Identities in upper triangular tropical matrix semigroups and the bicyclic monoid

Marianne Johnson
(Joint work with Laure Daviaud and Mark Kambites)

York Semigroup Seminar, May 2017

Semigroup identities and varieties of semigroups

Let Σ be an alphabet, and consider two words $w, v \in \Sigma^+$. Say that $w = v$ is a **semigroup identity** for S if $\varphi(w) = \varphi(v)$ for all semigroup morphisms $\varphi : \Sigma^+ \to S$.

A **variety of semigroups** is a class of semigroups defined by satisfaction of a particular set of identities.

Semigroup identities and varieties of semigroups

Let Σ be an alphabet, and consider two words $w, v \in \Sigma^+$. Say that $w = v$ is a **semigroup identity** for S if $\varphi(w) = \varphi(v)$ for all semigroup morphisms $\varphi : \Sigma^+ \to S$.

A **variety of semigroups** is a class of semigroups defined by satisfaction of a particular set of identities.

Birkhoff's theorem: A class of semigroups is a variety if and only if it is closed under taking subsemigroups, direct products and homomorphic images.

Semigroup identities and varieties of semigroups

Let Σ be an alphabet, and consider two words $w, v \in \Sigma^+$. Say that $w = v$ is a **semigroup identity** for S if $\varphi(w) = \varphi(v)$ for all semigroup morphisms $\varphi : \Sigma^+ \to S$.

A **variety of semigroups** is a class of semigroups defined by satisfaction of a particular set of identities.

Birkhoff's theorem: A class of semigroups is a variety if and only if it is closed under taking subsemigroups, direct products and homomorphic images.

By the **variety generated by a semigroup S** we mean the variety defined by the set of all identities satisfied by S.

General Questions

Given a semigroup S we may ask...

- **Does S satisfy a non-trivial semigroup identity?**

General Questions

Given a semigroup S we may ask...

- **Does S satisfy a non-trivial semigroup identity?**

Golubchik and Mikhalev, 1978: For $n > 1$, K an infinite field, $M_n(K)$ does not satisfy a non-trivial semigroup identity.

General Questions

Given a semigroup S we may ask...

- **Does S satisfy a non-trivial semigroup identity?**

Golubchik and Mikhalev, 1978: For $n > 1$, K an infinite field, $M_n(K)$ does not satisfy a non-trivial semigroup identity.

- **Can we "describe" the identities satisfied by S?**

General Questions

Given a semigroup S we may ask...

- **Does S satisfy a non-trivial semigroup identity?**

Golubchik and Mikhalev, 1978: For $n > 1$, K an infinite field, $M_n(K)$ does not satisfy a non-trivial semigroup identity.

- **Can we "describe" the identities satisfied by S?**
  This can be difficult...

General Questions

Given a semigroup S we may ask...

- **Does S satisfy a non-trivial semigroup identity?**

Golubchik and Mikhalev, 1978: For $n > 1$, K an infinite field, $M_n(K)$ does not satisfy a non-trivial semigroup identity.

- **Can we "describe" the identities satisfied by S?**
  This can be difficult...

There are finite semigroups which are not finitely based!
Perkins, 1969: Six element example using 0-1 matrices.
Sapir/Volkov, 1980s: $M_n(K)$ where $n > 1$ and K finite.

General Questions

Given a semigroup S we may ask...

- **Does S satisfy a non-trivial semigroup identity?**

Golubchik and Mikhalev, 1978: For $n > 1$, K an infinite field, $M_n(K)$ does not satisfy a non-trivial semigroup identity.

- **Can we "describe" the identities satisfied by S?**
  This can be difficult...

There are finite semigroups which are not finitely based!
Perkins, 1969: Six element example using 0-1 matrices.
Sapir/Volkov, 1980s: $M_n(K)$ where $n > 1$ and K finite.

- **Does S generate the same semigroup variety as some other interesting/well-studied semigroup?**

The basics of tropical arithmetic

Let $\mathbb{T}$ denote the **tropical semiring** $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where $\oplus$ and $\otimes$ denote two binary operations defined by:
$x \oplus y := \max(x, y)$, $x \otimes y := x + y$.

- $x \oplus x = x$;
- $x \otimes 0 = 0 \otimes x = x$.

