Classification of variants of partial Brauer monoids

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The York Semigroup Seminar
2nd June 2021

Overview

Variants

Let

- S be a semigroup, and
- fix an element $a \in S$.

Consider the semigroup $S^a = (S, \star_a)$, where

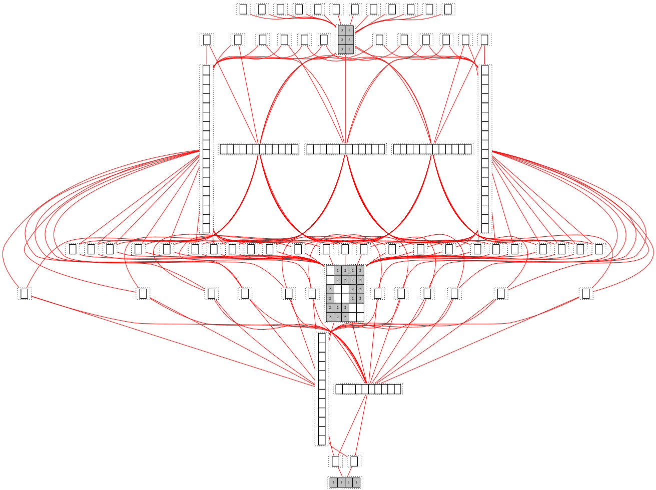
$$x \star_a y = xay, \qquad \text{for } x, y \in S.$$

Then, S^a is the *variant* of S with respect to a.

Example

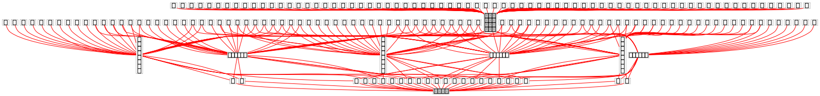
$\mathcal{T}_X^a$; $\mathcal{M}_n^A(\mathbb{F})$; G^a, where G is a group, ...

Examples

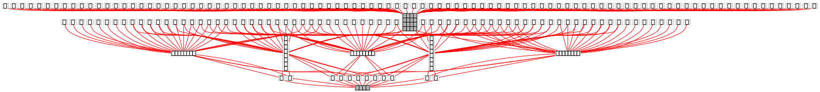


Egg-box diagram of the variant $\mathcal{T}_4^a$, where $a = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{smallmatrix}\right)$.

Examples



Egg-box diagram of the variant $\mathcal{T}_4^b$, where $b = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 2 \end{smallmatrix}\right)$.

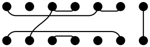


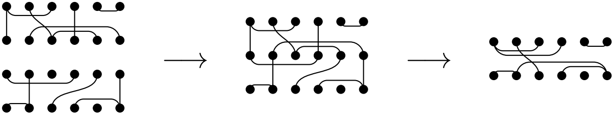
Egg-box diagram of the variant $\mathcal{T}_4^c$, where $c = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{smallmatrix}\right)$.

Source: I. Dolinka and J. East. Variants of finite full transformation. *Internat. J. Algebra Comput.*, 25(8): 1187–1222, 2015.

Partitions

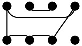
Consider the semigroup

- of partitions of $[n] \cup [n]'$ (example:  $\in \mathcal{P}_7$),
- with the composition

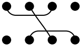


We are interested in variants of form $\mathcal{P}_n^\sigma$, as well as:

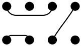
Planar $\mathscr{P}\mathcal{P}_n^\sigma$



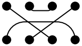
Partial Brauer $\mathcal{PB}_n^\sigma$



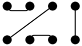
Motzkin $\mathscr{M}_n^\sigma$



Brauer $\mathcal{B}_n^\sigma$



Temperley-Lieb $\mathcal{TL}_n^\sigma$



Motivation

Variants

- were first studied by Hickey in his 1983 and 1986 papers, (they were used to provide a natural interpretation of the Nambooripad partial order on a regular semigroup);
- arise naturally in relation to Rees matrix semigroups [Khan and Lawson, 2001];
- were used as a means for introducing an alternative to the group of units in some classes of non-monoidal regular semigroups [Khan and Lawson, 2001);
- merit a whole chapter in the monograph *Classical finite transformation semigroups* by Ganyushkin and Mazorchuk (2009).

