CATEGORICAL AND SEMIGROUP-THEORETIC DESCRIPTIONS OF BASS-SERRE THEORY

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Abstract. Self-similar group actions may be encoded by a class of left cancellative monoids called left Rees monoids. This connection was discovered by Perrot and the first author who subsequently generalized it to self-similar groupoid actions and a class of categories called left Rees categories. In this paper, we prove that the theory of Rees categories, that is the left Rees categories which are actually cancellative, may be viewed as a generalization of the classical theory of graphs of groups as developed by Serre and the groupoid approach to that theory by Philip Higgins. Using a standard construction, we also show that the theory of graphs of groups may be viewed as part of the theory of inverse semigroups. This enables us to prove that the Serre tree associated with a graph of groups can be constructed using Ehresmann's maximum enlargement theorem. This shows the close connection that exists between the theory of graphs of groups and McAlister's classical P-theorem within inverse semigroup theory.

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1. Introduction

The theory of self-similar groups has two aspects: a group-theoretical and a monoid-theoretical. The group-theoretical is well-known, being the subject of the 2005 book by Nekrashevych [31], and originating in the 1980's. The monoid-theoretical is less well-known. The first author showed [20] that self-similar groups were also defined in the 1972 thesis of J.-F. Perrot [33] (see also [34]). In [20], the first author established correspondences between three classes of mathematical structures:

(1) Self-similar group actions defined in full generality without the assumption that the action be faithful.
(2) A class of left cancellative monoids, called left Rees monoids.
(3) A class of 0-bisimple inverse monoids with zero, referred to in this paper as 0-*bisimple Perrot monoids*.

The correspondence between (1) and (2) was investigated in more detail in the authors' paper [24] and yielded an unexpected connection between the theory of Rees monoids and HNN extensions of groups; it was this connection that was the catalyst for this paper. Our main goal is to show that by replacing Rees *monoids* by Rees *categories*, we may in fact develop a theory that subsumes not only HNN extensions and free products with amalgamation but all of the classical theory of graphs of groups. Our work was additionally strongly influenced by Cohn [6], the work of von Karger [38, 39], a paper by Krieger [14], the first author's paper [21] and Philip Higgins's groupoid approach to graphs of groups described in [8]. Once we have achieved our main goal, it is then relatively routine to construct an inverse semigroup theoretical approach to graphs of groups.

Acknowledgements This paper was inspired by a few pages in the first edition of Cohn's book [6] where he develops a theory of group embeddings of a class of cancellative monoids he calls 'rigid'. The theory reminded us of Bass-Serre theory but defined for cancellative monoids rather than groups.

2. Terminology

Any undefined terms from category theory we use may be found in [28]. For us, categories are small and objects are replaced by identities; thus we view them as 'monoids with many identities'. The elements of a category C are called *arrows* and the set of identities of C is denoted by C_o. Each arrow a has a *domain*, denoted by $\mathbf{d}(a)$, and a *codomain* denoted by $\mathbf{r}(a)$, both of which are identities and $a = \mathbf{d}(a)a = a\mathbf{r}(a)$. The product ab exists if and only if $\mathbf{r}(a) = \mathbf{d}(b)$.[1] Thus our products should be conceived thus

$$e \xrightarrow{a} f \xrightarrow{b} i$$

where e, f and i are identities. We shall sometimes write $\exists ab$ to mean that the product ab exists in the category. Given identities e and f, the set of arrows eCf is called a *hom-set* and eCe is a monoid called the *local monoid at* e. The group of units of the local monoid at e is called the *local group at* e. We say that arrows x and y are *parallel* if they belong to the same hom-set. A category is called *left cancellative* if whenever $ax = ay$ we have that $x = y$. We define *right cancellative* categories dually. A *cancellative category* is one which is both left and right cancellative. An arrow a is *invertible* or an *isomorphism* if there is an arrow a^{-1}, called an *inverse* and perforce unique, such that $aa^{-1} = \mathbf{d}(a)$ and $a^{-1}a = \mathbf{r}(a)$. An element $a \in C$ of a category is said to be an *atom* if it is not invertible and if $a = bc$ then either b or c is invertible.

Lemma 2.1. *If a is an atom in an arbitrary category, g is invertible and $\exists ga$ then ga is an atom.*

Proof. Suppose that $ga = bc$. Then $a = (g^{-1}b)c$. Thus $g^{-1}b$ or c is invertible; that is, b or c is invertible. It follows that ga is also an atom. □

A category in which every arrow is invertible is called a *groupoid*. If a groupoid is just a disjoint union of its local groups then we say that it is *totally disconnected*. The set of invertible elements of a category forms a groupoid with the same set of identities. Later, we shall deal only with categories C having the following property: any isomorphism belongs to a local monoid. We shall say that such categories are *skeletal*. The following is well-known.

Lemma 2.2. *Every category is equivalent to a skeletal category.*

[1] This is the reverse of the way that the first author usually treats category products but is the most natural one in the light of the applications we have in mind.

Proof. Let C be a category. Let $\{e_i : i \in I\}$ be a transversal of the isomorphism classes of the identities. Thus if $i \neq j$ then e_i is not isomorphic to e_j and every identity in C is isomorphic to some e_i. Let C' be the full subcategory determined by the e_i. Then C' is a skeletal category equivalent to C. $\square$

Given any directed graph D, we may construct the *free category on D* denoted by D^*. This has one identity for each vertex and consists of all finite directed paths in the graph. Multiplication is concatenation of paths. The elements of D^* will be written $a_1 \cdot \ldots \cdot a_m$ where the a_i are edges of the graph such that $\xrightarrow{a_1}\xrightarrow{a_2}\ldots\xrightarrow{a_m}$.

With each category C, we may associate its *universal* or *fundamental groupoid* $U(C)$ [27]. To construct this groupoid, first regard C as a directed graph. For each edge $e \xrightarrow{a} f$ attach a new edge $f \xrightarrow{a^{-1}} e$. Form the free category $(C \cup C^{-1})^*$ on the set $C \cup C^{-1}$. We denote elements of this category by $a_1 \cdot \ldots \cdot a_n$ where $a_i \in C \cup C^{-1}$ and $\mathbf{r}(a_i) = \mathbf{d}(a_{i+1})$ for $i = 1, 2, \ldots, n-1$. Now define a congruence $\equiv$ on this free category generated by

$$a \cdot a^{-1} \equiv \mathbf{d}(a), \quad a^{-1} \cdot a \equiv \mathbf{r}(a), \quad a \cdot b \equiv ab \text{ if } \exists ab$$

where $a, b \in C$. The first two identifications ensure that the quotient category is a groupoid and the third ensures that there will be a functor from C to the quotient. This gives us our groupoid $U(C)$ with an associated functor $\iota \colon C \to U(G)$. We prove that this has the correct universal property. Let $\theta \colon C \to G$ be a functor to a groupoid. We may extend θ to a function $\bar{\theta} \colon C \cup C^{-1} \to G$ where we define $\bar{\theta}(a) = \theta(a)$ and $\bar{\theta}(a^{-1}) = \theta(a)^{-1}$. We may therefore extend $\bar{\theta}$ to a functor θ' from the free category $(C \cup C^{-1})^*$ to G. Observe that that if $x \equiv y$ then $\bar{\theta}(x) \equiv \bar{\theta}(y)$. Thus we may define a functor Θ from $U(C)$ to G. The uniqueness property comes from the fact that $U(C)$ is generated by C.