The basics of tropical arithmetic

Let $\mathbb{T}$ denote the **tropical semiring** $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where $\oplus$ and $\otimes$ denote two binary operations defined by:
$x \oplus y := \max(x, y)$, $x \otimes y := x + y$.

- $x \oplus x = x$;
- $x \otimes 0 = 0 \otimes x = x$.

The **semigroup** $M_n(\mathbb{T})$ is the set of all $n \times n$ **tropical matrices**, with multiplication $\otimes$ defined in the obvious way:

$$\begin{pmatrix} 2 & 1 \\ 0 & 19 \end{pmatrix} \otimes \begin{pmatrix} -1 & -1 \\ -20 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & 23 \end{pmatrix}$$

The basics of tropical arithmetic

Let $\mathbb{T}$ denote the **tropical semiring** $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where $\oplus$ and $\otimes$ denote two binary operations defined by:
$x \oplus y := \max(x, y)$, $x \otimes y := x + y$.

- $x \oplus x = x$;
- $x \otimes 0 = 0 \otimes x = x$.

The **semigroup** $M_n(\mathbb{T})$ is the set of all $n \times n$ **tropical matrices**, with multiplication $\otimes$ defined in the obvious way:

$$\begin{pmatrix} 2 & 1 \\ 0 & 19 \end{pmatrix} \otimes \begin{pmatrix} -1 & -1 \\ -20 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & 23 \end{pmatrix}$$

Variants: Let $\mathcal{T} = \mathbb{R}, \mathbb{Q}, \mathbb{Z}$ or $\mathbb{N}_0$. Can also consider:
$\bar{\mathcal{T}}_{\max} := (\mathcal{T} \cup \{-\infty\}, \oplus, \otimes)$, or $\mathcal{T}_{\max} := (\mathcal{T}, \oplus, \otimes)$, or
$\bar{\mathcal{T}}_{\min} := (\mathcal{T} \cup \{+\infty\}, \boxplus, \otimes)$, or $\mathcal{T}_{\min} := (\mathcal{T}, \boxplus, \otimes)$,
where $x \boxplus y = \min(x, y)$.

Tropical matrix identities

Does $M_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

Tropical matrix identities

Does $M_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

Izhakian and Margolis, 2010: For $n = 2$, yes!

Tropical matrix identities

Does $M_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?
Izhakian and Margolis, 2010: For $n = 2$, yes!

They proved that...

- The bicyclic monoid $\mathcal{B}$ embeds into the monoid $UT_2(\mathbb{T})$ of upper triangular tropical matrices.
  So, any identity satisfied by $UT_2(\mathbb{T})$ is also satisfied by $\mathcal{B}$.

Tropical matrix identities

Does $M_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?
Izhakian and Margolis, 2010: For $n = 2$, yes!

They proved that...

- The bicyclic monoid $\mathcal{B}$ embeds into the monoid $UT_2(\mathbb{T})$ of upper triangular tropical matrices.
  So, any identity satisfied by $UT_2(\mathbb{T})$ is also satisfied by $\mathcal{B}$.

- $UT_2(\mathbb{T})$ satisfies **Adjan's identity**:

$$abba\ ab\ abba = abba\ ba\ abba,$$

  thus exhibiting a minimal length identity for $UT_2(\mathbb{T})$.

Tropical matrix identities

Does $M_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?
Izhakian and Margolis, 2010: For $n = 2$, yes!

They proved that...

- The bicyclic monoid $\mathcal{B}$ embeds into the monoid $UT_2(\mathbb{T})$ of upper triangular tropical matrices.
  So, any identity satisfied by $UT_2(\mathbb{T})$ is also satisfied by $\mathcal{B}$.

- $UT_2(\mathbb{T})$ satisfies **Adjan's identity**:

$$abba\ ab\ abba = abba\ ba\ abba,$$

  thus exhibiting a minimal length identity for $UT_2(\mathbb{T})$.

- $M_2(\mathbb{T})$ satisfies the identity:

$$a^2b^2b^2a^2 \quad a^2b^2 \quad a^2b^2b^2a^2 = a^2b^2b^2a^2 \quad b^2a^2 \quad a^2b^2b^2a^2.$$

Upper triangular tropical matrix identities

- **Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?**

Upper triangular tropical matrix identities

- **Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?**

  Izhakian 2014, Okninski 2015, Taylor 2016: Yes - different constructions.

