PROPER RESTRICTION SEMIGROUPS – SEMIDIRECT PRODUCTS AND W-PRODUCTS

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Abstract. Fountain and Gomes have shown that any proper left ample semigroup embeds into a so-called W-product, which is a subsemigroup of a reverse semidirect product $T \ltimes \mathcal{Y}$ of a semilattice $\mathcal{Y}$ by a monoid T, where the action of T on $\mathcal{Y}$ is injective with images of the action being order ideals of $\mathcal{Y}$. Proper left ample semigroups are proper left restriction, the latter forming a much wider class. The aim of this paper is to give necessary and sufficient conditions on a proper left restriction semigroup such that it embeds into a W-product. We also examine the complex relationship between W-products and semidirect products of the form $\mathcal{Y} \rtimes T$.

Introduction

Left restriction semigroups arise from many sources.[1] They are a class of unary semigroups (that is, semigroups equipped with an additional unary operation) that are precisely the unary semigroups isomorphic to unary subsemigroups of partial tranformation semigroups $\mathcal{PT}_X$, where the unary operation is $\alpha \mapsto I_{\operatorname{dom}\alpha}$. The reader can consult [10] or the unpublished notes [8] for history and further details.

Left restriction semigroups form a variety of unary semigroups, that is, semigroups equipped with an additional unary operation, here denoted by $^+$. The identities additional to associativity that define left restriction semigroups are:

$$x^+x = x,\ x^+y^+ = y^+x^+,\ (x^+y)^+ = x^+y^+,\ xy^+ = (xy)^+x.$$

For any left restriction semigroup S, we put

$$E = \{x^+ : x \in S\}.$$

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[1] They also have equally many names, including *weakly left E-ample*. We remark that the adjective 'left' is sometimes dropped in the literature; moreover some authors refer to what we would call *right restriction* semigroups as *restriction semigroups*.

It is easy to see that E is a semilattice under the semigroup multiplication, the *semilattice of projections* of S (also known as the *distinguished semilattice* of S).

Right restriction semigroups are defined dually, where in this case we use * for the unary operation. A bi-unary semigroup is *restriction* if it is both left and right restriction such that the semilattices of projections coincide, which latter condition is equivalent to the identities

$$(x^+)^* = x^+ \text{ and } (x^*)^+ = x^*$$

being satisfied.

Every inverse semigroup is left restriction with $a^+ = aa^{-1}$ (and also right restriction with $a^* = a^{-1}a$), so that, as left restriction semigroups form a variety, every subsemigroup of an inverse semigroup that is closed under $^+$ is left restriction. Not every left restriction semigroup is obtained in this way. Those that are, are precisely the *left ample* (formerly, left type A) semigroups, forming a quasi-variety determined by the addition of the quasi-identity

$$xz = yz \to xz^+ = yz^+. \tag{1}$$

A left restriction semigroup is *reduced* if $|E| = 1$. In this case, it is a monoid with identity the single projection. On the other hand, it is easy to see that any monoid M is left restriction, where we declare $a^+ = 1$, for every $a \in M$. An inverse semigroup, regarded as a left restriction semigroup, is reduced if and only if it is a group. We view left restriction semigroups as being natural extensions of inverse semigroups, obtained by dropping the condition of regularity, and, indeed, they have many analogous properties. A central theme has been to describe them in terms of reduced left restriction semigroups (i.e. monoids) and semilattices, echoing the approach to inverse semigroups which uses groups and semilattices as its building blocks. There is naturally a similar theme for (two-sided) restriction semigroups.

We denote the semidirect product of a semilattice $\mathcal{Y}$ by a monoid T acting on the *left* of $\mathcal{Y}$ by $\mathcal{Y} \rtimes T$ and the reverse semidirect product of a semilattice $\mathcal{Y}$ by a monoid T acting on the *right* by $T \ltimes \mathcal{Y}$; semigroups $\mathcal{Y} \rtimes T$ ($T \ltimes \mathcal{Y}$) are left (right) restriction, and more than this, are *proper* [6, 1]. Here 'proper' is the appropriate analogue of the notion of an E-unitary, or proper, inverse semigroup. If $\mathcal{Y}$ has an identity, and T acts on the left and right of $\mathcal{Y}$ satisfying the *compatibility conditions*, then a subsemigroup $\mathcal{Y} *_m T$ of $\mathcal{Y} \rtimes T$ is proper as a (two-sided) restriction semigroup [5]. The final construction we use here is that of a W-product $W(T, \mathcal{Y})$ of a semilattice $\mathcal{Y}$ by a monoid T, which is a subsemigroup of a semigroup of the form $T \ltimes \mathcal{Y}$, where the action of T satisfies some special properties; W-products are again proper restriction semigroups [7, 14].

The relationship between semidirect products of semilattices by monoids, semigroups of the form $\mathcal{Y} *_m T$ and W-products, is complex. Our first aim is to show that any W-semigroup embeds as a restriction semigroup into a semigroup of the form $\mathcal{Y} *_m T$ which, in our particular case, is the full semidirect product $\mathcal{Y} \rtimes T$.

This we do in Section 2. On the other hand, it is known that any proper left ample semigroup embeds into a W-product [4]. In Section 3 we find necessary and sufficient conditions on a proper left restriction semigroup such that it embeds into a W-product. Since these conditions involve the least right cancellative congruence ω_S on a semigroup S, we investigate ω_S in Section 4.

In Section 1 we briefly define the tools needed for the rest of the paper. We refer the reader to [11] for general semigroup background and [8] for further details concerning restriction semigroups and related classes.

1. Preliminaries

We emphasise that we always regard left restriction semigroups as algebras possessing two basic operations; as such, substructures, morphisms and congruences must respect both. Similarly, we regard restriction semigroups as algebras with three basic operations.

Let S be a left (right) restriction semigroup. We recall that S is partially ordered by $\leq$ where for any $a, b \in S$ we have $a \leq b$ if and only if $a = a^+b$ ($a = ba^*$) [8]. The relation $\leq$ is compatible on both sides with multiplication in S, and with the unary operation. If S is restriction, then it is easy to see that $a = a^+b$ if and only if $a = ba^*$, so that $\leq$ is unambiguously defined.

Result 1.1. [8] *Let S be a left restriction semigroup. Then for any $a, b \in S$:*

(i) $(a^+)^+ = a^+$;
(ii) $(ab)^+ \leq a^+$;
(iii) $(ab^+)^+ = (ab)^+$.

Let S be a left restriction semigroup. The relation σ_S on S is the least congruence identifying all the elements of E. As explained in [8, Section 8], we can regard σ_S as either a semigroup congruence or as a unary semigroup congruence.[2]

Result 1.2. [8, Lemma 8.1] *Let S be a left restriction semigroup. Then for any $a, b \in S$, we have that $a \,\sigma_S\, b$ if and only if $ea = eb$ for some $e \in E$.*

Clearly, S/σ_S is reduced for any left restriction semigroup S, and is the greatest reduced image of S. We also have call here to consider ω_S, the least right cancellative congruence on S. Since right cancellative semigroups have at most one idempotent, certainly $\sigma_S \subseteq \omega_S$.

Lemma 1.3. *Let S be a left restriction semigroup and let κ be a congruence contained in ω_S. Then*

(i) $S/\omega_S \cong (S/\kappa)/(\omega_S/\kappa)$;
(ii) $\omega_{S/\kappa} = \omega_S/\kappa$.

[2]From another perspective, a left restriction semigroup is a semigroup (rather than a unary semigroup) possessing particular properties related to a semilattice E of idempotents. In view of this, the relation σ_S is also referred to as σ_E in the literature.

In particular, $\omega_{S/\sigma_S} = \omega_S/\sigma_S$.