3. Equidivisible categories

The following definition is generalized from semigroup theory [30]. A category C is said to be *equidivisible* if for every commutative square

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "c"'] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, "d"'] & \cdot \end{tikzcd}$$

we either have an arrow u making the following diagram commute

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "c"'] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, "d"'] \arrow[ur, "u"] & \cdot \end{tikzcd}$$

or one, v, making the following diagram commute

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "c"'] & \cdot \arrow[d, "b"] \arrow[dl, "v"'] \\ \cdot \arrow[r, "d"'] & \cdot \end{tikzcd}$$

A *length functor* is a functor $\lambda\colon C \to \mathbb{N}$ from a category C to the additive monoid of natural numbers satisfying the following conditions:

(LF1): If xy is defined then $\lambda(xy) = \lambda(x) + \lambda(y)$.
(LF2): $\lambda^{-1}(0)$ consists of all and only the invertible elements of C.
(LF3): $\lambda^{-1}(1)$ consists of all and only the atoms of C.

Sometimes we shall write λ_C if we wish to emphasize the fact that the length functor belongs to C.

A *principal right ideal* in a category C is a subset of the form aC where $a \in C$. We may similarly define *principal right ideals* and *principal ideals*. Greens relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ can also be defined in categories. Thus, for example, in the category C we define $a \mathscr{L} b$ if and only if $Ca = Cb$. Let $aC \subseteq bC$. We use the notation $[aC, bC]$ to mean the set of principal right ideals xC such that $aC \subseteq xC \subseteq bC$. The proof of the following is straightforward.

Lemma 3.1. *Let C be a category equipped with a length functor having G as its groupoid of invertible elements.*

(1) $a \mathscr{L} b \Leftrightarrow Ga = Gb$. In particular, $\mathbf{r}(a) = \mathbf{r}(b)$.
(2) $a \mathscr{R} b \Leftrightarrow aG = bG$. In particular, $\mathbf{d}(a) = \mathbf{d}(b)$.
(3) $a \mathscr{J} b \Leftrightarrow GaG = GbG$.

Principal right ideals of the form eC where e is an identity are maximal such ideals because if $eC \subseteq aC$ then $e = \mathbf{d}(a)$ and $aC = eaC \subseteq eC$ giving $eC = aC$. A principal right ideal aC is said to be *submaximal* if $aC \neq \mathbf{d}(a)C$ and there are no proper principal right ideals between aC and $\mathbf{d}(a)C$.

Lemma 3.2. *Let C be a category equipped with a length functor λ.*

(1) The element a is an atom if and only if aC is submaximal if and only if Ca is submaximal.
(2) Each non-invertible element a of C can be written $a = a'b$ where a' is an atom.
(3) Every non-invertible element of C can be written as a product of atoms.

Proof. (1). Suppose that a is an atom and that $aC \subseteq bC$. Then $a = bc$. But a is an atom and so either b is invertible or c is invertible. Suppose that c is invertible then $aC = bC$. Suppose that b is invertible then $bC = \mathbf{d}(b)C$. We have proved that aC is submaximal.

Conversely, suppose that aC is submaximal. Let $a = bc$. Then $aC \subseteq bC$. It follows that either $aC = bC$ or $bC = \mathbf{d}(b)C$. It the latter occurs then b is invertible. If the former occurs then $\lambda(a) = \lambda(b)$. It follows that $\lambda(c) = 0$ and so c is invertible.

(2). Let a be a non-invertible element. If a is an atom then we are done. If not, then $a = a_1 b_1$ for some b_1 and a_1 where neither a_1 nor b are invertible. Observe that $\lambda(a_1) < \lambda(a)$. If a_1 is an atom then we are done, otherwise we may write $a_1 = a_2 b_2$

again where neither a_2 and b_2 are atoms. Observe that $\lambda(a_2) < \lambda(a_1)$. This process can only continue in a finite number of steps and will end with $a = a_n b'$ for some b' where a_n is an atom.

(3). If a is an atom then there is nothing to prove. Otherwise by (2), we may write $a = a_1 b_1$ where a_1 is an atom and $\lambda(b_1) = \lambda(a) - 1$. If b_1 were invertible then a would have been an atom. If b_1 is an atom then we are done. Otherwise, we may repeat the above procedure with b_1. □

In categories equipped with length functors, we can write every element in terms of atoms and invertible elements. The obvious next question is what kind of uniqueness we can expect. Under the additional assumption of equidivisibility, we can retrieve a kind of uniqueness.

Lemma 3.3. *Let C be an equidivisible category equipped with a length functor and suppose that*

$$x = a_1 \dots a_m = b_1 \dots b_n$$

where the a_i and b_j are atoms.

(1) $m = n$.

(2) There are invertible elements $g_1, \dots, g_{n-1}$ such that

$$a_1 = b_1 g_1, \quad a_2 = g_1^{-1} b_2 g_2, \quad \dots \quad a_n = g_{n-1}^{-1} b_n.$$

This data is best presented by means of the following interleaving diagram.

$$\begin{tikzcd}
 & {} \arrow[r, "a_2"] & {} \arrow[r, "a_3"] & {} & \cdots & {} \arrow[r, "a_{n-1}"] & {} \arrow[dr, "a_n"] & \\
{} \arrow[ur, "a_1"] \arrow[dr, "b_1"'] & & & & & & & {} \\
 & {} \arrow[uu, "g_1"] \arrow[r, "b_2"'] & {} \arrow[uu, "g_2"] \arrow[r, "b_3"'] & {} \arrow[uu, "g_3"] & \cdots & {} \arrow[uu, "g_{n-2}"] \arrow[r, "b_{n-1}"'] & {} \arrow[uu, "g_{n-1}"] \arrow[ur, "b_n"'] &
\end{tikzcd}$$

(3) If C is in addition skeletal then a_i is parallel to b_i for $i = 1, \dots, m$.