Upper triangular tropical matrix identities

- **Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?**

  Izhakian 2014, Okninski 2015, Taylor 2016: Yes - different constructions.

- **Can we "describe" the identities satisfied by $UT_n(\mathbb{T})$?**

Upper triangular tropical matrix identities

- **Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?**

  Izhakian 2014, Okninski 2015, Taylor 2016: Yes - different constructions.

- **Can we "describe" the identities satisfied by $UT_n(\mathbb{T})$?**

  Daviaud, J, Kambites, 2017: We give a necessary and sufficient condition for an identity to hold in $UT_n(\mathbb{T})$.

Upper triangular tropical matrix identities

- **Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?**

  Izhakian 2014, Okninski 2015, Taylor 2016: Yes - different constructions.

- **Can we "describe" the identities satisfied by $UT_n(\mathbb{T})$?**

  Daviaud, J, Kambites, 2017: We give a necessary and sufficient condition for an identity to hold in $UT_n(\mathbb{T})$.

- **Do $UT_2(\mathbb{T})$ and $\mathcal{B}$ generate the same variety?**

Upper triangular tropical matrix identities

- **Does $UT_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?**

  Izhakian 2014, Okninski 2015, Taylor 2016: Yes - different constructions.

- **Can we "describe" the identities satisfied by $UT_n(\mathbb{T})$?**

  Daviaud, J, Kambites, 2017: We give a necessary and sufficient condition for an identity to hold in $UT_n(\mathbb{T})$.

- **Do $UT_2(\mathbb{T})$ and $\mathcal{B}$ generate the same variety?**

  Daviaud, J, Kambites, 2017: Yes! Several variants of $UT_2(\mathbb{T})$ and $\mathcal{B}$ also generate the same variety.

The case $n = 2$

Consider morphisms $\varphi : \Sigma^+ \to UT_2(\mathbb{T})$ of the form:

$$(\bigstar) \quad \varphi(s) = \begin{pmatrix} x_s & x'_s \\ -\infty & 0 \end{pmatrix}, \text{ for all } s \in \Sigma.$$

The case $n = 2$

Consider morphisms $\varphi : \Sigma^+ \to UT_2(\mathbb{T})$ of the form:

$$(\bigstar) \quad \varphi(s) = \begin{pmatrix} x_s & x'_s \\ -\infty & 0 \end{pmatrix}, \text{ for all } s \in \Sigma.$$

- For all $w \in \Sigma^+$, can see that:

$$\varphi(w)_{1,1} \; = \; \sum_{t\in\Sigma} |w|_t x_t, \quad \varphi(w)_{2,1} = -\infty, \quad \varphi(w)_{2,2} = 0$$

The case $n = 2$

Consider morphisms $\varphi : \Sigma^+ \to UT_2(\mathbb{T})$ of the form:

$$(\bigstar) \quad \varphi(s) = \begin{pmatrix} x_s & x'_s \\ -\infty & 0 \end{pmatrix}, \text{ for all } s \in \Sigma.$$

- For all $w \in \Sigma^+$, can see that:

$$\varphi(w)_{1,1} = \sum_{t \in \Sigma} |w|_t x_t, \quad \varphi(w)_{2,1} = -\infty, \quad \varphi(w)_{2,2} = 0$$

$$\varphi(w)_{1,2} = \max_{s \in \Sigma} \left(\max_{w=usv} \left(\sum_{t \in \Sigma} |u|_t x_t + x'_s + 0 \right) \right)$$

The case $n = 2$

Consider morphisms $\varphi : \Sigma^+ \to UT_2(\mathbb{T})$ of the form:

$$(\bigstar) \quad \varphi(s) = \begin{pmatrix} x_s & x'_s \\ -\infty & 0 \end{pmatrix}, \text{ for all } s \in \Sigma.$$

- For all $w \in \Sigma^+$, can see that:

$$\begin{aligned}
\varphi(w)_{1,1} &= \sum_{t\in\Sigma} |w|_t x_t, \quad \varphi(w)_{2,1} = -\infty, \quad \varphi(w)_{2,2} = 0 \\
\varphi(w)_{1,2} &= \max{}_{s\in\Sigma}\left(\max{}_{w=usv}\left(\sum_{t\in\Sigma}|u|_t x_t + x'_s + 0\right)\right) \\
&= \max{}_{s\in\Sigma}\left(x'_s + \max{}_{w_i=s}\left(\sum_{t\in\Sigma}\lambda_t^w(i-1)x_t\right)\right)
\end{aligned}$$

where $|w|_t$ = the number of occurrences of the letter t in w, and $\lambda_t^w(k) = \#t$'s in the first k letters of w.