Proof. (i) is a standard algebraic fact. To show (ii), let $\nu_\kappa \colon S \mapsto S/\kappa$ and $\nu_{\omega_S} \colon S \mapsto S/\omega_S$ be the natural maps, and let $\nu_{\omega_S/\kappa} \colon S/\kappa \to S/\omega_S$ be the morphism $s\kappa \mapsto s\omega_S$ such that

$$\begin{tikzcd}
S \arrow[r, "\nu_\kappa"] \arrow[d, "\nu_{\omega_S}"'] & S/\kappa \arrow[dl, "\nu_{\omega_S/\kappa}"'] \\
S/\omega_S &
\end{tikzcd}$$

commutes. Clearly S/ω_S is a right cancellative image of S/κ. On the other hand, if $\theta \colon S/\kappa \to T$ is a morphism, where T is right cancellative, then there must be a morphism $\phi \colon S/\omega_S \to T$ such that $\nu_\kappa\theta = \nu_{\omega_S}\phi$. Then

$$(s\kappa)\nu_{\omega_S/\kappa}\phi = s\nu_\kappa\nu_{\omega_S/\kappa}\phi = s\nu_{\omega_S}\phi = s\nu_\kappa\theta = (s\kappa)\theta,$$

so that the diagram below commutes.

$$\begin{tikzcd}
S \arrow[r, "\nu_\kappa"] \arrow[d, "\nu_{\omega_S}"'] & S/\kappa \arrow[r, "\theta"] \arrow[dl, "\nu_{\omega_S/\kappa}"'] & T \\
S/\omega_S \arrow[urr, "\phi"'] & &
\end{tikzcd}$$

Clearly $\omega_S/\kappa = \ker \nu_{\omega_S/\kappa}$. $\square$

Result 1.4. [3, 8] *If S is a left ample semigroup then $\omega_S = \sigma_S$, and every idempotent is a projection in S.*

To see the second statement, let $s \in S$ such that $s^2 = s$. Then $s^2 = s^+s$, and so $ss^+ = s^+s^+ = s^+$ follows by (1). Thus

$$s^+ = ss^+ = (ss^+)^+s = (s^+)^+s = s^+s = s,$$

making use of the defining identities and Result 1.1.

Definition 1.5. A left restriction semigroup is *proper* if

$$a^+ = b^+ \text{ and } a\,\sigma_S\,b \text{ implies that } a = b.$$

The dual definition holds for right restriction semigroups. A restriction semigroup is *proper* if it is proper as both a left and as a right restriction semigroup.

Lemma 1.6. *Let S be a unary subsemigroup of a proper left restriction semigroup U. Then S is proper and*

$$\sigma_U \cap (S \times S) = \sigma_S.$$

Proof. It is clear that $\sigma_S \subseteq \sigma_U \cap (S \times S)$ and so S is proper. Suppose that $a, b \in S$ and $a \,\sigma_U\, b$. Then $a^+b \,\sigma_U\, b^+a$ and it follows from Result 1.1 that

$$(a^+b)^+ = (a^+b^+)^+ = a^+b^+ = b^+a^+ = (b^+a^+)^+ = (b^+a)^+,$$

so that as U is proper, $a^+b = b^+a$ and $a \,\sigma_S\, b$.

□

We remark that if S is a proper left restriction semigroup, then E is a σ_S-class, but the converse need not be true [3, Example 3]. However, it is well known that an inverse semigroup is proper if and only if it is E-unitary, that is, if and only if $E(S)$ forms a σ_S-class.

McAlister's 'P-theorem', giving the structure of E-unitary inverse semigroups, has analogues for proper left restriction semigroups. Each approach involves monoids acting on semilattices, a notion recapped below.

Definition 1.7. Let T be a monoid and let $\mathcal{Y}$ be a semilattice, with binary operation of meet denoted by $\wedge$. Then T *acts* on $\mathcal{Y}$ (on the left) by morphisms if there is a map $T \times \mathcal{Y} \to \mathcal{Y}$, $(t, a) \mapsto {}^t a$, such that for all $a, b \in \mathcal{Y}, s, t \in T$ we have

$${}^1 a = a,\ {}^{st} a = {}^s({}^t a) \text{ and } {}^s(a \wedge b) = {}^s a \wedge {}^s b.$$

Suppose now that the monoid T acts by morphisms on the left of a semilattice $\mathcal{Y}$. We denote by $\mathcal{Y} \rtimes T$ the *semidirect product* of $\mathcal{Y}$ by T, so that

$$\mathcal{Y} \rtimes T = \mathcal{Y} \times T \text{ and } (e, s)(f, t) = (e \wedge {}^s f, st)$$

for all $(e, s), (f, t) \in \mathcal{Y} \times T$.

We remark that the right action of a monoid T on a semilattice $\mathcal{Y}$ by morphisms is defined dually to that in Definition 1.7, where we write a^t for the right action of $t \in T$ on $a \in \mathcal{Y}$. The *reverse semidirect product* $T \ltimes \mathcal{Y}$ is then dual to the construction above.

Result 1.8. [5, Lemma 6.1] *Let T be a monoid acting by morphisms on the left of a semilattice $\mathcal{Y}$. Then $\mathcal{Y} \rtimes T$ is proper left restriction with $(e, s)^+ = (e, 1)$, so that the semilattice of projections of $\mathcal{Y} \rtimes T$ is $\{(e, 1) : e \in \mathcal{Y}\}$ and is isomorphic to $\mathcal{Y}$. Further, $(e, s) \,\sigma_{\mathcal{Y} \rtimes T}\, (f, t)$ if and only if $s = t$, so that $\mathcal{Y} \rtimes T / \sigma_{\mathcal{Y} \rtimes T} \cong T$.*

We note that if in Result 1.8, the monoid T is right cancellative, then it is easy to check that $\mathcal{Y} \rtimes T$ is left ample.

There are various approaches to constructing a 'P-theorem' for left restriction semigroups and their specialisations (see [3, 12, 6, 1]). Here we describe that of [6, 1].

Let T be a monoid acting on the left of a semilattice $\mathcal{X}$ via morphisms. Suppose that $\mathcal{X}$ has subsemilattice $\mathcal{Y}$ with upper bound ε (where if $\mathcal{Y}$ is a monoid, then ε is the identity of $\mathcal{Y}$) such that:

(a) for all $t \in T$ there exists $e \in \mathcal{Y}$ such that $e \leq {}^t\varepsilon$;

(b) if $e \leq {}^t\varepsilon$ then for all $f \in \mathcal{Y}$, $e \wedge {}^t f \in \mathcal{Y}$.

Then $(T, \mathcal{X}, \mathcal{Y})$ is a *strong left M-triple.*

For a strong left M-triple $(T, \mathcal{X}, \mathcal{Y})$ we put

$$\mathcal{M}(T, \mathcal{X}, \mathcal{Y}) = \{(e, t) \in \mathcal{Y} \times T : e \leq {}^t\varepsilon\} \subseteq \mathcal{X} \rtimes T.$$

From [1, Lemma 7.1], $\mathcal{M} = \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ is a unary subsemigroup of $\mathcal{X} \rtimes T$ and is proper left restriction. We say that $\mathcal{M}$ is the *strong $\mathcal{M}$-semigroup* associated with $(T, \mathcal{X}, \mathcal{Y})$. Part (iii) of the result below follows easily from Lemma 1.6 and Result 1.8.

Result 1.9. [1, Theorem 7.2] *Let S be a left restriction semigroup. Then the following conditions are equivalent:*

(i) *S is proper;*

(ii) *S is isomorphic to an $\mathcal{M}$-semigroup $\mathcal{M} = \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$;*

(iii) *S embeds into a semidirect product of a semilattice by a monoid T.*

If these conditions hold, then we may take $T = S/\sigma_S$ in (ii) *and* (iii) *and $\mathcal{Y} = E$ in* (ii).

We note that corresponding results exist for proper restriction semigroups, but in this case one requires partial actions of a monoid on a semilattice [2].

In view of the comment following Result 1.8, we can deduce the following result for left ample semigroups. The original version of the equivalence of (i) and (ii), using a construction slightly more akin to the P-theorem, appears in [12].

Result 1.10. [6] *Let S be a left restriction semigroup. Then the following conditions are equivalent:*

(i) *S is proper left ample;*

(ii) *S is isomorphic to an $\mathcal{M}$-semigroup $\mathcal{M} = \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ where T is right cancellative;*

(iii) *S embeds into a semidirect product of a semilattice by a right cancellative monoid T.*

If these conditions hold, then we may take $T = S/\sigma_S$ in (ii) *and* (iii) *and $\mathcal{Y} = E$ in* (ii).