Proof. (1). This is immediate from the properties of length functors.

(2). We bracket as follows

$$a_1(a_2 \dots a_m) = b_1(b_1 \dots b_m).$$

By equidivisibility, $a_1 = b_1 u$ and $b_2 \dots b_m = u a_2 \dots a_m$ for some u or $b_1 = a_1 v$ and $a_2 \dots a_m = v b_2 \dots b_m$ for some v. In either case, u and v are invertible since both a_1 and b_1 are atoms using the length function. Thus $a_1 = b_1 g_1$, where $v = g_1$ is an isomorphism, and $b_2 \dots b_m = g_1 a_2 \dots a_m$.

We now repeat this procedure bracketing thus

$$b_2(b_3 \dots b_m) = g_1 a_2 (a_3 \dots a_m).$$

By the same argument as above, we get that $g_1 a_2 = b_2 g_2$ for some isomorphism g_2 and $b_3 \dots b_m = g_2 a_3 \dots a_m$.

The process continues and we obtain the result.

(3). The result is immediate from the assumption that the category is also skeletal. □

We shall be interested in *left cancellative* equidivisible categories equipped with length functors and ultimately those that are also skeletal. But we shall start with a slightly different definition and show that it is equivalent to this one. A category C is said to be *right rigid* if $aC \cap bC \neq \emptyset$ implies that $aC \subseteq bC$ or $bC \subseteq aC$. A *left Rees category* is a left cancellative, right rigid category in which each principal right ideal is properly contained in only finitely many distinct principal right ideals.

Example 3.4. Free categories are left Rees categories: the atoms are the edges; the length functor simply counts the number of edges in a path; the groupoid of invertible elements is trivial.

Lemma 3.5. *Let C be a left cancellative category with groupoid of invertible elements G.*

(1) $a \mathscr{R} b$ if and only if $aG = bG$.
(2) C is a right rigid category if and only if it is equidivisible.
(3) If $e = xy$ is an identity then x is invertible with inverse y.

Proof. (1). Only one direction needs proving. Suppose that $a \mathscr{R} b$. Then $a = bx$ and $b = ay$ for some $x, y \in C$. Thus $a = ayx$ and $b = bxy$. By left cancellation both xy and yx are identities and so x is invertible with inverse y.

(2). Only one direction needs proving. Suppose that $ab = cd$. Then $aC \cap cC \neq \emptyset$. Without loss of generality, we may suppose by right rigidity that $aC \subseteq cC$. Thus $a = cu$ But then $cub = cd$ and so $ub = d$, as required.

(3). We have that $xyx = x$ and by left cancellation this shows that yx is an identity and so x is invertible with inverse y. □

In the light of the lemma above, the following says that left cancellative equidivisible categories equipped with length functors are left Rees categories.

Lemma 3.6. *Let C be a right rigid, left cancellative category equipped with a length functor λ. Then C is a left Rees category.*

Proof. Let $a \in C$ be any element with $e = \mathbf{d}(a)$. We need to prove that the set $[aC, eC]$ is finite. Let $bC \in [aC, eC]$. Then $a = bx$ for some x and so $\lambda(a) \geq \lambda(b)$. There is therefore an upper bound on the lengths of those elements b such that $aC \subseteq bC$. Let $b_1C, b_2C \in [aC, eC]$ and suppose that $\lambda(b_1) = \lambda(b_2)$. By right rigidity, we may assume, without loss of generality that $b_1C \subseteq b_2C$. Thus $b_1 = b_2x$ for some x. But b_1 and b_2 have the same length and so x must have length zero. It follows that x is invertible. Hence $b_1C = b_2C$. It follows that the set $[aC, eC]$ is finite, as claimed. □

We shall now prove the converse to the above result. This involves proving that every left Rees category is equipped with a length functor. Let $a \in C$ be an element of a left Rees category. We shall define the *length*, $\lambda(a)$, of a. Put $\mathbf{d}(a) = e$. By assumption, the set $[aC, eC]$ is finite and linearly ordered. The proofs of the following are straightforward.

Lemma 3.7. *Let C be a left Rees category.*

(1) The set $[aC, eC]$ contains one element if and only if a is invertible.
(2) The set $[aC, eC]$ contains two elements if and only if a is an atom.

If the set $[aC, eC]$ contains n elements, where $n \geq 1$ always, define $\lambda(a) = n - 1$.

Lemma 3.8. *Let C be a left Rees category and let $a, b \in C$ such that ab is defined.*

(1) $a[bC, \mathbf{d}(b)C] = [abC, aC]$.
(2) $[abC, \mathbf{d}(a)C] = a[bC, \mathbf{d}(b)C] \cup [aC, \mathbf{d}(a)C]$.

Proof. (1). Let $xC \in [bC, \mathbf{d}(b)C]$. Then $bC \subseteq xC \subseteq \mathbf{d}(b)C$. It is immediate that $abC \subseteq axC \subseteq aC$. Thus the lefthand side is contained in the righthand side. Now let $abC \subseteq yC \subseteq aC$. Then $y = au$ for some $u \in C$. Thus $yC = auC = a(uC)$. Now $ab = yv$ for some v. Hence $ab = auv$ and so by left cancellation, we have that $b = uv$. Thus $bC \subseteq uC \subseteq \mathbf{d}(b)C$.

(2). Let $c = ab$. Then $cC \subseteq aC \subseteq \mathbf{d}(a)C$. Let $xC \in [cC, \mathbf{d}(c)C]$. Since this set is linearly ordered either $xC \subseteq aC$ or $aC \subseteq xC$. It follows that the lefthand side is contained in the righthand side. The proof of the reverse containment is immediate. □

The proof of the following is now immediate.

Lemma 3.9. *Let C be a left Rees category.*

(1) $\lambda(ab) = \lambda(a) + \lambda(b)$.
(2) $\lambda(a) = 0$ *if and only if a is invertible.*
(3) $\lambda(a) = 1$ *if and only if a is an atom.*

We have therefore proved the following.

Proposition 3.10. *A left cancellative, right rigid category is a left Rees category if and only if it is equipped with a length functor. In other words, the left Rees categories are precisely the left cancellative equidivisible categories equipped with length functors.*

The following result is worth noting here.