The case $n = 2$

The latter equation can be written tropically as:

$$\varphi(w)_{1,2} \;=\; \bigoplus_{s\in\Sigma}\left(x'_s \otimes \bigoplus_{w_i=s}\bigotimes_{t\in\Sigma} x_t^{\otimes\lambda_t^w(i-1)}\right) = \bigoplus_{s\in\Sigma}\left(x'_s \otimes f_s^w(\underline{x})\right)$$

The case $n = 2$

The latter equation can be written tropically as:

$$\varphi(w)_{1,2} \quad = \quad \bigoplus_{s \in \Sigma} \left(x'_s \otimes \bigoplus_{w_i = s} \bigotimes_{t \in \Sigma} x_t^{\otimes \lambda_t^w (i-1)} \right) = \bigoplus_{s \in \Sigma} \left(x'_s \otimes f_s^w(\underline{x}) \right)$$

Theorem 1: Let $w, v \in \Sigma^+$.
The identity $w = v$ holds on $UT_2(\mathbb{T})$ if and only if
$f_s^w(\underline{x}) = f_s^v(\underline{x})$ for all $s \in \Sigma$ and all $\underline{x} \in \mathbb{R}^\Sigma$.

The case $n = 2$

The latter equation can be written tropically as:

$$\varphi(w)_{1,2} \;=\; \bigoplus_{s\in\Sigma}\left(x'_s \otimes \bigoplus_{w_i=s}\bigotimes_{t\in\Sigma} x_t^{\otimes \lambda_t^w(i-1)}\right) = \bigoplus_{s\in\Sigma}\left(x'_s \otimes f_s^w(\underline{x})\right)$$

Theorem 1: Let $w, v \in \Sigma^+$.
The identity $w = v$ holds on $UT_2(\mathbb{T})$ if and only if $f_s^w(\underline{x}) = f_s^v(\underline{x})$ for all $s \in \Sigma$ and all $\underline{x} \in \mathbb{R}^\Sigma$.

- If $f_s^w(\underline{x}) \neq f_s^v(\underline{x})$ for some $s \in \Sigma$ and $\underline{x} \in \mathbb{R}^\Sigma$, then there is a morphism $(\bigstar)$ falsifying the identity in $UT_2(\mathbb{T})$.

The case $n = 2$

The latter equation can be written tropically as:

$$\varphi(w)_{1,2} = \bigoplus_{s \in \Sigma} \left(x'_s \otimes \bigoplus_{w_i = s} \bigotimes_{t \in \Sigma} x_t^{\otimes \lambda_t^w (i-1)} \right) = \bigoplus_{s \in \Sigma} \left(x'_s \otimes f_s^w(\underline{x}) \right)$$

Theorem 1: Let $w, v \in \Sigma^+$.
The identity $w = v$ holds on $UT_2(\mathbb{T})$ if and only if $f_s^w(\underline{x}) = f_s^v(\underline{x})$ for all $s \in \Sigma$ and all $\underline{x} \in \mathbb{R}^\Sigma$.

- If $f_s^w(\underline{x}) \neq f_s^v(\underline{x})$ for some $s \in \Sigma$ and $\underline{x} \in \mathbb{R}^\Sigma$, then there is a morphism $(\bigstar)$ falsifying the identity in $UT_2(\mathbb{T})$.
- If $f_s^w(\underline{x}) = f_s^v(\underline{x})$ for all $s \in \Sigma$ and all $\underline{x} \in \mathbb{R}^\Sigma$ can show that w and v must have the **same content**. The result then follows from the following (technical) reductions...

