifield and their action upon vectors.

Tropical matrices

Consider the set $M_n(\mathbb{FT})$ of all $n \times n$ matrices over $\mathbb{FT}$.
We define multiplication $\otimes$ of tropical matrices as follows:

$$(A \otimes B)_{i,j} = \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{FT}).$$

It is easy to see that $(M_n(\mathbb{FT}), \otimes)$ is a **semigroup**.

Example.

$$\begin{pmatrix} 0 & 1 & 2 \\ 7 & 19 & 3 \\ -5 & 2 & 6 \end{pmatrix} \otimes \begin{pmatrix} -1 & -1 & -2 \\ -20 & 4 & 5 \\ 1 & 2 & 9 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 11 \\ 6 & 23 & 24 \\ 7 & 8 & 15 \end{pmatrix}$$

Tropical vectors

We write $\mathbb{FT}^n$ to denote the set of all n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{FT}$ and extend the **addition** $\oplus$ to $\mathbb{FT}^n$ componentwise:

$$(x \oplus y)_i = x_i \oplus y_i.$$

We also define a **scaling** action of $\mathbb{FT}$ on $\mathbb{FT}^n$:

$$(\lambda \otimes x)_i = \lambda \otimes x_i, \text{ for all } \lambda \in \mathbb{FT}.$$

Thus $\mathbb{FT}^n$ has the structure of an **$\mathbb{FT}$-module**.

Tropical polytopes

Consider the $\mathbb{FT}$-submodules of $\mathbb{FT}^n$
(i.e. subsets $X \subseteq \mathbb{FT}^n$ that are closed under $\oplus$ and scaling.)

A **tropical polytope** is a finitely generated submodule of $\mathbb{FT}^n$.

Example.
Let $A \in M_n(\mathbb{FT})$. We define the **row space** $R(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the rows of A.

Similarly, we define the **column space** $C(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the columns of A.

Green's relations in $M_n(\mathbb{FT})$

Theorem 1. [Various authors]
Let $A, B \in M_n(\mathbb{FT})$. Then

- $A \leqslant_{\mathcal{L}} B$ if and only if $R(A) \subseteq R(B)$;
- $A\mathcal{L}B$ if and only if $R(A) = R(B)$;
- $A \leqslant_{\mathcal{R}} B$ if and only if $C(A) \subseteq C(B)$;
- $A\mathcal{R}B$ if and only if $C(A) = C(B)$;
- $A\mathcal{H}B$ if and only if $R(A) = R(B)$ and $C(A) = C(B)$;

HK, 2010: $A\mathcal{D}B$ if and only if $R(A) \cong R(B)$;
$A\mathcal{D}B$ if and only if $C(A) \cong C(B)$;

JK, 2011: $\mathcal{D} = \mathcal{J}$.

Projectivity, idempotents and regularity

A module P is called **projective** if for every morphism $f : P \to M$ and every surjective morphism $g : N \twoheadrightarrow M$ there exists a morphism $h : P \to N$ such that $f = g \circ h$.

Lemma 2. [IJK]
A tropical polytope $X \subseteq \mathbb{FT}^n$ is projective if and only if it is isomorphic to the image of an idempotent in $M_n(\mathbb{FT})$.

Corollary 3. [IJK]
A is regular $\Leftrightarrow$ $R(A)$ is projective $\Leftrightarrow$ $C(A)$ is projective.

Proof:

- A is regular $\Leftrightarrow$ it is $\mathcal{D}$-related to an idempotent.
- HK, 2010: A is $\mathcal{D}$-related to an idempotent E if and only if $C(A) \cong C(E)$.
- Thus, by Lemma 2, A is regular if and only if $C(A)$ is projective.

Projectivity, idempotents and regularity

Example.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} -2 & -2 & -2 \\ -2 & -2 & -4 \\ 0 & -2 & -2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix}$$

1. Can we give a **geometric** characterisation of the projective tropical polytopes (and hence, the regular matrices in $M_n(\mathbb{FT})$)?
2. Given a projective polytope X, is there a *unique* idempotent E with $X = C(E)$?

Dimensions of tropical polytopes

Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope.

The **tropical dimension** of X is the maximum topological dimension of X regarded as a subset of $\mathbb{R}^n$.

We say that X has **pure tropical dimension** k if every open subset of X has topological dimension k.

The **generator dimension** of X is the minimum cardinality of a generating set for X.

The **dual dimension** of X is the minimum k such that X embeds linearly into $\mathbb{FT}^k$.

Lemma 4. [IJK] The dual dimension of X is equal to the generator dimension of $C(A)$, where A is any matrix satisfying $X = R(A)$.

Geometric characterisation of projectivity

Theorem 5. [IJK] Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope. Then
X is projective $\Leftrightarrow$ X has pure tropical dimension and trop. dim = gen. dim = dual dim.

Corollary 6. [IJK] Let $A \in M_n(\mathbb{FT})$. Then
A is regular $\Leftrightarrow$ $R(A)$ and $C(A)$ have the same pure tropical dimension equal to their generator dimension.

Proof:

- Corollary 3: A regular $\Leftrightarrow R(A)$ projective $\Leftrightarrow C(A)$ projective.
- Theorem 5: $R(A)$ projective $\Leftrightarrow R(A)$ has pure tropical dim. and tropical dim. = generator dim. = dual dim.
- Lemma 4: The dual dimension of $R(A)$ is equal to the generator dimension of $C(A)$.

The number of idempotents in an $\mathcal{R}$-class

Recall we have shown that A is regular if and only if $C(A)$ is projective.
In other words, the $\mathcal{R}$-class corresponding to A contains an idempotent if and only if $C(A)$ is projective.

Given a projective polytope X, is there a *unique* idempotent E with $X = C(E)$?

Theorem 7. [JK] Let $X \subseteq \mathbb{FT}^n$ be a projective polytope.

- If X has tropical dimension n, then there is a unique idempotent $E \in M_n(\mathbb{FT})$ such that $X = C(E)$;
- Otherwise, X has tropical dimension $k < n$ and there are continuum many idempotents $E \in M_n(\mathbb{FT})$ such that $X = C(E)$

Idempotents and finite metrics

Let $[n] = \{1, \ldots, n\}$ and let $d : [n] \times [n] \to \mathbb{R}$ be a metric. Consider the $n \times n$ matrix E with $E_{i,j} = -d(i,j)$.
Then

- E is symmetric;
- $E \otimes E = E$;
- $C(E)$ has tropical dimension n.

Theorem 8. **[JK]** The columns of E with respect to d_H form a metric space isometric to $([n], d)$.

Corollary 9. **[JK]** Every n-point metric space embeds in $\mathbb{FT}^n$ as the (tropical) vertices of a pure n-dimensional polytope.