Motivation

Partitions

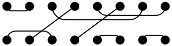
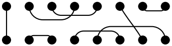
- arise in representation theory (see the survey *Diagram categories, representation theory, statistical mechanics* by Martin in Noncommutative rings, group rings, diagram algebras and their applications, 2008);
- arise in statistical mechanics and
- in knot theory (e.g. see the works of Jones from 1983, 1987, 1994 and Kauffman from 1987, 1990, 1997);
- arise in invariant theory [Lehrer and Zhang, 2012 and 2015];
- Partial Brauer algebras and semigroups were investigated by Kudryavtseva and Mazorchuk, (2006), Martin and Mazorchuk (2014), Dolinka, Gray, and East (2017), East and Ruškuc (to appear) and others.

Classification of variants of other types

Properties which determine the isomorphism class of

- the variants of form $\mathcal{T}_n^a$: size n and the structure of the kernel of the sandwich element a [Tsyaputa, 2003];

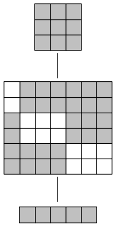
  example: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 2 & 4 & 1 & 2 & 1 & 4 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 8 & 6 & 5 & 8 & 3 & 3 & 5 \end{pmatrix}$.
- the variants of form $\mathcal{PT}_n^a$: size n and the structure of the kernel of the sandwich element a [Tsyaputa, 2004];
- the variants of form $\mathcal{B}_n^a$: size n and the rank of the sandwich element a [Dolinka, Đurđev, and East, to appear];

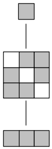
  example:  and 

WHAT DO WE KNOW?

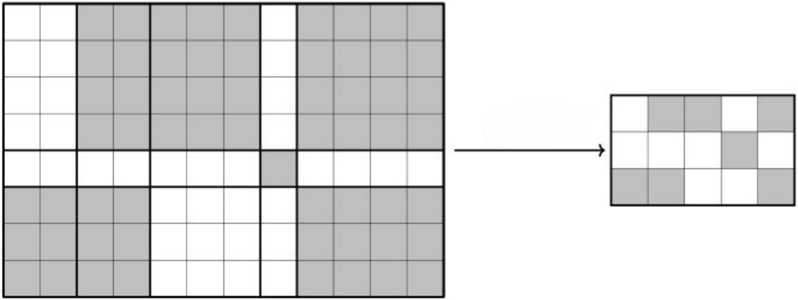
In *Sandwich semigroups in diagram categories* (by Dolinka, Đurđev, and East), we have proved that

- the regular elements of the variants of $\mathcal{P}_n$, $\mathscr{P}\mathcal{P}_n$, $\mathcal{B}_n$, $\mathcal{TL}_n$, $\mathcal{PB}_n$ and $\mathscr{M}_n$ form a subsemigroup;
- for $\mathcal{K} \in \{\mathcal{P}, \mathscr{P}\mathcal{P}, \mathcal{PB}, \mathcal{B}\}$ and any $\sigma \in \mathcal{K}_n$, the regular $\mathscr{D}$-classes in $\mathcal{K}_n^\sigma$ are $\mathrm{D}_q^\sigma = \{\alpha \in \mathrm{Reg}(\mathcal{K}_n^\sigma) : \mathrm{rank}(\alpha) = q\}$, for $0 \leq q \leq \mathrm{rank}(\sigma)$;
- there exists a homomorphism $\phi : \mathrm{Reg}(\mathcal{K}_n^a) \to \mathcal{K}_{\mathrm{rank}(a)}$.





A CLOSE-UP



Left: A $\mathscr{D}$-class in $\mathrm{Reg}(\mathcal{K}_n^a)$.
Right: The corresponding $\mathscr{D}$-class in $\mathcal{K}_{\mathrm{rank}(a)}$.