Reductions

$UT_n(\mathbb{T})$ and $UT_n(\mathbb{R}_{\max})$ satisfy the same identities.

Reductions

$UT_n(\mathbb{T})$ and $UT_n(\mathbb{R}_{\max})$ satisfy the same identities.

If $\rho : \Sigma^+ \to UT_n(\mathbb{T})$ is a morphism falsifying an identity, then construct from this a morphism $\rho_x : \Sigma^+ \to UT_n(\mathbb{R}_{\max})$ where any $-\infty$ entries on or above the diagonal in $\rho(s)$ are replaced by $x \ll 0$ in $\rho_x(s)$.

Reductions

$UT_n(\mathbb{T})$ and $UT_n(\mathbb{R}_{\max})$ satisfy the same identities.

If $\rho : \Sigma^+ \to UT_n(\mathbb{T})$ is a morphism falsifying an identity, then construct from this a morphism $\rho_x : \Sigma^+ \to UT_n(\mathbb{R}_{\max})$ where any $-\infty$ entries on or above the diagonal in $\rho(s)$ are replaced by $x \ll 0$ in $\rho_x(s)$.

Let $w, v \in \Sigma^+$ be words of the **same content**.
If $\varphi(w) = \varphi(v)$ for each morphism $\varphi : \Sigma^+ \to UT_n(\mathbb{R}_{\max})$ satisfying $\varphi(s)_{n,n} = 0$ for all $s \in \Sigma$, then $w = v$ is an identity in $UT_n(\mathbb{T})$.

Reductions

$UT_n(\mathbb{T})$ and $UT_n(\mathbb{R}_{\max})$ satisfy the same identities.

If $\rho : \Sigma^+ \to UT_n(\mathbb{T})$ is a morphism falsifying an identity, then construct from this a morphism $\rho_x : \Sigma^+ \to UT_n(\mathbb{R}_{\max})$ where any $-\infty$ entries on or above the diagonal in $\rho(s)$ are replaced by $x \ll 0$ in $\rho_x(s)$.

Let $w, v \in \Sigma^+$ be words of the **same content**.
If $\varphi(w) = \varphi(v)$ for each morphism $\varphi : \Sigma^+ \to UT_n(\mathbb{R}_{\max})$ satisfying $\varphi(s)_{n,n} = 0$ for all $s \in \Sigma$, then $w = v$ is an identity in $UT_n(\mathbb{T})$.

Follows from the above together with the fact that if ψ and ϕ are morphisms related by $\psi(s) = \mu_s \otimes \phi(s)$ for some $\mu_s \in \mathbb{R}_{\max}$,

$$\phi(w) = \phi(v) \Leftrightarrow \psi(w) = \psi(v).$$

The case $n = 2$: Example

By **Theorem 1:** The identity
$w := abba\ ab\ abba = abba\ ba\ abba =: v$ holds on $UT_2(\mathbb{T})$ if and only if $f_s^w(\underline{x}) = f_s^v(\underline{x})$ for $s = a, b$ and all $\underline{x} \in \mathbb{R}^\Sigma$, where

$$f_s^u(\underline{x}) = \bigoplus_{u_i = s} \bigotimes_{t \in \Sigma} x_t^{\otimes \lambda_t^u(i-1)}$$

The case $n = 2$: Example

By **Theorem 1:** The identity
$w := abba\ ab\ abba = abba\ ba\ abba =: v$ holds on $UT_2(\mathbb{T})$ if and only if $f_s^w(\underline{x}) = f_s^v(\underline{x})$ for $s = a, b$ and all $\underline{x} \in \mathbb{R}^\Sigma$, where

$$f_s^u(\underline{x}) = \bigoplus_{u_i = s} \bigotimes_{t \in \Sigma} x_t^{\otimes \lambda_t^u(i-1)}$$

So for example,

$$f_a^w(x_a, x_b) \quad = \quad \max(0, x_a + 2x_b, 2x_a + 2x_b, 3x_a + 3x_b, 4x_a + 5x_b)$$

The case $n = 2$: Example

By **Theorem 1:** The identity
$w := abba\ ab\ abba = abba\ ba\ abba =: v$ holds on $UT_2(\mathbb{T})$ if and only if $f_s^w(\underline{x}) = f_s^v(\underline{x})$ for $s = a, b$ and all $\underline{x} \in \mathbb{R}^\Sigma$, where

$$f_s^u(\underline{x}) = \bigoplus_{u_i = s} \bigotimes_{t \in \Sigma} x_t^{\otimes \lambda_t^u(i-1)}$$