Lemma 3.11. *Let C be a left Rees category. Then each local monoid of C is a left Rees monoid.*

Proof. Let $S = eCe$ be a local monoid. It is immediate that it is a left cancellative monoid. Let $aS \cap bS \neq \emptyset$ where $a, b \in S$. Then clearly $aC \cap bC \neq \emptyset$. Without loss of generality, it follows that $aC \subseteq bC$. Thus $a = bc$ for some $c \in C$. Now $eae = a$ and $b = ebe$ so that $a = b(ece)$ and $c = ece$. We have therefore proved that $aS \subseteq bS$. Thus S is right rigid. We denote by $[aS, bS]$ the obvious set of principal right ideals *in* S. The set $[aS, eS]$ is linearly ordered. We shall prove that it is finite. Let $aS \subseteq b_1S \subset b_2S \subseteq eS$ where $b_1, b_2 \in S$. We shall prove that $b_1C \neq b_2C$. Suppose on the contrary that $b_1C = b_2C$. Then $b_1 = b_2g$ for some invertible element g. then $b_1 = b_2(ege)$ and $ege = g$. Thus g is an invertible element in S and so $b_1S = b_2S$, a contradiction. Since, by assumption, the set $[aC, bC]$ is finite it follows that the set $[aS, bS]$ is finite. □

Remark 3.12. Let C be a left Rees category. It is important to observe that although $S = eCe$ is a left Rees monoid, its length function need not be the restriction of the one in C. The length of the element $a \in S$, viewed as an element of the monoid S, is defined to be one less than the number of elements of $[aS, eS]$. The length of the element $a \in S$ viewed as an element of C is defined to be one less than the number of elements in $[aC, eC]$. But the latter set may contain more elements than the former. Thus $\lambda_S(a) \leq \lambda_C(a)$. Consider the following example.

Let C be the free category defined by the following directed graph

$$\begin{tikzcd} e \arrow[r, bend left, "a"] & f \arrow[l, bend left, "b"] \end{tikzcd}$$

The local monoid at e is just $S = (ab)^*$, the free monoid on one generator ab. Here $\lambda_S(ab) = 1$ but $\lambda_C(ab) = 2$ since both a and b are atoms in C.

4. Constructing equidivisible categories

The goal of this section is to show that each equidivisible category with a length functor is isomorphic to a tensor category over what we call a bimodule: that is, sets on which a given groupoid acts on the left and the right in such a way that the two actions associate.

Let G be a groupoid and let X be a set equipped with two functions

$$G_0 \xleftarrow{\mathbf{s}} X \xrightarrow{\mathbf{t}} G_o.$$

We suppose that there is a left groupoid action $G \times X \to X$ and a right groupoid action $X \times G \to X$ such that the two actions associate meaning $(gx)h = g(xh)$ when defined. We write $\exists gx$ and $\exists xg$ if the actions are defined. Observe that $\exists gx$ iff $\mathbf{r}(g) = \mathbf{s}(x)$ and $\exists xg$ iff $\mathbf{t}(x) = \mathbf{d}(g)$. We call the structure (G, X, G) a *bimodule* or a *(G,G)-bimodule*. If whenever $\exists xg$ and $xg = x$ we have that g is an identity, then we say the action is *right free*. A bimodule which is right free is called a *covering bimodule*. We define *left free* dually. A bimodule which is both left and right free is said to be *bifree*. We define *homomorphisms* and *isomorphisms* between (G,G)-bimodules in the usual way.

Our first result shows that bimodules arise naturally from our categories. The proof is straightforward; in particular, the fact that the actions are well-defined follows from Lemma 2.1.

Lemma 4.1. *Let C be an equidivisible category with length functor λ. Denote by X the set of all atoms of C equipped with the maps $\mathbf{d}, \mathbf{r} \colon X \to C_0$. Denote by G the groupoid of invertible elements of C. Define a bimodule (G, X, G) where the left and right actions are defined via multiplication in C when defined. We obtain a covering bimodule if C is left cancellative and a bifree bimodule if C is cancellative*

We call (G, X, G) constructed as in the above lemma, the *bimodule associated with C* or the *bimodule of atoms of C*.

Our goal now is to show that from each bimodule we may construct a suitable category. Our tool for this will be tensor products and the construction of a suitable tensor algebra: see Chapter 6 of [37], for example. We recall the key definitions and results we need first.

Let G be a groupoid that acts on the set X on the right and the set Y on the left. We consider the set $X * Y$ consisting of those pairs (x, y) where $\mathbf{t}(x) = \mathbf{s}(x)$. A function $\alpha \colon X * Y \to Z$ to a set Z is called a *bi-map* or a *2-map* if $\alpha(xg, y) = (x, gy)$ for all $(xg, y) \in X * Y$ where $g \in G$. We may construct a universal such bimap $\lambda \colon X * Y \to X \otimes Y$ in the usual way [7]. However, there is a simplification in the theory due to the fact that we are acting by means of a groupoid. The element $x \otimes y$ in $X \otimes Y$ is the equivalence class of $(x, y) \in X * Y$ under the relation $\sim$ where

$(x,y) \sim (x',y')$ if and only if $(x',y') = (xg^{-1}, gy)$ for some $g \in G$. Observe that we may define $\mathbf{s}(x \otimes y) = \mathbf{s}(x)$ and $\mathbf{t}(x \otimes y) = \mathbf{t}(y)$ unambiguously.

Suppose now that X is a (G,G)-bimodule. We may therefore define the tensor product $X \otimes X$ as a set. This set is equipped with maps $\mathbf{s}, \mathbf{t} \colon X \otimes X \to G_o$. We define $g(x \otimes y) = gx \otimes y$ and $(x \otimes y)g = x \otimes yg$ when this makes sense. Observe that $x \otimes y = x' \otimes y'$ implies that $gx \otimes y = gx' \otimes y'$, and dually. It follows that $X \otimes X$ is a also a bimodule. Put $X^{\otimes 2} = X \otimes X$. More generally, we may define $X^{\otimes n}$ for all $n \geq 1$ using n-maps, and we define $X^{\otimes 0} = G$ where G acts on itself by multiplication on the left and right. The proof of the following lemma is almost immediate from the definition and the fact that we are acting by a groupoid.

Lemma 4.2. *Let $n \geq 2$. Then*

$$x_1 \otimes \ldots \otimes x_n = y_1 \otimes \ldots \otimes y_n$$

if and only if there are elements $g_1, \ldots, g_{n-1} \in G$ such that $y_1 = x_1 g_1$, $y_2 = g_1^{-1} x_2 g_2$, $y_3 = g_2^{-1} x_3 g_3$, $\ldots$ $y_n = g_{n-1}^{-1} x_n$.

Define

$$\mathsf{T}(X) = \bigcup_{n=0}^{\infty} X^{\otimes n}.$$

We shall call this the *tensor category* associated with the bimodule (G,X,G). Observe that we may regard X as a subset of $\mathsf{T}(X)$. The justification for this terminology will follow from (1) below.

Theorem 4.3.