For $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{H}\}$, and $a, b \in \mathrm{Reg}(\mathcal{K}_n^a)$ we define

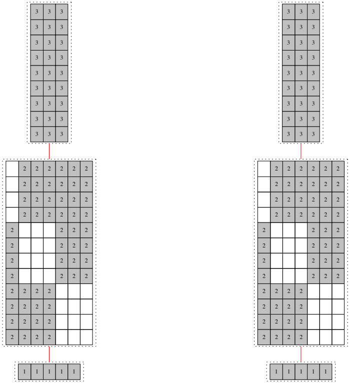
$$a \,\widehat{\mathscr{K}}\, b \quad \Leftrightarrow \quad \phi(a)\, \mathscr{K}\, \phi(b).$$

Isomorphisms of variants

Proposition

Let i, j be objects and let $a \in S_i$ and $c \in S_j$ be sandwich-regular elements of a locally small category S. If $\phi : S_i^a \to S_j^c$ is an isomorphism, then it preserves the relations $\widehat{\mathscr{R}}$, $\widehat{\mathscr{L}}$ and $\widehat{\mathscr{H}}$.

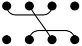
Isomorphisms of variants



The regular parts of two isomorphic variants.

Source: I. Dolinka and J. East. Variants of finite full transformation. *Internat. J. Algebra Comput.*, 25(8): 1187–1222, 2015.

PARTITIONS: NOTATION AND NOTIONS

- An example: $\alpha =$  $\in \mathcal{PB}_4$
  - $\mathrm{dom}(\alpha) = \{2\}$,
  - $\mathrm{codom}(\alpha) = \{3\}$,
  - $\ker(\alpha) = \{\{1,3\},\{2\},\{4\}\}$,
  - $\mathrm{coker}(\alpha) = \{\{1\},\{2,4\},\{3\}\}$,
  - $\mathrm{rank}(\alpha) = 1$,

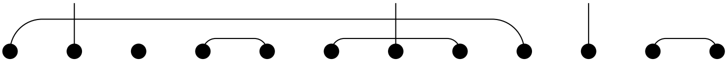
Lemma (Mazorchuk, 1998)

For $\alpha, \beta \in \mathcal{PB}_n$, in the monoid $\mathcal{PB}_n$ we have

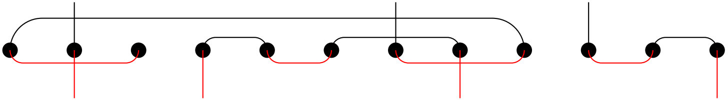
- $\alpha \mathscr{R} \beta \Leftrightarrow \mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ *and* $\ker(\alpha) = \ker(\beta)$,
- $\alpha \mathscr{L} \beta \Leftrightarrow \mathrm{codom}(\alpha) = \mathrm{codom}(\beta)$ *and* $\mathrm{coker}(\alpha) = \mathrm{coker}(\beta)$.

PB-pairs

- PB-pair (ε, X) on n:
  - ε – an equivalence on the set $[n] = \{1, \dots, n\}$,
  - $X \subseteq [n]$,
  - there exists $\alpha \in \mathcal{PB}_n$ such that $\ker(\alpha) = \varepsilon$ and $\mathrm{dom}(\alpha) = X$.



- The join of two PB pairs on n:



the result:

$$\Big(\{1,3,7,9 \mid 2 \mid 4,5,6,8 \mid 10,11,12\} \ , \ \{(2,2),(10,12)\} \Big)$$

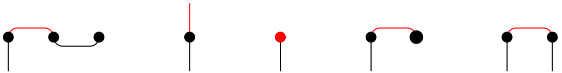
The number of $\mathscr{L}$-classes

- D_q^σ = the $\mathscr{D}$-class of $\mathcal{PB}_n^\sigma$ containing all regular elements of rank q;
- $|\,\mathrm{D}_q^\sigma\,/\,\mathscr{L}\,|$ = the number of PB-pairs (ε, X) on $\{1, \ldots, n\}$ with
  - $|X| = q$,
  - the join of (ε, X) and $(\ker(\sigma), \mathrm{dom}(\sigma))$ has rank q.
- This value depends on:
  - the number of vertices (n),
  - the number of singletons in $\ker(\sigma)$,
  - the rank of σ,
  - the value q.