So for example,

$$\begin{aligned} f_a^w(x_a, x_b) &= \max(0, x_a + 2x_b, 2x_a + 2x_b, 3x_a + 3x_b, 4x_a + 5x_b) \\ &= \max(0, x_a + 2x_b, 3x_a + 3x_b, 4x_a + 5x_b) \end{aligned}$$

The case $n = 2$: Example

By **Theorem 1:** The identity
$w := abba\ ab\ abba = abba\ ba\ abba =: v$ holds on $UT_2(\mathbb{T})$ if and only if $f_s^w(\underline{x}) = f_s^v(\underline{x})$ for $s = a, b$ and all $\underline{x} \in \mathbb{R}^{\Sigma}$, where

$$f_s^u(\underline{x}) = \bigoplus_{u_i = s} \bigotimes_{t \in \Sigma} x_t^{\otimes \lambda_t^u(i-1)}$$

So for example,

$$\begin{aligned} f_a^w(x_a, x_b) &= \max(0, x_a + 2x_b, 2x_a + 2x_b, 3x_a + 3x_b, 4x_a + 5x_b) \\ &= \max(0, x_a + 2x_b, 3x_a + 3x_b, 4x_a + 5x_b) \end{aligned}$$

$$\begin{aligned} f_a^v(x_a, x_b) &= \max(0, x_a + 2x_b, 2x_a + 3x_b, 3x_a + 3x_b, 4x_a + 5x_b) \\ &= \max(0, x_a + 2x_b, 3x_a + 3x_b, 4x_a + 5x_b) \end{aligned}$$

The bicyclic monoid

Recall that $\mathcal{B} = \langle p, q : pq = 1 \rangle = \{q^i p^j : i, j \in \mathbb{N}_0\}$.

- Izhakian and Margolis: $\mathcal{B} \hookrightarrow UT_2(\mathbb{T})$ via

$$q^i p^j \mapsto \begin{pmatrix} i-j & i+j \\ -\infty & j-i \end{pmatrix}$$

Theorem 2: $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety.

The bicyclic monoid

Recall that $\mathcal{B} = \langle p, q : pq = 1 \rangle = \{q^i p^j : i, j \in \mathbb{N}_0\}$.

- Izhakian and Margolis: $\mathcal{B} \hookrightarrow UT_2(\mathbb{T})$ via

$$q^i p^j \mapsto \begin{pmatrix} i-j & i+j \\ -\infty & j-i \end{pmatrix}$$

Theorem 2: $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety.

- We show that if $w \neq v$ in $UT_2(\mathbb{T})$ then we can construct a morphism from Σ^+ to the image of $\mathcal{B}$ in $UT_2(\mathbb{T})$ that falsifies the identity.
- Easy to see that this can be done if $\mathrm{cont}(w) \neq \mathrm{cont}(v)$.

The bicyclic monoid

Suppose that $w \neq v$ in $UT_2(\mathbb{T})$, where $\mathrm{cont}(w) = \mathrm{cont}(v)$.

The bicyclic monoid

Suppose that $w \neq v$ in $UT_2(\mathbb{T})$, where $\mathrm{cont}(w) = \mathrm{cont}(v)$.

- Can assume that $f_t^w(\underline{x}) > f_t^v(\underline{x})$ for some $t \in \Sigma$, $\underline{x} \in \mathbb{R}^\Sigma$.

The bicyclic monoid

Suppose that $w \neq v$ in $UT_2(\mathbb{T})$, where $\mathrm{cont}(w) = \mathrm{cont}(v)$.

- Can assume that $f_t^w(\underline{x}) > f_t^v(\underline{x})$ for some $t \in \Sigma$, $\underline{x} \in \mathbb{R}^\Sigma$.

- The functions are **piecewise linear**. Inequality above holds in some open neighbourhood of $\underline{x}$, so can choose $\underline{x} \in \mathbb{Q}^\Sigma$.