We now remind the reader of the so-called 'W-products', originally introduced by Fountain and Gomes [4] to describe proper left ample monoids.

Let T be a monoid acting by morphisms on the *right* of a semilattice $\mathcal{Y}$ such that for all $a, b \in \mathcal{Y}$ and $t \in T$:

(a) $a^t = b^t \Rightarrow a = b$;

(b) $a \leq b^t \Rightarrow a = c^t$ for some $c \in \mathcal{Y}$.

We say that $(T, \mathcal{Y})$ is a *W-pair*. Notice that it is a consequence of (a) that for any $a, b \in \mathcal{Y}$ and $t \in T$, if $a^t \leq b^t$, then $a \leq b$.

Let $(T, \mathcal{Y})$ be a W-pair. We put

$$W = W(T, \mathcal{Y}) := \{(t, a^t) : t \in T, a \in \mathcal{Y}\} \subseteq T \ltimes \mathcal{Y}$$

and define $^+$ and * on W by

$$(t, a^t)^+ = (1, a) \text{ and } (t, a^t)^* = (1, a^t).$$

Result 1.11. [4, 7, 14] *Let $(T, \mathcal{Y})$ be a W-pair. Then $W = W(T, \mathcal{Y})$ is a subsemigroup of $T \ltimes \mathcal{Y}$ that is proper restriction, with semilattice of projections*

$$\{(1, a) : a \in \mathcal{Y}\}$$

isomorphic to $\mathcal{Y}$. For any $(t, a^t), (s, b^s) \in W$ we have

$$(t, a^t)\, \sigma_W\, (s, b^s) \textit{ if and only if } t = s,$$

so that $W/\sigma_W \cong T$.

Further, W is left ample if and only if T is right cancellative.

We say that $W(T, \mathcal{Y})$ above is the *W-product* associated with the W-pair $(T, \mathcal{Y})$.

Lemma 1.12. *Let $W(T, \mathcal{Y})$ be a W-product for a W-pair $(T, \mathcal{Y})$. Then $(T/\omega_T, \mathcal{Y})$ is a W-pair where the action of T/ω_T on $\mathcal{Y}$ is given by $a^{t\omega_T} = a^t$, and the map*

$$\eta \colon W(T, \mathcal{Y}) \to W(T/\omega_T, \mathcal{Y}),\ (t, a^t) \mapsto (t\omega_T, a^{t\omega_T})$$

is a surjective morphism.

Proof. Let $\alpha \colon T \to \operatorname{End} \mathcal{Y}$ be the morphism that corresponds to the action of T on $\mathcal{Y}$. Notice that the image of α is contained in the monoid of injective mappings of $\mathcal{Y}$, which is right cancellative. Hence $\omega_T \subseteq \ker \alpha$ and there is a morphism $\overline{\alpha} \colon T/\omega_T \to \operatorname{End} \mathcal{Y}$ given by $(t\omega_T)\overline{\alpha} = t\alpha$. This mapping defines an action of T/ω_T on $\mathcal{Y}$. It is then easy to see that $(T/\omega_T, \mathcal{Y})$ is a W-pair with respect to this action, and η is a surjective morphism. $\square$

Corollary 1.13. *Let $W(T, \mathcal{Y})$ be a W-product. Then the relation*

$$\{\big((t, a^t), (s, b^s)\big) : a = b \textit{ and } t\, \omega_T\, s\}$$

is a projection separating congruence on $W(T, \mathcal{Y})$.

Proof. The relation above is $\ker \eta$, where η is given in Lemma 1.12, and clearly is projection separating. $\square$

Notice that if T is right cancellative in Corollary 1.13, then it follows from the definition of η in Lemma 1.12 that the given relation is equality.

The construction of $W(T, \mathcal{Y})$ with T a right cancellative monoid is introduced in [4] as a construction of a proper left ample semigroup. In [7] (see also [14]), it is generalized for any monoid T, and it is noticed that there is a natural unary operation * on $W(T, \mathcal{Y})$, stemming from the oversemigroup $T \ltimes \mathcal{Y}$, making $W(T, \mathcal{Y})$

a proper restriction semigroup. Similarly to the usual notion of an almost factorisable inverse semigroup, a notion of an almost left factorisable restriction semigroup is introduced, and it is established that the almost left factorisable restriction semigroups are just the (projection separating) homomorphic images of W-products. Moreover, it is shown that a restriction semigroup is proper and almost left factorisable if and only if it is isomorphic to a W-product.

2. W-products embed into semidirect products

Let $W = W(T, \mathcal{Y})$ be a W-product. From Result 1.11 we know that W is proper restriction, and hence from Result 1.9, W embeds as a unary semigroup into a semidirect product of a semilattice by a monoid, where we can take the monoid to be T. However, the embedding provided in [1] to prove Result 1.9 is far from transparent, and moreover is argued only in the context of unary semigroups. As we have already remarked, W is proper (two-sided) restriction. It is true that semidirect products of semilattices by monoids are not normally restriction, but in certain cases, explained below, they are. The semidirect product in which we embed our W is one of this kind, and W embeds into it as a bi-unary semigroup. Our construction is short and direct, avoiding the machinery of [1].

We say that a monoid T *acts doubly* on a semilattice $\mathcal{Y}$ with identity ϵ, if T acts by morphisms on the left and right of $\mathcal{Y}$ and the *compatibility conditions* hold, that is

$$({}^te)^t = \epsilon^t \wedge e \text{ and } {}^t(e^t) = e \wedge {}^t\epsilon.$$

for all $t \in T, e \in Y$.

Proposition 2.1. [5] *Let T be a monoid acting doubly on a semilattice $\mathcal{Y}$ with identity ϵ. Then*

$$\mathcal{Y} *_m T = \{(e, t) : e \leq {}^t\epsilon\} \subseteq \mathcal{Y} \rtimes T$$

is a proper restriction monoid with identity $(\epsilon, 1)$ such that

$$(e, t)^+ = (e, 1) \text{ and } (e, t)^* = (e^t, 1).$$

If T is left (right) cancellative, then $\mathcal{Y} *_m T$ is right (left) ample.

Suppose that T acts doubly on $\mathcal{Y}$ as above, with the additional property that ${}^t\epsilon = \epsilon$ for all $t \in T$. Then

$$\mathcal{Y} *_m T = \{(e, t) : e \leq {}^t\epsilon\} = \mathcal{Y} \rtimes T$$

so that $\mathcal{Y} \rtimes T$ is proper restriction with $(e, t)^+ = (e, 1)$ and $(e, t)^* = (e^t, 1)$.

If $\mathcal{P}$ is a partially ordered set, then we denote the smallest order ideal containing $P \subseteq \mathcal{P}$ by $\langle P \rangle$, abbreviated to $\langle p \rangle$ where $P = \{p\}$ is a singleton.

Proposition 2.2. *Let $W(T, \mathcal{Y})$ be a W-product and denote by $\overline{\mathcal{Y}}$ the semilattice of order ideals of $\mathcal{Y}$ under intersection*[3]*. Then T acts doubly on $\overline{\mathcal{Y}}$ such that ${}^t\mathcal{Y} = \mathcal{Y}$ for all $t \in T$, and $W(T, \mathcal{Y})$ embeds (as a bi-unary semigroup) into $\overline{\mathcal{Y}} \rtimes T$.*

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[3] We allow $\emptyset$ to be an order ideal.

Proof. For $t \in T$ and $I \in \overline{\mathcal{Y}}$, define

$$^{t}I = \{a : a^t \in I\}.$$

Notice that if $a \in {}^{t}I$ and $b \leq a$, then we have $a^t \in I$ and $b^t \leq a^t$. As I is an order ideal, we have $b^t \in I$ so that $b \in {}^{t}I$. Hence ${}^{t}I$ is an order ideal of $\mathcal{Y}$. It is easy to check that this operation produces an action of T on $\overline{\mathcal{Y}}$ by morphisms. Moreover, $\mathcal{Y}$ is the identity element of $\overline{\mathcal{Y}}$, and ${}^{t}\mathcal{Y} = \mathcal{Y}$ for all $t \in T$.