1. *$\mathsf{T}(X)$ is an equidivisible category equipped with a length functor whose associated bimodule is (G,X,G).*
2. *The category $\mathsf{T}(X)$ is left cancellative if and only if (G,X,G) is right free, and dually.*
3. *Let C be a category whose groupoid of invertible elements is G. Regard C as a (G,G)-bimodule under left and right multiplication. Let $\theta \colon X \to C$ be any bimodule morphism to C. Then there is a unique functor $\Theta \colon \mathsf{T}(X) \to C$ extending θ.*
4. *Every equidivisible category equipped with a length functor is isomorphic to the tensor category of its associated bimodule.*

Proof. (1). The identities of the category are the same as the identities of G. The element $x_1 \otimes \ldots \otimes x_n$ has domain $\mathbf{d}(x_1)$ and codomain $\mathbf{r}(x_n)$. Multiplication is tensoring of sequences that begin and end in the right places and left and right actions by elements of G. We define $\lambda(g) = 0$ where $g \in G$ and $\lambda(x_1 \otimes \ldots \otimes x_n) = n$. Formally, we are using the fact that there is a canonical isomorphism

$$X^{\otimes p} \otimes X^{\otimes q} \cong X^{\otimes (p+q)}.$$

The proof of equidivisibility is essentially the same as that of Proposition 5.6 of [20]. The elements of length 0 are precisely the elements of G and so the invertible elements; the elements of length 1 are precisely the elements of X.

(2). Suppose that the category is left cancellative and that $xg = x = x\mathbf{d}(g)$ in the bimodule. But this can also be interpreted as a product in the category and so $g = \mathbf{d}(g)$, as required. Conversely, suppose that the bimodule is right free. Let $\mathbf{x} \otimes \mathbf{y} = \mathbf{x} \otimes \mathbf{z}$. From Lemma 4.2 and the fact that lengths match, we have that

$(\mathbf{x}, \mathbf{y}) = (\mathbf{x}g, g^{-1}\mathbf{z})$ for some $g \in G$. But using the fact that the action is right free, we get that g is an identity and so $\mathbf{y} = \mathbf{z}$, as required.

(3). Define $\Theta(g) = g$ when $g \in G$ and $\Theta(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = \theta(x_1) \otimes \theta(x_2) \otimes \ldots \otimes \theta(x_n)$. This is well-defined by Lemma 4.2. It is routine to check that this defines a functor.

(4). This now follows from (3) above, part (3) of Lemma 3.2, and part (2) of Lemma 3.3. □

Remark 4.4. It follows by the above theorem that left Rees categories are described by covering bimodules.

5. Left Rees categories

We shall describe the structure of arbitrary left Rees categories in terms of free categories using Zappa-Szép products generalized to categories; see [3]. This will show that they can be regarded as the categories associated with self-similar groupoid actions. The material in this section can be regarded as a special case of [21]. However, we have included it for the sake of completeness.

Let G be a groupoid with set of identities G_0 and let C be a category with set of identities C_o. We shall suppose that there is a bijection between G_0 and C_o and, to simplify notation, we shall identify these two sets. Denote by $G * C$ the set of pairs (g, x) such that $\mathbf{r}(g) = \mathbf{d}(x)$. We shall picture such pairs as follows:

$$\begin{tikzcd} \cdot \arrow[d, "g"'] & \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

We suppose that there is a function

$$G * C \to C \text{ denoted by } (g, x) \mapsto g \cdot x,$$

which gives a left action of G on C and a function

$$G * C \to G \text{ denoted by } (g, x) \mapsto g|_x,$$

which gives a right action of C on G such that these two functions satisfy the following conditions:

(C1): $\mathbf{d}(g \cdot x) = \mathbf{d}(g)$.
(C2): $\mathbf{r}(g \cdot x) = \mathbf{d}(g|_x)$.
(C3): $\mathbf{r}(x) = \mathbf{r}(g|_x)$.

This information is summarized by the following diagram

$$\begin{tikzcd} \cdot \arrow[r, "g \cdot x"] \arrow[d, "g"'] & \cdot \arrow[d, "g|_x"] \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

We also require that the following axioms be satisfied:

(SS1): $\mathbf{d}(x) \cdot x = x$. Observe that this is the action, not the category product.
(SS2): If gh is defined then $(gh) \cdot x = g \cdot (h \cdot x)$.

(SS3): $\mathbf{d}(g) = g \cdot \mathbf{r}(g)$.
(SS4): $\mathbf{d}(x)|_x = \mathbf{r}(x)$.
(SS5): $g|_{\mathbf{r}(g)} = g$.
(SS6): If xy is defined and $\mathbf{r}(g) = \mathbf{d}(x)$ then $g|_{xy} = (g|_x)|_y$.
(SS7): If gh is defined and $\mathbf{r}(h) = \mathbf{d}(x)$ then $(gh)|_x = g|_{h\cdot x}h|_x$.
(SS8): If xy is defined and $\mathbf{r}(g) = \mathbf{d}(x)$ then $g \cdot (xy) = (g \cdot x)(g|_x \cdot y)$.

If there are maps $(g, x) \mapsto g \cdot x$ and $(g, x) \mapsto g|_x$ satisfying (C1)–(C3) and (SS1)–(SS8) then we say that there is a *self-similar action of G on C.* Put

$$C \bowtie G = \{(x, g) \in C \times G \colon \mathbf{r}(x) = \mathbf{d}(g)\}.$$

We represent (x, g) by the diagram

$$\begin{tikzcd} \cdot \arrow[r, "x"] & \cdot \arrow[d, "g"] \\ & \cdot \end{tikzcd}$$

Given elements (x, g) and (y, h) satisfying $\mathbf{r}(g) = \mathbf{d}(y)$ we then have the following diagram

$$\begin{tikzcd} \cdot \arrow[r, "x"] & \cdot \arrow[r, "g\cdot y"] \arrow[d, "g"] & \cdot \arrow[d, "g|_y"] \\ & \cdot \arrow[r, "y"] & \cdot \arrow[d, "h"] \\ & & \cdot \end{tikzcd}$$

Completing the square, as shown, enables us to define a partial binary operation on $C \bowtie G$ by

$$(x, g)(y, h) = (x(g \cdot y), g|_y h).$$

Lemma 5.1. *Let G be a groupoid having a self-similar action on the category C.*

(1) If y is an invertible element of C then so too is $g \cdot y$.
(2) If $g \in G$ and $x \in X$ is an atom then $g \cdot x$ is an atom.
(3) If C is a left Rees category with length function λ then $\lambda(g \cdot x) = \lambda(x)$.