The bicyclic monoid

Suppose that $w \neq v$ in $UT_2(\mathbb{T})$, where $\mathrm{cont}(w) = \mathrm{cont}(v)$.

- Can assume that $f_t^w(\underline{x}) > f_t^v(\underline{x})$ for some $t \in \Sigma$, $\underline{x} \in \mathbb{R}^\Sigma$.

- The functions are **piecewise linear**. Inequality above holds in some open neighbourhood of $\underline{x}$, so can choose $\underline{x} \in \mathbb{Q}^\Sigma$.

- All linear expressions involved are **homogeneous**. Multiplying all entries of $\underline{x}$ by a positive integer does not change the inequality, so can choose $\underline{x} \in (2\mathbb{Z})^\Sigma$.

The bicyclic monoid

Suppose that $w \neq v$ in $UT_2(\mathbb{T})$, where $\mathrm{cont}(w) = \mathrm{cont}(v)$.

- Can assume that $f_t^w(\underline{x}) > f_t^v(\underline{x})$ for some $t \in \Sigma$, $\underline{x} \in (2\mathbb{Z})^\Sigma$.

The bicyclic monoid

Suppose that $w \neq v$ in $UT_2(\mathbb{T})$, where $\mathrm{cont}(w) = \mathrm{cont}(v)$.

- Can assume that $f_t^w(\underline{x}) > f_t^v(\underline{x})$ for some $t \in \Sigma$, $\underline{x} \in (2\mathbb{Z})^\Sigma$.

- For each $s \in \Sigma$ choose a positive even integer $x'_s > x_s$ in such a way that $x'_s \ll x'_t$ for $s \neq t$.

The bicyclic monoid

Suppose that $w \neq v$ in $UT_2(\mathbb{T})$, where $\mathrm{cont}(w) = \mathrm{cont}(v)$.

- Can assume that $f_t^w(\underline{x}) > f_t^v(\underline{x})$ for some $t \in \Sigma$, $\underline{x} \in (2\mathbb{Z})^\Sigma$.
- For each $s \in \Sigma$ choose a positive even integer $x'_s > x_s$ in such a way that $x'_s \ll x'_t$ for $s \neq t$.
- With these choices, the morphism $(\star)$ falsifies the identity: $\varphi(w)_{1,2} = x'_t \otimes f_t^w(\underline{x}) \neq x'_t \otimes f_t^v(\underline{x}) = \varphi(v)_{1,2}$.

The bicyclic monoid

Suppose that $w \neq v$ in $UT_2(\mathbb{T})$, where $\mathrm{cont}(w) = \mathrm{cont}(v)$.

- Can assume that $f_t^w(\underline{x}) > f_t^v(\underline{x})$ for some $t \in \Sigma$, $\underline{x} \in (2\mathbb{Z})^\Sigma$.
- For each $s \in \Sigma$ choose a positive even integer $x'_s > x_s$ in such a way that $x'_s \ll x'_t$ for $s \neq t$.
- With these choices, the morphism $(\bigstar)$ falsifies the identity: $\varphi(w)_{1,2} = x'_t \otimes f_t^w(\underline{x}) \neq x'_t \otimes f_t^v(\underline{x}) = \varphi(v)_{1,2}$.
- Setting $i_s = \frac{x'_s}{2}, j_s = \frac{x'_s - x_s}{2} \in \mathbb{N}_0$ then gives

$$\varphi(s) = (i_s - j_s) \otimes \begin{pmatrix} i_s - j_s & i_s + j_s \\ -\infty & j_s - i_s \end{pmatrix}$$

The bicyclic monoid

Suppose that $w \neq v$ in $UT_2(\mathbb{T})$, where $\mathrm{cont}(w) = \mathrm{cont}(v)$.

- Can assume that $f_t^w(\underline{x}) > f_t^v(\underline{x})$ for some $t \in \Sigma$, $\underline{x} \in (2\mathbb{Z})^\Sigma$.
- For each $s \in \Sigma$ choose a positive even integer $x'_s > x_s$ in such a way that $x'_s \ll x'_t$ for $s \neq t$.
- With these choices, the morphism ($\bigstar$) falsifies the identity: $\varphi(w)_{1,2} = x'_t \otimes f_t^w(\underline{x}) \neq x'_t \otimes f_t^v(\underline{x}) = \varphi(v)_{1,2}$.
- Setting $i_s = \frac{x'_s}{2}, j_s = \frac{x'_s - x_s}{2} \in \mathbb{N}_0$ then gives

$$\varphi(s) = (i_s - j_s) \otimes \begin{pmatrix} i_s - j_s & i_s + j_s \\ -\infty & j_s - i_s \end{pmatrix}$$

- Take $\psi(s) = (j_s - i_s) \otimes \varphi(s)$.