We now define a right action of T on $\overline{\mathcal{Y}}$ by

$$I^t = \{a^t : a \in I\}.$$

If I is an ideal and $b \leq a^t$ where $a \in I$, then as $(T, \mathcal{Y})$ is a W-pair, we have $b = c^t$ for some $c \in \mathcal{Y}$. From $c^t \leq a^t$ we have $c \leq a$, so that $c \in I$ and $b \in I^t$. Thus I^t is indeed in $\overline{\mathcal{Y}}$.

It is clear that T acts on the right on $\overline{\mathcal{Y}}$ and the action is order preserving. To see that the action is by morphisms, suppose that $I, J \in \overline{\mathcal{Y}}$, and $t \in T$. As the action is order preserving, certainly $(I \cap J)^t \subseteq I^t \cap J^t$. On the other hand, if $x \in I^t \cap J^t$, then $x = a^t = b^t$ for some $a \in I$ and $b \in J$. As the right action of T on $\mathcal{Y}$ is injective, we deduce that $a = b \in I \cap J$ so that $x \in (I \cap J)^t$ and $(I \cap J)^t = I^t \cap J^t$. We now show that T acts doubly on $\overline{\mathcal{Y}}$. Let $t \in T$ and $I \in \overline{\mathcal{Y}}$. If $a \in ({}^{t}I)^t$, then $a = b^t$ for some $b \in {}^{t}I$. By definition of ${}^{t}I$ we have $b^t = a \in I$, so that $a \in I \cap \mathcal{Y}^t$. Conversely, if $c \in I \cap \mathcal{Y}^t$, then $c = d^t$ for some $d \in \mathcal{Y}$, so that $d \in {}^{t}I$ and $c \in ({}^{t}I)^t$. Hence $({}^{t}I)^t = I \cap \mathcal{Y}^t$.

We have shown that the first compatibility condition holds. For the second, we wish to show that ${}^{t}(I^t) = I \cap {}^{t}\mathcal{Y}$ for any $I \in \overline{\mathcal{Y}}$ and $t \in T$. But as ${}^{t}\mathcal{Y} = \mathcal{Y}$, this is equivalent to showing that ${}^{t}(I^t) = I$. Let $a \in {}^{t}(I^t)$. Then $a^t \in I^t$, so that $a^t = b^t$ for some $b \in I$, giving that $a = b \in I$. The converse is clear, so that ${}^{t}(I^t) = I$.

Given that T acts doubly on $\overline{\mathcal{Y}}$ and ${}^{t}\mathcal{Y} = \mathcal{Y}$ for all $t \in T$, Proposition 2.1 and the succeeding remark tell us that $\overline{\mathcal{Y}} \rtimes T$ is proper restriction.

Define $\phi \colon W(T, \mathcal{Y}) \to \overline{\mathcal{Y}} \rtimes T$ by

$$(t, a^t)\phi = (\langle a \rangle, t).$$

Since $(t, a^t)^+ = (1, a)$, it is clear this is a well-defined mapping that is injective.

We show that ϕ is a morphism. Let $(t, a^t), (s, b^s) \in W(T, \mathcal{Y})$. Consider $a^t \wedge b$. Certainly $a^t \wedge b \leq a^t$ so that as $(T, \mathcal{Y})$ is a W-pair, $a^t \wedge b = c^t$ for some $c \in \mathcal{Y}$ and again by definition of W-pair, c is uniquely defined. Let $x \in \mathcal{Y}$ and we calculate:

$$\begin{aligned}
x \in \langle c \rangle &\Leftrightarrow x \leq c \\
&\Leftrightarrow x^t \leq c^t \\
&\Leftrightarrow x^t \leq a^t \wedge b \\
&\Leftrightarrow x^t \leq a^t \text{ and } x^t \leq b \\
&\Leftrightarrow x \leq a \text{ and } x^t \in \langle b \rangle \\
&\Leftrightarrow x \in \langle a \rangle \text{ and } x \in {}^{t}\langle b \rangle \\
&\Leftrightarrow x \in \langle a \rangle \cap {}^{t}\langle b \rangle.
\end{aligned}$$

It follows that

$$\big((t,a^t)(s,b^s)\big)\phi = (ts, a^{ts}\wedge b^s)\phi = (ts,(a^t\wedge b)^s)\phi = (ts,c^{ts})\phi =$$
$$(\langle c\rangle, ts) = (\langle a\rangle \cap {}^t\langle b\rangle, ts) = (\langle a\rangle, t)(\langle b\rangle, s) = (t,a^t)\phi(s,b^s)\phi.$$

It is easy to check that

$$((t,a^t)\phi)^+ = (\langle a\rangle, t)^+ = (\langle a\rangle, 1) = (1,a^1)\phi = (t,a^t)^+\phi.$$

Finally we show that ϕ preserves *. For any $a\in\mathcal{Y}$ and $t\in T$ we have

$$\begin{array}{rcl} b\in\langle a^t\rangle & \Leftrightarrow & b\le a^t \\ & \Leftrightarrow & b=c^t \text{ for some } c\in\mathcal{Y} \text{ with } c\le a \\ & \Leftrightarrow & b\in\langle a\rangle^t. \end{array}$$

Let $(t,a^t)\in W$. Then

$$(t,a^t)^*\phi = (1,a^t)\phi = (\langle a^t\rangle, 1) = (\langle a\rangle^t, 1) = (\langle a\rangle, t)^* = ((t,a^t)\phi)^*.$$

This completes the proof that ϕ is an embedding. □

We further add to the subtle connection between W-products and semidirect products by the final result of this section.

Proposition 2.3. *Let T be a monoid acting doubly on a semilattice $\mathcal{Y}$ with identity ϵ, such that ${}^t\epsilon=\epsilon$ for all $t\in T$. Then $(T,\mathcal{Y})$ is a W-pair and the semidirect product $\mathcal{Y}\rtimes T$ is isomorphic to $W(T,\mathcal{Y})$.*

Proof. The compatibility conditions tell us that for all $t\in T, e\in\mathcal{Y}$,

$$({}^te)^t = e\wedge\epsilon^t \text{ and } {}^t(e^t) = e\wedge {}^t\epsilon = e,$$

using the assumption ${}^t\epsilon=\epsilon$.

If $a,b\in\mathcal{Y}$ and $t\in T$, then if $a^t=b^t$ we have $a={}^t(a^t)={}^t(b^t)=b$. Moreover, if $c\le a^t$, then as certainly $c\le\epsilon^t$, we have that $({}^tc)^t = c\wedge\epsilon^t = c$. Hence $(T,\mathcal{Y})$ is a W-pair.

Define $\theta\colon \mathcal{Y}\rtimes T\to W(T,\mathcal{Y})$ by

$$(a,t)\theta = (t,a^t).$$

Let $(a,t),(b,s)\in\mathcal{Y}\rtimes T$. If $(a,t)\theta=(b,s)\theta$, then $(t,a^t)=(s,b^s)$, so that by the properties of W-pair we have $(a,t)=(b,s)$. It is clear that θ is onto and hence a bijection.

We calculate:

$$\begin{array}{rcl} \big((a,t)(b,s)\big)\theta & = & (a\wedge {}^tb, ts)\theta \\ & = & (ts,(a\wedge {}^tb)^{ts}) \\ & = & (ts,(a^t\wedge({}^tb)^t)^s) \\ & = & (ts,(a^t\wedge\epsilon^t\wedge b)^s) \\ & = & (ts,(a^t\wedge b)^s) \\ & = & (ts,a^{ts}\wedge b^s) \\ & = & (t,a^t)(s,b^s) \\ & = & (a,t)\theta(b,s)\theta. \end{array}$$

Further,

$$(a,t)^+\theta = (a,1)\theta = (1,a) = (t,a^t)^+ = ((a,t)\theta)^+$$

and

$$(a,t)^*\theta = (a^t,1)\theta = (1,a^t) = (t,a^t)^* = ((a,t)\theta)^*.$$

Hence θ is an isomorphism. □

3. Embedding proper left restriction semigroups into W-products

Let S be a left restriction semigroup. By considering the intersection of all the proper left ample congruences on S, that is, congruences τ such that S/τ is proper left ample, it is clear that a least such congruence exists. We denote this by ρ_S. In general we cannot find a closed formula for ρ_S, although a description can be constructed along the lines of that of ω_S provided in Section 4. However, if S is embeddable into a W-product then we will see that ρ_S can be explicitly defined in terms of ω_S. Indeed, setting

$$\tau_S = \{(a,b) \in S \times S : a^+ = b^+ \text{ and } a\,\omega_S\,b\}$$

we will see that $\rho_S = \tau_S$ in this case. It is clear that τ_S is an equivalence. By Result 1.1, τ_S is projection separating.