Proof. (1). We prove first that if y is an invertible element of C then so too is $g \cdot y$. Suppose that $e \xrightarrow{y} f$. We have that $g \cdot e = \mathbf{d}(g)$, by (SS3), and $e = yy^{-1}$. Thus $\mathbf{d}(g) = (g \cdot y)(g|_y \cdot y^{-1})$. It follows that $g \cdot y$ is invertible with inverse $g|_y \cdot y^{-1}$.

(2). Let $g \cdot x = uv$, where $u, v \in C$. By axioms (SS1), (SS2) and (SS8), we have that

$$x = (g^{-1} \cdot u)(g^{-1}|_u \cdot v).$$

By assumption, x is an atom and so at least one of the elements in the product is invertible. Suppose that $g^{-1} \cdot u$ is invertible. Then by our result above $g \cdot (g^{-1} \cdot u) = u$ is invertible. Suppose now that $g^{-1}|_u \cdot v$ is invertible then we may deduce that v is invertible. We have therefore proved that $g \cdot x$ is an atom.

(3). Write $x = x_1 \dots x_n$ a product of atoms where $n = \lambda(x)$. Now use (SS8). $\square$

Proposition 5.2. *Let G be a groupoid having a self-similar action on the category C.*

(1) *$C \bowtie G$ is a category.*
(2) *$C \bowtie G$ contains copies C' and G' of C and G respectively such that each element of $C \bowtie G$ can be written as a product of a unique element from C' followed by a unique element from G'.*
(3) *If C is left cancellative then so too is $C \bowtie G$.*
(4) *If C is left cancellative then the set of invertible elements of $C \bowtie G$ consists of all those elements (x,g) where x is invertible in C*
(5) *If C is left cancellative then the set of atoms in $C \bowtie G$ consists of all those elements (x,g) where x is an atom in C.*
(6) *If C is left cancellative and right rigid then so too is $C \bowtie G$.*
(7) *If C is a left Rees category then so too is $C \bowtie G$.*

Proof. (1) Define $\mathbf{d}(x,g) = (\mathbf{d}(x), \mathbf{d}(x))$ and $\mathbf{r}(x,g) = (\mathbf{r}(g), \mathbf{r}(g))$. The condition for the existence of $(x,g)(y,h)$ is that $\mathbf{r}(x,g) = \mathbf{d}(y,h)$. Axioms (C1),(C2) and (C3) then guarantee the existence of $(x(g \cdot y), g|_y h)$. We therefore have a partially defined multiplication. We next locate the identities. Suppose that (u,a) is an element such that if $(u,a)(x,g)$ is defined then $(u,a)(x,g) = (x,g)$. Now $(u,a)(\mathbf{r}(a), \mathbf{r}(a))$ is defined. We deduce that $\mathbf{r}(a) = u(a \cdot \mathbf{r}(a))$ and $\mathbf{r}(a) = a|_{\mathbf{r}(a)}\mathbf{r}(a)$. By (SS5), we have that $a = \mathbf{r}(a)$ and by (SS3) that $a \cdot \mathbf{r}(a) = \mathbf{d}(a)$. Thus $(u,a) = (\mathbf{r}(a), \mathbf{r}(a))$. By (SS1) and (SS4), we deduce that the identities are the elements of the form (e,e) where $e \in C_o = G_o$. Observe that $\mathbf{d}[(x,g)(y,h)] = \mathbf{d}(x,g)$ and $\mathbf{r}[(x,g)(y,h)]\mathbf{r}(y,h)$. It remains only to prove associativity. Suppose first that

$$[(x,g)(y,h)](z,k)$$

exists. The product $(x,g)(y,h)$ exists and so we have the following diagram

$$\begin{tikzcd}
{} \arrow[r, "x"] & {} \arrow[r, "g\cdot y"] \arrow[d, "g"] & {} \arrow[d, "g|_y"] \\
& {} \arrow[r, "y"] & {} \arrow[d, "h"] \\
& & {}
\end{tikzcd}$$

similarly $[(x,g)(y,h)](z,k)$ exists and so we have the following diagram

$$\begin{tikzcd}
{} \arrow[r, "x(g\cdot y)"] & {} \arrow[r, "(g|_y h)\cdot z"] \arrow[d, "g|_y h"] & {} \arrow[d, "(g|_y h)|_z"] \\
& {} \arrow[r, "z"] & {} \arrow[d, "k"] \\
& & {}
\end{tikzcd}$$

resulting in the product

$$(x(g\cdot y)[(g|_y h)\cdot z], (g|_y h)|_z k).$$

By assumption, $x(g\cdot y)[(g|_y h)\cdot z]$ exists and so $(g\cdot y)[(g|_y h)\cdot z]$ exists. We now use (SS8) and (SS4) and (SS7), to get that $y(h\cdot z)$ exists and we use (SS7) and (SS6) to show that

$$(g|_y h)|_z k = g|_{y(h\cdot z)} h|_z k.$$

By (SS2),

$$x(g\cdot y)[(g|_y h)\cdot z] = (x(g\cdot y)((g|_y \cdot (h\cdot z)).$$

It now follows that

$$(y,h)(z,k) = (y(h\cdot z), h|_z k)$$

exists. It also follows that $(x,g)[(y,h)(z,k)]$ exists and is equal to

$$[(x,g)(y,h)](z,k).$$

Next suppose that

$$(x,g)[(y,h)(z,k)]$$

exists. This multiplies out to give $(x[g\cdot(y(h\cdot z))], g|_{y(h\cdot z)} h|_z k)$. By (SS6) and (SS7) we get that

$$g|_{y(h\cdot z)} h|_z k = (g|_y h)|_z k,$$

and by (SS8) and (SS2) we get that $x[g\cdot(y(h\cdot z))] = x(g\cdot y)[(g|_y h)\cdot z]$. This completes the proof that $C \bowtie G$ is a category.

(2) Define $\iota_C\colon C \to C \bowtie G$ by $\iota_C(x) = (x, \mathbf{r}(x))$. This is well-defined. Suppose that xy exists. Then in particular $\mathbf{r}(x) = \mathbf{d}(y)$. It is easy to check using (SS4) and (SS1) that $\iota_C(x)\iota_C(y) = \iota_C(xy)$. In fact, $\iota_C(x)\iota_C(y)$ exists iff xy exists. Thus the categories C and C' are isomorphic.

Now define $\iota_G\colon G \to C \bowtie G$ by $\iota_G(g) = (\mathbf{r}(g), g)$. Then once again the categories G and G' are isomorphic.