Variants

- Let $\mathcal{T} = \mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$ and define the semigroup $\mathcal{B}_{\mathcal{T}} := \mathcal{T} \times \mathcal{T}$ via the product

$$(a, b) \cdot (c, d) = (a - b + \max(b, c), \;\; d - c + \max(b, c)).$$

- For each $\mathcal{T}$ as above we have

$$\mathcal{B} \hookrightarrow \mathcal{B}_{\mathcal{T}} \hookrightarrow UT_2(\mathcal{T}_{\max}) \hookrightarrow UT_2(\bar{\mathcal{T}}_{\max}) \hookrightarrow UT_2(\mathbb{T}).$$

- Since $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same semigroup variety, it follows that each of the intermediate variants above must satisfy exactly the same semigroup identities as these two.

Generalisation

$[n] := \{1, \ldots, n\}$, $\Gamma \subseteq \{(i,j) : i \neq j\}$ a directed graph.

If Γ is **transitive**, then

$$\Gamma(\mathbb{T}) = \{A \in M_n(\mathbb{T}) : A_{i,j} \neq -\infty \Rightarrow i = j \text{ or } (i,j) \in \Gamma\}$$

is a semigroup.

Theorem: Let Γ be **non-empty, transitive, and acyclic**. The semigroup identity $w = v$ holds in $\Gamma(\mathbb{T})$ if and only if for all pairs $u \in \Sigma^+$ and ρ a path of length $|u|$ through Γ we have equality of tropical polynomial functions $f^w_{u,\rho} = f^v_{u,\rho}$.

Corollary: Let $L(\Gamma)$ denote the maximum length of any directed path in Γ. Then $\Gamma(\mathbb{T})$ satisfies the same semigroup identities as $UT_{L(\Gamma)+1}(\mathbb{T})$.

Example: The free monogenic inverse monoid

$$\begin{aligned} \mathcal{I} &= \langle x, x^{-1} : xx^{-1}x = x, x^{-1}xx^{-1} = x^{-1} \rangle \\ &\cong \{(i,j,k) \in \mathbb{Z}^3 : i, j \geqslant 0, -j \leq k \leq i\}. \end{aligned}$$

with product

$$(i,j,k) \cdot (i',j',k') = (\max(i,\ i'+k),\ \ \max(j,\ j'-k),\ \ k+k')$$

via $x \mapsto (1,0,1)$, $x^{-1} \mapsto (0,1,-1)$,

- For $\Gamma = \{(1,3),(2,3)\}$ can show that $\mathcal{I} \hookrightarrow \Gamma(\mathbb{T})$

$$(i,j,k) \mapsto \begin{pmatrix} k & -\infty & i \\ -\infty & -k & j \\ -\infty & -\infty & 0 \end{pmatrix}.$$

- $\Gamma(\mathbb{T})$ satisfies the same semigroup identities as $UT_2(\mathbb{T})$, and hence $\mathcal{B}$.
- Follows that $\mathcal{I}$ satisfies all identities satisfied by $\mathcal{B}$.

Tropical matrix identities

- Does $M_n(\mathbb{T})$ satisfy a non-trivial semigroup identity?

  Izhakian and Margolis: Identity of length 20 for $M_2(\mathbb{T})$.
  Shitov: Identity of length 1,795,308 for $M_3(\mathbb{T})$.
  Open for $n > 3$.

- Does $M_2(\mathbb{T})$ satisfy a shorter semigroup identity?

  Daviaud, J, 2017: Yes!
  (Although not much shorter - minimal length is 17.)

- Can we "describe" the identities satisfied by $M_2(\mathbb{T})$?

  Daviaud and J, 2017: Necessary conditions for $n = 2$; these may not yet be sufficient. (Work in progress.)