For later purposes we now define two conditions on a left restriction semigroup S:

(C): for any $a,b \in S$, if $a^+ = b^+$ and $a\,\omega_S\,b$, then $(ae)^+ = (be)^+$ for all $e \in E$;

(D): for any $r,s,t,x \in S$, if $x^+ = r^+$, $rs^+ = r$, $xt^+ = x$ and $rs\,\omega_S\,x$, then $r(st)^+ = r$.

Lemma 3.1. *Let S be a left restriction semigroup. Then*

(i) *$\tau_S \subseteq \rho_S$;*

(ii) *τ_S is a congruence on S if and only if Condition (C) holds.*

Proof. (i) Clearly $\rho_S \subseteq \omega_S$ so that using Lemma 1.3 we have that $\omega_{S/\rho_S} = \omega_S/\rho_S$, and as S/ρ_S is left ample, $\omega_{S/\rho_S} = \sigma_{S/\rho_S}$. Let $a,b \in S$ with $a\,\tau_S\,b$. Then $a^+ = b^+$ and $a\,\omega_S\,b$ whence $(a\rho_S)^+ = (b\rho_S)^+$ and $(a\rho_S)\,\omega_{S/\rho_S}\,(b\rho_S)$. As $\omega_{S/\rho_S} = \sigma_{S/\rho_S}$ and S/ρ_S is proper, it follows that $a\rho_S = b\rho_S$, that is, $a\,\rho_S\,b$. Hence $\tau_S \subseteq \rho_S$.

(ii) Suppose that τ_S is a congruence. Let $a,b \in S$ be such that $a^+ = b^+$ and $a\,\omega_S\,b$, and let $e \in E$. Then $a\,\tau_S\,b$ so that as τ_S is a congruence we have $(ae)^+\,\tau_S\,(be)^+$. Since τ_S is projection separating we must have $(ae)^+ = (be)^+$ so that (C) holds.

Conversely, suppose that (C) holds and $a\,\tau_S\,b$. By definition, $a^+ = b^+$ and $a\,\omega_S\,b$. Let $c \in S$. Clearly $ca\,\omega_S\,cb$ and $ac\,\omega_S\,bc$. Moreover by Result 1.1, we have

$$(ca)^+ = (ca^+)^+ = (cb^+)^+ = (cb)^+,$$

giving $ca\,\tau_S\,cb$. By (C) we have that

$$(ac)^+ = (ac^+)^+ = (bc^+)^+ = (bc)^+,$$

and so $ac\,\tau_S\,bc$ also. Finally it is clear that $a\,\tau_S\,b$ implies that $a^+\,\tau_S\,b^+$, since $(a^+)^+ = a^+ = b^+ = (b^+)^+$ and $a^+\,\omega_S\,b^+$. □

We need the following technical lemma for the proof of (v) implies (i) in Theorem 3.3.

Lemma 3.2. *Let S be a left restriction semigroup satisfying (D). Then $z\,\omega_S\,z^+$ implies that $(zf)^+ = z^+f$, for any $z \in S$ and $f \in E$.*

Proof. Suppose that $z\,\omega_S\,z^+$. Put

$$r = (zf)^+,\ x = zf,\ s = z^+ \text{ and } t = f.$$

Then by Result 1.1, $r^+ = x^+$ and $rs^+ = (zf)^+z^+ = (zf)^+ = r$. Also, $xt^+ = zff = zf = x$ and $rs = (zf)^+z^+\,\omega_S\,(zf)^+z = zf = x$. Hence by (D) we have that

$$(zf)^+(z^+f) = (zf)^+(z^+f)^+ = r(st)^+ = r = (zf)^+$$

so that $(zf)^+ \leq z^+f$.

A similar calculation, with $r = z^+f = x$, $s = z$ and $t = f$ gives that $z^+f \leq (zf)^+$ and so we have equality as claimed. □

Theorem 3.3. *The following are equivalent for a left restriction semigroup S:*

(i) *S is embeddable into a W-product;*
(ii) *S is proper, and the relation τ_S is a congruence on S;*
(iii) *S is proper, and ρ_S, the least proper left ample congruence on S, is projection separating;*
(iv) *S is proper and satisfies Condition (C);*
(v) *S is proper and satisfies Condition (D).*

If any (each) of these conditions is satisfied, then $\tau_S = \rho_S$.

Proof. (i)⇒ (ii) We assume that S is a unary subsemigroup of some $W(T,\mathcal{Y})$ so that by Lemma 1.6, S is proper. Denote by π_1 the first projection of $W(T,\mathcal{Y})$ onto T. By replacing T with $S\pi_1$ if necessary, we can assume that $T = S\pi_1$. Then $\pi_1|_S\colon S \to T$ is a surjective morphism and again by Lemma 1.6, $\ker \pi_1|_S = \sigma_S$. Hence $S/\sigma_S \cong T$.

By Lemma 1.3 we see that, for any $(t,a^t),(u,b^u) \in S$, $(t,a^t)\,\omega_S\,(u,b^u)$ if and only if $t\,\omega_T\,u$. Thus

$$\begin{aligned}
\tau_S &= \{(t,a^t),(u,b^u) \in S\times S : (t,a^t)^+ = (u,b^u)^+ \text{ and } (t,a^t)\,\omega_S\,(u,b^u)\}\\
&= \{(t,a^t),(u,b^u) \in S\times S : (1,a) = (1,b) \text{ and } t\,\omega_T\,u\}\\
&= \{(t,a^t),(u,b^u) : a = b \text{ and } t\,\omega_T\,u\} \cap (S\times S).
\end{aligned}$$

Corollary 1.13 implies that τ_S is a (projection separating) congruence on S.

(ii) ⇒ (iii) Suppose that τ_S is a congruence on S. We have observed that τ_S is projection separating. Now we check that S/τ_S is proper left ample.

To show that S/τ_S is left ample, by (1) it is enough to show that if $a\tau_S, x\tau_S, y\tau_S \in S/\tau_S$ with $(x\tau_S)(a\tau_S) = (y\tau_S)(a\tau_S)$, then $(x\tau_S)(a\tau_S)^+ = (y\tau_S)(a\tau_S)^+$. With a, x, y

as given, we have that $xa\,\tau_S\,ya$ and so $(xa)^+ = (ya)^+$ and $xa\,\omega_S\,ya$. Using Result 1.1 and the fact that S/ω_S is right cancellative, we deduce that $(xa^+)^+ = (ya^+)^+$ and $xa^+\,\omega_S\,ya^+$, that is, $xa^+\,\tau_S\,ya^+$. It follows that $(x\tau_S)(a\tau_S)^+ = (y\tau_S)(a\tau_S)^+$.

Since $\tau_S \subseteq \omega_S$, Lemma 1.3 gives that $\omega_{S/\tau_S} = \omega_S/\tau_S$. However, S/τ_S is left ample, therefore $\omega_{S/\tau_S} = \sigma_{S/\tau_S}$.

In order to check that S/τ_S is proper, let $a, b \in S$ such that $(a\tau_S)^+ = (b\tau_S)^+$ and $a\tau_S\,\sigma_{S/\tau_S}\,b\tau_S$. As τ_S is projection separating, the equality implies that $a^+ = b^+$. The latter relation implies $a\,\omega_S\,b$ by the observation in the previous paragraph. Hence $a\,\tau_S\,b$ follows from the definition of τ_S and so $a\tau_S = b\tau_S$. This completes the proof that S/τ_S is proper.

From Lemma 3.1 we have that $\tau_S \subseteq \rho_S$ and we have just shown that $\rho_S \subseteq \tau_S$. Hence $\rho_S = \tau_S$ and ρ_S is projection separating.