THE RECURRENCE RELATION

For $n, k, r, q \in \mathbb{N}_0$, we define $\mu(n,k,r,q)$:

(i) $$\begin{aligned}\mu(n,k,r,q) =&(n-k)\mu(n-2,k,r,q)+\\ &\mu(n-1,k-1,r-1,q-1)+\\ &\mu(n-1,k-1,r-1,q)+\\ &(k-r)\mu(n-2,k-2,r-1,q)+\\ &(r-1)\mu(n-2,k-2,r-2,q)\end{aligned}$$
if $n \geq k \geq r \geq q > 0$ and $n \equiv k \pmod 2$.



(ii) $\mu(n,k,r,0) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}(2i-1)!!$
if $n \geq k \geq r$ and $n \equiv k \pmod 2$.

(iii) $\mu(n,k,r,q) = 0$, otherwise.

The crucial results

Proposition

Let $\sigma \in \mathcal{PB}_n$, where $\ker(\sigma)$ has k singletons and $\operatorname{rank}(\sigma) = r$. For $0 \leq q \leq r$, in $\mathcal{PB}_n^\sigma$ holds:

$$|\,\mathrm{D}_q^\sigma \,/\, \mathscr{L}\,| = \mu(n, k, r, q).$$

Proposition

Let $n, k, r, q \in \mathbb{N}_0$ with $n \geq k \geq r \geq q \geq \mathbf{1}$ and $n \equiv k (\mathrm{mod}\, 2)$. If $n \geq k + 2$, then

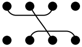
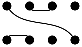
$$\mu(n, k, r, q) > \mu(n, k + 2, r, q).$$

Case 1: sandwich element with non-zero rank

Theorem

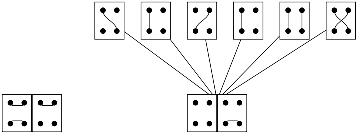
Let $m, n \in \mathbb{N}$, and let $\sigma \in \mathcal{PB}_m$ and $\tau \in \mathcal{PB}_n$ with $r = \operatorname{rank}(\sigma) \geq 1$ and $s = \operatorname{rank}(\tau) \geq 1$. In addition, write k and l for the number of singleton classes in $\ker(\sigma)$ and $\ker(\tau)$, respectively. Similarly, write p and w for the number of singleton classes in $\operatorname{coker}(\sigma)$ and $\operatorname{coker}(\tau)$, respectively. Then $\mathcal{PB}_m^\sigma \cong \mathcal{PB}_n^\tau$ if and only if $m = n$, $k = l$, $p = w$, and $r = s$.

Example

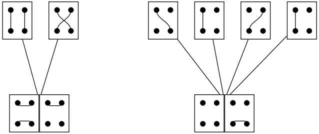
For $\alpha =$  and $\beta =$ , the variants $\mathcal{PB}_4^\alpha$ and $\mathcal{PB}_4^\beta$ are isomorphic.

Case 2: sandwich element of rank zero

- We have $\mathrm{Reg}(\mathcal{PB}_n^\sigma) = \{\alpha \in \mathcal{PB}_n : \mathrm{rank}(\alpha) = 0\} = \mathrm{D}_0^\sigma = \mathrm{J}_{\mathbf{0}}$. The remaining elements form singleton $\mathscr{J}$-classes, all unrelated and above $\mathrm{J}_{\mathbf{0}}$.
- However, these variants are not necessarily isomorphic! For $\sigma =$  and $\tau =$ , consider $\leq_{\mathscr{R}}$ in $\mathcal{PB}_2^\sigma$ and $\mathcal{PB}_2^\tau$:



In $\mathcal{PB}_2^\sigma$



In $\mathcal{PB}_2^\tau$

Open Problem

Classification of variants of $\mathcal{PB}_n$ having sandwich elements of rank 0.

Directions for further investigation

- Classification of variants of other partition monoids.
- Classification of sandwich semigroups of transformations.
- Classification of sandwich semigroups of partitions.

Thank you!