Finally, pick an arbitrary non-zero element (x,g). Then $\mathbf{r}(x) = \mathbf{d}(g)$. We may write $(x,g) = (x, \mathbf{r}(x))(\mathbf{d}(g), g)$ using the fact that $\mathbf{r}(x)\cdot\mathbf{d}(g) = \mathbf{d}(g)$ by (SS1) and $\mathbf{r}(x)|_{\mathbf{d}(g)} = \mathbf{d}(g)$ by (SS4).

(3) Suppose that C is left cancellative. We prove that $C \bowtie G$ is left cancellative. Suppose that $(x,g)(y,h) = (x,g)(z,k)$. Then $x(g\cdot y) = x(g\cdot z)$ and $g|_y h = g|_z k$. By left cancellation in C it follows that $g\cdot y = g\cdot z$ and by (SS1) we deduce that $y = z$. Hence $h = k$. We have therefore proved that $(y,h) = (z,k)$, as required.

(4) We know by (3), that the resulting category is left cancellative. Suppose that (x,g) is invertible. Then there is an element (y,h) such that $(\mathbf{d}(x), \mathbf{d}(x)) =$

$(x,g)(y,h)$. In particular, $\mathbf{d}(x) = x(g \cdot y)$ and so x is invertible. Conversely, if x is invertible, it can be verified that

$$(x,g)^{-1} = (g^{-1} \cdot x^{-1}, (g|_{g^{-1} \cdot x^{-1}}))^{-1}.$$

(5). Suppose that x is an atom. Let $(x,g) = (u,h)(v,k)$. Then $x = u(h \cdot v)$ and $g = h|_v k$. If u is invertible then (u,h) is invertible, whereas if $h \cdot v$ is invertible then v is invertible and so (v,k) is invertible. It follows that (x,g) is an atom. To prove the converse, suppose that (x,g) is an atom. It is immediate that x is not invertible. Suppose that x is not an atom. Then we may write $x = uv$ where neither u nor v is invertible. But then $(x, \mathbf{r}(x)) = (u, \mathbf{r}(u))(v, \mathbf{r}(x))$ and this leads to a non-trivial factorization of (x,g), which is a contradiction.

(6) Suppose now that C is left cancellative and right rigid. By (3), we know that $C \bowtie G$ is left cancellative so it only remains to be proved that $C \bowtie G$ is right rigid. Suppose that

$$(x,g)(y,h) = (u,k)(v,l)$$

From the definition of the product it follows that $x(g \cdot y) = u(k \cdot v)$ and $g|_y h = k|_v l$. From the first equation see that $xC \cap uC \neq \emptyset$. Without loss of generality, suppose that $x = uw$. Then by left cancellation $w(g \cdot y) = k \cdot v$. Observe that $k^{-1} \cdot (k \cdot v)$ is defined and so $k^{-1} \cdot (w(g \cdot y))$ is defined by (SS2). Thus by (SS8), $k^{-1} \cdot w$ is defined. It is now easy to check that

$$(x,g) = (u,k)(k^{-1} \cdot w, (k|_{k^{-1} \cdot w})^{-1} g).$$

(7) Let C be a left Rees category. It is enough to prove that $C \bowtie G$ is equipped with a length function. Let the length function on C be denoted by λ. Let $(x,g) \in C \bowtie G$. Define $\mu(x,g) = \lambda(x)$. By the above, $\mu(x,g) = 0$ if and only if (x,g) is invertible, and $\mu(x,g) = 1$ if and only if x is an atom. The fact that μ is a functor follows from the fact that $\lambda(g \cdot y) = \lambda(y)$. □

We call $C \bowtie G$ the *Zappa-Szép product* of the category C by the groupoid G.

Let C be a left Rees category. A transversal of the generators of the submaximal principal right ideals is called a *basis* for the category. From Section 3, a basis is therefore a subset of the set of atoms of C.

Theorem 5.3. *A category is a left Rees category if and only if it is isomorphic to the Zappa-Szép product of a free category by a groupoid.*

Proof. We shall sketch the proof. Let C be the left Rees category. Choose a basis X for C. Every element of C can be written uniquely as a product of elements of X followed by an invertible element. The subcategory X^* generated by X is free. Thus $C = X^* G$. The Zappa-Szép product representation then readily follows. □

The following theorem describes the precise circumstances under which Zappa-Szép products arise.

Theorem 5.4. *Let A be a category with subcategories B and C. We suppose that $A_o = B_o = C_o$ and that $C = AB$, uniquely. Then C is isomorphic to the Zappa-Szép product $A \bowtie B$.*

Proof. We sketch out the proof. Suppose that ba is defined where $b \in B$ and $a \in A$. Then $ba = a'b'$ for uniquely determined elements $a' \in A$ and $b' \in B$. We define

$$ba = (b \cdot a) b|_a.$$

It is immediate that axioms (C1), (C2) and (C3) hold. The equality $a = \mathbf{d}(a)a$ yields the axioms (SS1) and (SS4); the equality $b = b\mathbf{r}(b)$ yields axioms (SS3) and (SS5); the equality $b_1(b_2a) = (b_1b_2)a$ yields axioms (SS2) and (SS7); the equality $b(a_1a_2) = (ba_1)a_2$ yields axioms (SS6) and (SS8). We may therefore construct the category $A \bowtie B$. We define a map $C \to A \bowtie B$ by $c \mapsto (a, b)$ if $c = ab$. It is now straightforward to check that this determines an isomorphism of categories. □

The following result shows that the theory in this paper can be regarded as a generalization of the theory of free categories.

Theorem 5.5. *The free categories are precisely the left Rees categories with a trivial groupoid of invertible elements and precisely the equidivisible categories with length functors having trivial groupoids of invertible elements.*

Proof. We need only prove that an equidivisible category with length functor having a trivial groupoid of invertible elements is left cancellative. But this essentially follows from Lemma 3.3. □

Remark 5.6. Cancellative equidivisible monoids with trivial groups of units do not have to be free; see, for example, Example 1.8 of Chapter 5 of [15]. This example also suggests that studying equidivisible categories with more general kinds of length functors may be interesting.

6. Diagrams of partial homomorphisms

From now on, the groupoids involved in bimodules will always be totally disconnected.

Let D be a directed graph. An edge x from the vertex e to the vertex f will be written $e \xrightarrow{x} f$. With each vertex e of D we associate a group G_e, called the *vertex group*, and with each edge $e \xrightarrow{x} f$, we associate a surjective homomorphism $\phi_x \colon (G_e)^+_x \to (G_f)^-_x$ where $(G_e)^+_x \leq G_e$ and $(G_f)^-_x \leq G_f$. In other words, with each edge $e \xrightarrow{x} f$, we associate a partial homomorphism ϕ_x from G_e to G_f. We call this structure a *diagram of partial homomorphisms*. If all the ϕ_x are *isomorphisms* then we shall speak of a *diagram of partial isomorphisms*. For brevity, we shall say that D is the diagram of partial homomorphisms though of course it is defined only by the totality of data.