(iii) $\Rightarrow$ (iv) Suppose that $a, b \in S$, with $a^+ = b^+$ and $a\,\omega_S\,b$, and $e \in E$. As $\tau_S \subseteq \rho_S$ we have that $a\,\rho_S\,b$ so that $(ae)^+\,\rho_S\,(be)^+$ and as ρ_S is projection separating, $(ae)^+ = (be)^+$.

(iv) $\Rightarrow$ (v) Suppose that S is proper and (C) holds. Let $r, s, t, x \in S$ with $x^+ = r^+, rs^+ = r, xt^+ = x$ and $rs\,\omega_S\,x$. Then $(rs)^+ = (rs^+)^+ = r^+ = x^+$. By (C)

$$(r(st)^+)^+ = (rst)^+ = (rst^+)^+ = (xt^+)^+ = x^+ = r^+,$$

and as $r(st)^+\,\sigma_S\,r$ and S is proper, $r(st)^+ = r$ as required.

(v) $\Rightarrow$ (i) Suppose that S is proper and satisfies (D). Let $X = E \times S/\omega_S$. We define a relation $\preceq$ on X by

$$(e, u\omega_S) \preceq (f, v\omega_S) \Leftrightarrow \exists r \in S \text{ with } u\omega_S = (r\omega_S)(v\omega_S),\ e = r^+ \text{ and } rf = r.$$

Lemma 3.4. *The relation $\preceq$ is a pre-order on X.*

Proof. For $(e, u\omega_S) \in X$,

$$u\omega_S = (e\omega_S)(u\omega_S),\ e = e^+ \text{ and } ee = e,$$

so that $(u, \omega_S) \preceq (u, \omega_S)$ and $\preceq$ is reflexive.

Suppose that $(e, u\omega_S) \preceq (f, v\omega_S) \preceq (g, w\omega_S)$, so that we have $r, s \in S$ with

$$u\omega_S = (r\omega_S)(v\omega_S),\ e = r^+,\ rf = r,\ v\omega_S = (s\omega_S)(w\omega_S),\ f = s^+ \text{ and } sg = s.$$

Clearly

$$u\omega_S = ((rs)\omega_S)(w\omega_S), e = r^+ = (rf)^+ = (rs^+)^+ = (rs)^+ \text{ and } rsg = rs,$$

so that $(e, u\omega_S) \preceq (g, w\omega_S)$.

$\square$

Lemma 3.5. *For any $(e, u\omega_S), (f, u\omega_S) \in X$, we have*

$$(e, u\omega_S) \preceq (f, u\omega_S) \Leftrightarrow e \leq f.$$

Proof. If $(e, u\omega_S) \preceq (f, u\omega_S)$, then there exists $r \in S$ with

$$u\omega_S = (r\omega_S)(u\omega_S),\ e = r^+ \text{ and } rf = r.$$

Since S/ω_S is right cancellative, the first equality implies $r\,\omega_S\,r^+$. By Lemma 3.2, we have that

$$e = r^+ = (rf)^+ = r^+f \leq f.$$

Conversely, if $e \leq f$ then with $r = e$ it is easy to see that $(e, u\omega_S) \preceq (f, u\omega_S)$. $\square$

We now define an action of S/ω_S on the right of X by

$$(e, u\omega_S)^{t\omega_S} = (e, (u\omega_S)(t\omega_S)).$$

Lemma 3.6. *The action of S/ω_S on X preserves $\preceq$. Further,*

$$(e, u\omega_S) \preceq (f, v\omega_S) \Leftrightarrow (e, u\omega_S)^{t\omega_S} \preceq (f, v\omega_S)^{t\omega_S},$$

for any $(e, u\omega_S), (f, v\omega_S) \in X$ and $t \in S$.

Proof. Let $(e, u\omega_S), (f, v\omega_S) \in X$ and $t \in S$. Using the fact that S/ω_S is right cancellative we calculate:

$$\begin{array}{rcl}(e, u\omega_S) \preceq (f, v\omega_S) & \Leftrightarrow & \exists r \in S \text{ with } u\omega_S = (r\omega_S)(v\omega_S),\ e = r^+,\ rf = f \\ & \Leftrightarrow & \exists r \in S \text{ with } (u\omega_S)(t\omega_S) = (r\omega_S)(v\omega_S)(t\omega_S), \\ & & e = r^+,\ rf = f \\ & \Leftrightarrow & (e, (u\omega_S)(t\omega_S)) \preceq (f, (v\omega_S)(t\omega_S)) \\ & \Leftrightarrow & (e, u\omega_S)^{t\omega_S} \preceq (f, v\omega_S)^{t\omega_S}.\end{array}$$

$\square$

Let $\equiv$ be the equivalence relation on X associated with $\preceq$; by the previous lemma note that for any $\alpha, \beta \in X$ and $t \in S$ we have that $\alpha \equiv \beta$ if and only if $\alpha^{t\omega_S} \equiv \beta^{t\omega_S}$. Let $\mathcal{X} = X/\equiv$, so that $\mathcal{X}$ is partially ordered by $\leq$ where, denoting the equivalence class of $\alpha \in X$ by $[\alpha]$, we have $[\alpha] \leq [\beta]$ if and only if $\alpha \preceq \beta$. Clearly S/ω_S acts on $\mathcal{X}$ via $[\alpha]^{t\omega_S} = [\alpha^{t\omega_S}]$, and this action preserves the order in $\mathcal{X}$.

Let $\mathcal{Y}$ denote the set of order ideals of $\mathcal{X}$ (including $\emptyset$). Clearly, $\mathcal{Y}$ is a semilattice under intersection. Define an action of S/ω_S on $\mathcal{Y}$, for which we use a slightly different notation to usual to prevent confusion, by

$$I \cdot t\omega_S = \langle I^{t\omega_S} \rangle,$$

that is, $I \cdot t\omega_S$ is the order ideal generated by $I^{t\omega_S} = \{[\alpha]^{t\omega_S} : [\alpha] \in I\}$.

An entirely routine argument, not specific to our given partially ordered set $\mathcal{X}$ nor our given action, shows that $\cdot$ is an action of S/ω_S on $\mathcal{Y}$ and is order preserving.

Lemma 3.7. *The monoid S/ω_S acts by morphisms on the semilattice $\mathcal{Y}$.*

Proof. Let $I, J \in \mathcal{Y}$ and $s\omega_S \in S/\omega_S$. Since the action is order preserving, clearly

$$(I \cap J) \cdot s\omega_S \subseteq (I \cdot s\omega_S) \cap (J \cdot s\omega_S).$$

Conversely, let $[\beta] \in (I \cdot s\omega_S) \cap (J \cdot s\omega_S)$. Then there exists $[\gamma] \in I$ and $[\delta] \in J$ such that $[\beta] \leq [\gamma]^{s\omega_S}$ and $[\beta] \leq [\delta]^{s\omega_S}$, so that $\beta \preceq \gamma^{s\omega_S}$ and $\beta \preceq \delta^{s\omega_S}$.

Let $\beta = (e, u\omega_S), \gamma = (f, v\omega_S)$ and $\delta = (g, w\omega_S)$. By definition of $\preceq$, there exist $r, z \in S$ with

$$\begin{array}{rclrclrcl} u\omega_S & = & (r\omega_S)(v\omega_S)(s\omega_S), & e & = & r^+, & rf & = & r, \\ u\omega_S & = & (z\omega_S)(w\omega_S)(s\omega_S), & e & = & z^+, & zg & = & z. \end{array}$$

Since S/ω_S is right cancellative we have that $(r\omega_S)(v\omega_S) = (z\omega_S)(w\omega_S)$. Now let $\mu = (e, (r\omega_S)(v\omega_S))$. Then $\beta = \mu^{s\omega_S}$ and it is easy to check that $\mu \preceq \gamma$ and $\mu \preceq \delta$, so that $[\mu] \leq [\gamma]$ and $[\mu] \leq [\delta]$ and consequently, $[\mu] \in I \cap J$. Hence $[\beta] \in (I \cap J)^{s\omega_S} \subseteq (I \cap J) \cdot s\omega_S$. It follows that $(I \cap J) \cdot s\omega_S \supseteq (I \cdot s\omega_S) \cap (J \cdot s\omega_S)$ and so we deduce that $(I \cap J) \cdot s\omega_S = (I \cdot s\omega_S) \cap (J \cdot s\omega_S)$ and we have an action as claimed. $\square$

Lemma 3.8. *The action of S/ω_S on $\mathcal{Y}$ produces a W-pair $(\mathcal{Y}, S/\omega_S)$.*

Proof. Let $I, J \in \mathcal{Y}$ and suppose that $I \cdot t\omega_S \subseteq J \cdot t\omega_S$. Let $[\beta] \in I$. Then $[\beta]^{t\omega_S} \in I^{t\omega_S} \subseteq I \cdot t\omega_S \subseteq J \cdot t\omega_S$ and so $[\beta]^{t\omega_S} \leq [\alpha]^{t\omega_S}$ for some $[\alpha] \in J$. Consequently, $\beta^{t\omega_S} \preceq \alpha^{t\omega_S}$ so that by Lemma 3.6 we have $\beta \preceq \alpha$, giving that $[\beta] \leq [\alpha]$ and hence $[\beta] \in J$. We have shown that $I \subseteq J$.

Suppose now that $I \subseteq J \cdot t\omega_S$. Let

$$K = \{[\alpha] : [\alpha] \in J \text{ and } [\alpha]^{t\omega_S} \in I\}.$$

It is easy to check that $K \in \mathcal{Y}$. Moreover, $K^{t\omega_S} \subseteq I$ so that as I is an order ideal, $K \cdot t\omega_S \subseteq I$. We show the reverse inclusion.

Let $[\beta] \in I$. Then $[\beta] \leq [\gamma]^{t\omega_S}$ for some $[\gamma] \in J$ so that $\beta \preceq \gamma^{t\omega_S}$. Let $\beta = (e, u\omega_S)$ and $\gamma = (f, v\omega_S)$ so that $(e, u\omega_S) \preceq (f, (v\omega_S)(t\omega_S))$. Therefore there exists $r \in S$ with

$$u\omega_S = (r\omega_S)(v\omega_S)(t\omega_S),\ e = r^+ \text{ and } rf = r.$$

Let $\delta = (e, (r\omega_S)(v\omega_S))$. Then $\delta^{t\omega_S} = (e, (r\omega_S)(v\omega_S)(t\omega_S)) = (e, u\omega_S) = \beta$. By Lemma 3.6, $\delta \preceq \gamma$ so that $[\delta] \in J$ and $[\delta]^{t\omega_S} = [\delta^{t\omega_S}] = [\beta]$. It follows that $[\delta] \in K$ and $[\beta] \in K^{t\omega_S} \subseteq K \cdot t\omega_S$. Hence $I = K \cdot t\omega_S$ as required. $\square$

We now remark that S/σ_S acts on $\mathcal{Y}$ via $(I, s\sigma_S) \mapsto I^{s\sigma_S} = I \cdot s\omega_S$. It is clear that $(S/\sigma_S, \mathcal{Y})$ is a W-pair.

Define $\theta\colon S \to W = W(S/\sigma_S, \mathcal{Y})$ by

$$s\theta = (s\sigma_S, I(s^+)^{s\sigma_S}) = (s\sigma_S, I(s^+) \cdot s\omega_S)$$

where for any projection e, $I(e) = \big\langle [(e, 1_{S/\omega_S})] \big\rangle$. It is immediate that

$$s^+\theta = (s^+\sigma_S, I(s^+)^{s^+\sigma_S}) = (1_{S/\sigma_S}, I(s^+)) = (s\theta)^+.$$

If $s, t \in S$ and $s\theta = t\theta$, then

$$(s\sigma_S, I(s^+)^{s\sigma_S}) = (t\sigma_S, I(t^+)^{t\sigma_S}),$$

so that $s\sigma_S = t\sigma_S$ and $I(s^+) = I(t^+)$, since $(S/\sigma_S, \mathcal{Y})$ is a W-pair. But then

$$[(s^+, 1_{S/\omega_S})] \leq [(t^+, 1_{S/\omega_S})] \leq [(s^+, 1_{S/\omega_S})],$$

so that Lemma 3.5 gives $s^+ = t^+$ and consequently, $s = t$ as S is proper. Hence θ is one-one.

To see that θ preserves the binary operation, let $s, t \in S$, so that

$$\begin{aligned} s\theta t\theta &= (s\sigma_S, I(s^+) \cdot s\omega_S)(t\sigma_S, I(t^+) \cdot t\omega_S) \\ &= \big((st)\sigma_S, (I(s^+) \cdot s\omega_S)^{t\sigma_S} \cap (I(t^+) \cdot t\omega_S)\big) \\ &= \big((st)\sigma_S, (I(s^+) \cdot (st)\omega_S) \cap (I(t^+) \cdot t\omega_S)\big) \end{aligned}$$

and

$$(st)\theta = ((st)\sigma_S, I((st)^+) \cdot (st)\omega_S).$$

We must show that

$$(I(s^+) \cdot (st)\omega_S) \cap (I(t^+) \cdot t\omega_S) = I((st)^+) \cdot (st)\omega_S.$$

Let $[\beta] \in I((st)^+) \cdot (st)\omega_S$. Then $[\beta] \leq [\gamma]^{(st)\omega_S}$, for some $[\gamma] \in I((st)^+)$, so that $[\gamma] \leq \big[((st)^+, 1_{S/\omega_S})\big]$. Now $(st)^+ \leq s^+$, so that by Lemma 3.5,

$$((st)^+, 1_{S/\omega_S}) \preceq (s^+, 1_{S/\omega_S}).$$

It follows that

$$[\beta] \leq [\gamma]^{(st)\omega_S} \leq \big[((st)^+, 1_{S/\omega_S})\big]^{(st)\omega_S} \leq [(s^+, 1_{S/\omega_S})]^{(st)\omega_S}$$

and so $[\beta] \in I(s^+) \cdot (st)\omega_S$. Further, from the above inequality we have

$$[\beta] \leq [((st)^+, 1_{S/\omega_S})]^{(st)\omega_S} = [((st)^+, s\omega_S)]^{t\omega_S}.$$

Now

$$s\omega_S = (st^+)\omega_S\, 1_{S/\omega_S},\ (st)^+ = (st^+)^+,\ st^+t^+ = st^+,$$

so that

$$((st)^+, s\omega_S) \preceq (t^+, 1_{S/\omega_S}),$$

giving

$$[\beta] \leq [(t^+, 1_{S/\omega_S})]^{t\omega_S}$$

and so $[\beta] \in I(t^+) \cdot t\omega_S$.

Conversely, suppose that $[\beta] \in (I(s^+) \cdot (st)\omega_S) \cap (I(t^+) \cdot t\omega_S)$. This gives that $[\beta] \leq [\alpha]^{(st)\omega_S}$ for some $[\alpha] \in I(s^+)$, so that $[\alpha] \leq [(s^+, 1_{S/\omega_S})]$. Hence $[\beta] \leq [(s^+, (st)\omega_S)]$ and similarly, $[\beta] \leq [(t^+, t\omega_S)]$.

Let $\beta = (e, u\omega_S)$. Then there exist $r, x \in S$ such that

$$\begin{aligned} u\omega_S &= (r\omega_S)((st)\omega_S), & e &= r^+, & rs^+ &= r, \\ u\omega_S &= (x\omega_S)(t\omega_S), & e &= x^+, & xt^+ &= x. \end{aligned}$$

By right cancellation in S/ω_S we obtain that $rs\,\omega_S\,x$ and so by Condition (D) we have that $r(st)^+ = r$. This gives us that $\beta \preceq ((st)^+, (st)\omega_S)$ and so

$$[\beta] \leq [((st)^+, (st)\omega_S)] = [((st)^+, 1_{S/\omega_S})]^{(st)\omega_S}.$$

Consequently, $[\beta] \in I((st)^+) \cdot (st)\omega_S$.

We have completed the proof that θ is an embedding, so that (v) $\Rightarrow$ (i) holds. $\square$

The connection between Conditions (C) and (D) for a left restriction semigroup appears at first sight a little mysterious. In Theorem 3.3 we have shown that, given S is proper, (C) holds if and only if (D) holds. We proved the forward implication directly and think it is worthwhile here giving a short direct argument for the reverse implication.

Lemma 3.9. *Let S be a left restriction semigroup satisfying (D). Then S satisfies (C).*

Proof. Suppose that $a, b \in S$ with $a^+ = b^+$ and $a\,\omega_S\,b$. Take $e \in E$ and let

$$r = (ae)^+,\ x = ae,\ s = b \text{ and } t = e.$$

Then $r^+ = x^+$,

$$rs^+ = (ae)^+b^+ = (ae)^+a^+ = (ae)^+ = r,$$

$xt^+ = aee = ae = x$ and

$$rs = (ae)^+b\,\omega_S\,b\,\omega_S\,a\,\omega_S\,x.$$

By (D) we deduce that $r(st)^+ = r$ and so $(ae)^+(be)^+ = (ae)^+$. Reversing the roles of a and b now gives that $(ae)^+ = (be)^+$. $\square$

If S is proper left ample, then ρ_S is the identity relation and so certainly is projection separating. From (i) $\Leftrightarrow$ (iii) in Theorem 3.3 we deduce the following.

Corollary 3.10. [4] *Each proper left ample semigroup is embeddable into a W-product.*

In fact, our proof of (v) $\Rightarrow$ (i) in Theorem 3.3 uses a technique effectively established in [4]. In that paper the technique is couched in categorical language, which we have avoided above. The approach in [4] was inspired by an argument in [3] which is itself based on Munn's proof of McAlister's P-theorem in [13]. Without the categorical machinery, the use of Munn's ideas becomes more visible.

From Result 1.9 we know that any proper left restriction semigroup embeds into a semidirect product $\mathcal{Y} \rtimes T$, which is itself proper left restriction by Result 1.8.

Corollary 3.11. *Let $\mathcal{Y} \rtimes T$ be the semidirect product of a semilattice by a monoid. Then $\mathcal{Y} \rtimes T$ embeds into a W-product if and only if for all $e \in \mathcal{Y}$ and $s, t \in T$, $s\,\omega_T\,t$ implies that ${}^se = {}^te$.*

Proof. From Theorem 3.3, $\mathcal{Y} \rtimes T$ embeds into a W-product if and only if $\mathcal{Y} \rtimes T$ satisfies (C). Condition (C) holds if and only if for all $(e,s),(f,t) \in \mathcal{Y} \rtimes T$, if $(e,s)^+ = (f,t)^+$ and $(e,s)\, \omega_{\mathcal{Y}\rtimes T}\, (f,t)$, then for any $(g,1) \in \mathcal{Y}\rtimes T$, $((e,s)(g,1))^+ = ((f,t)(g,1))^+$. That is, for any $e,g \in \mathcal{Y}$ and $s,t \in T$ if $s\, \omega_T\, t$ then $e \wedge {}^s g = e \wedge {}^t g$. Taking $e = {}^s g$ and $e = {}^t g$ the latter is equivalent to the condition in the statement of the corollary. □

Example 3.12. Let $T = \mathcal{Y}$ be the two element semilattice $\{0,1\}$. Certainly T acts on $\mathcal{Y}$ (on the left) by multiplication. Clearly $0\, \omega_T\, 1$ but ${}^1 1 \neq {}^0 1$. Hence the semidirect product $\mathcal{Y} \rtimes T$ does not embed into a W-product.

4. The least right cancellative congruence on a semigroup

We recall from Result 1.2 that the relation σ_S on a left restriction semigroup S has a pleasant closed form. Given the importance of the role played by ω_S in Theorem 3.3, one might ask for a good description of ω_S. Unfortunately ω_S proves to be not as amenable as σ_S. However, for completeness, we give a defining set of relations for the least right cancellative congruence on *any* semigroup. Notice that for a left restriction semigroup, the least right cancellative *semigroup* congruence and the least right cancellative *unary semigroup* congruence coincide. The interested reader could perform a similar (although more complicated) procedure, this time certainly in the context of unary semigroups, to obtain ρ_S where S is left restriction.

Let S be a semigroup. We define a sequence $\omega_1, \omega_2, \ldots$ of congruences on S such that

$$\omega_1 \subseteq \omega_2 \subseteq \ldots$$

and $\bigcup_{i\in\mathbb{N}} \omega_i = \omega_S$.

Let

$$H_1 = \{(a,b) \in S \times S : \exists t \in S \text{ with } at = bt\},$$

and put

$$\omega_1 = H_1^\sharp,$$

the congruence generated by H_1. Assume now that we have defined subsets

$$H_1 \subseteq H_2 \subseteq \ldots \subseteq H_n$$

of $S \times S$ so that with $\omega_i = H_i^\sharp$,

$$\omega_1 \subseteq \omega_2 \subseteq \ldots \subseteq \omega_n.$$

Define

$$H_{n+1} = \{(a,b) \in S \times S : \exists t \in S \text{ with } at\, \omega_n\, bt\}.$$

Notice that as $\omega_n = H_n^\sharp$ we have that $H_n \subseteq H_{n+1}$ and so with $\omega_{n+1} = H_{n+1}^\sharp$ we have $\omega_n \subseteq \omega_{n+1}$. Further, if S/ω_i is right cancellative for some i, then $\omega_i = \omega_j$ for all $j \geq i$.

Let $\omega' = \bigcup_{i\in\mathbb{N}} \omega_i$ and let $H = \bigcup_{i\in\mathbb{N}} H_i$. If $(a,b) \in H^\sharp$ then as only finitely many members of H are needed to relate a to b, we must have that $(a,b) \in \omega_i$ for some $i \in \mathbb{N}$. It follows that $\omega' = H^\sharp$.

Proposition 4.1. *Let S be a semigroup. Then ω_S (the least right cancellative congruence on S) is the congruence ω' defined above.*

Proof. Suppose that $at\,\omega'\,bt$ for some $a,b,t \in S$. Then $at\,\omega_i\,bt$ for some $i \in \mathbb{N}$ and so $(a,b) \in H_{i+1} \subseteq H$. It follows that $a\omega' = b\omega'$, so that S/ω' is right cancellative.

Let T be a right cancellative monoid and suppose that $\phi\colon S \to T$ is a morphism. We show by induction that $\omega_i \subseteq \ker\phi$. If $at = bt$ for some $a,b,t \in S$, then $(a\phi)(t\phi) = (b\phi)(t\phi)$, so that as T is right cancellative, $a\phi = b\phi$. Hence $H_1 \subseteq \ker\phi$ and so $\omega_1 \subseteq \ker\phi$.

Suppose for induction that $\omega_i \subseteq \ker\phi$ and $(a,b) \in H_{i+1}$. Then $(a\omega_i)(t\omega_i) = (b\omega_i)(t\omega_i)$ for some $t \in S$, that is, $at\,\omega_i\,bt$. Since by assumption we have $\omega_i \subseteq \ker\phi$, we can deduce that $(at)\phi = (bt)\phi$ so that as ϕ is a morphism and T is right cancellative, $a\phi = b\phi$. Hence $H_{i+1} \subseteq \ker\phi$ and so $\omega_{i+1} \subseteq \ker\phi$. Applying induction we deduce that $\omega' \subseteq \ker\phi$.

It follows that ω' is the least right cancellative congruence on S, that is, $\omega' = \omega_S$. □

We note that if S is left ample, then $\omega_1 = \sigma_S = \omega_S$. For if $e,f \in E$, then $e(ef) = f(ef)$, so that $(e,f) \in H_1$ and so $\sigma_S \subseteq \omega_1$. On the other hand, if $a,b,t \in S$ and $at = bt$, then as S is left ample, $at^+ = bt^+$ so that

$$(at)^+a = at^+ = bt^+ = (bt)^+b,$$

whence $a\,\sigma_S\,b$. Thus $H_1 \subseteq \sigma_S$ and so $\omega_1 \subseteq \sigma_S$. We have already commented that for a left ample semigroup, $\omega_S = \sigma_S$.

If S is a commutative semigroup, then the description of ω_S, which is in this case the least cancellative congruence on S, simplifies considerably. In this case, $\omega_S = \omega_1$ by [9, Proposition II.2.3].

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