Let D_1 and D_2 be two diagrams of partial homomorphisms having the same vertex sets, and edge sets that differ only in labelling and the same vertex groups. We say these two diagrams of partial homomorphisms are *conjugate*, if for each edge $e \xrightarrow{x} f$ in D_1 and corresponding edge $e \xrightarrow{y} f$ in D_2 there are inner automorphisms $\alpha_{x,y} \colon G_e \to G_e$ and $\beta_{x,y} \colon G_f \to G_f$ such that $\alpha_{x,y}((G_e)^+_x) = (G_e)^-_y$ and $\beta_{x,y}((G_f)^+_x) = (G_f)^-_y$ and $\beta_{x,y}\phi_x = \phi_y\alpha_{x,y}$.

Remark 6.1. The above definition could be generalized a little by assuming only that the underlying graphs were isomorphic.

The goal of this section is to prove that covering bimodules and diagrams of partial homomorphisms are different ways of describing the same object.

Let (G, X, G) be a covering bimodule where $G = \bigcup_{e \in V} G_e$. Define a relation $\mathscr{C}$ on X by $x \mathscr{C} y$ if and only if $y = gxh$ for some $g, h \in G$.[2] Observe that because

[2] This relation was used by Paul Cohn whence the choice of notation.

G is a disjoint union of groups if $x \mathscr{C} y$ then x and y are parallel. Clearly, $\mathscr{C}$ is an equivalence relation.

Lemma 6.2. *Let (G, X, G) be a covering bimodule where $G = \bigcup_{e \in V} G_e$.*

(1) With each transversal of the $\mathscr{C}$-classes, we may associate a diagram of partial homomorphisms.
(2) Different transversals yield conjugate diagrams of partial homomorphisms.
(3) The bimodule is bifree if and only if the associated diagram is a diagram of partial isomorphisms.

Proof. (1). Choose a transversal E of the $\mathscr{C}$-classes. Define the directed graph D to have as vertices the set of identities of G and edges the elements of E. The group associated with the vertex e is the group G_e. Consider now the edge $e \xrightarrow{x} f$. We define

$$(G_e)^+_x = \{g \in G_e \colon gx = xh \text{ for some } h \in G_f\}$$

and

$$(G_f)^-_x = \{h \in G_f \colon gx = xh \text{ for some } g \in G_e\}$$

and $\phi_x(g) = h$ if $g \in (G_e)^+_x$ and $gx = xh$ where $h \in G_f$. In a covering bimodule the right action is free and so ϕ_x is a function and, in fact, a homomorphism. We have therefore constructed a diagram of partial homomorphisms.

(2). Let E' be another transversal. We denote the elements of E by x_i where $i \in I$ and the elements of E' by y_i where $i \in I$ and assume that $x_i \mathscr{C} y_i$. Choose $g_i, h_i \in G$ such that $x_i = g_i y_i h_i$. Let $e \xrightarrow{x_i} f$. Let $g \in (G_e)^+_{x_i}$. Then

$$(g_i^{-1} g g_i) y_i = y_i (h_i \phi_{x_i}(g) h_i^{-1}).$$

Thus

$$h_i \phi_{x_i}(g) h_i^{-1} = \phi_{y_i}(g_i^{-1} g g_i).$$

Define inner automorphisms $\beta_i(-) = h_i - h_i^{-1}$ and $\alpha_i(-) = g_i^{-1} - g_i$ where $\alpha_i \colon G_e \to G_e$ and $\beta_i \colon G_f \to G_f$. We therefore have $\beta_i \phi_{x_i} = \phi_{y_i} \alpha_i$ and $\alpha_i((G_e)^+_{x_i}) = (G_e)^+_{y_i}$ and $\beta_i((G_f)^+_{x_i} = (G_f)^+_{y_i}$. Thus the two diagrams are conjugate

(3). Suppose that the bimodule is bifree. By definition $gx = x\phi_x(g)$. Suppose that $g_1 x = g_2 x$. Then $x = (g_1^{-1} g_2) x$ and so by left freeness we have that $g_1^{-1} g_2$ is an identity and so $g_1 = g_2$. Thus ϕ_x is injective and so an isomorphism. The proof of the converse is straightforward. $\square$

We now show how to go in the opposite direction. Let D be a diagram of partial homomorphisms. Let $G = \bigcup_{e \in V} G_e$ be the disjoint union of the vertex groups regarded as a groupoid. Denote the set of edges by E. Let $G * E * G$ be the set of triples (g, x, h) where $e \xrightarrow{x} f$ and $g \in G_e$ and $h \in G_f$. We define a relation $\equiv$ on the set $G * E * G$ as follows: $(g_1, x_1, h_1) \equiv (g_2, x_2, h_2)$ if and only if $x_1 = x_2 = x$, say, $g_1^{-1} g_2 \in (G_e)^+_x$ and $\phi_x(g_1^{-1} g_2) = h_1 h_2^{-1}$.

Lemma 6.3. *With the above definition, $\equiv$ is an equivalence relation. Denote the $\equiv$-class containing (g, x, h) by $[g, x, h]$. Then we get a covering bimodule $\mathsf{B}(D)$ when we define $g[g_1, x, h_1] = [gg_1, x, h_1]$ when $\exists gg_1$ and $[g_1, x, h_1]h = [g_1, x, h_1 h]$ when $\exists h_1 h$. The diagram of partial homomorphisms associated with this covering module is conjugate to D.*

Proof. We shall just prove the last part since the other proofs are routine. We choose the transversal $[e, x, f]$ where $e \xrightarrow{x} f$ is an edge of the diagram D. It is now easy to check that we get back exactly the digram of partial homomorphisms we started with. $\square$

Lemma 6.4. *Let (G, X, G) be a covering bimodule, let E be a transversal of the $\mathscr{C}$-classes and let D be the associated diagram of partial homomomorphisms. Then the bimodule X is isomorphic to the bimodule $\mathsf{B}(D)$.*

Proof. Let $x \in X$. Then $x \mathscr{C} y$ for a unique $y \in E$. By definition, $x = g_1 y h_1$ for some $g_1, h_1 \in G$. Suppose also that $x = g_2 y h_2$. Then $g_2^{-1} g_1 y = y h_2 h_1^{-1}$. It follows that $(g_1, y, h_1) \equiv (g_2, y, h_2)$. We may therefore define a function $\theta \colon X \to \mathsf{B}(D)$ by $\theta(x) = [g_1, x, h_1]$. It remains to show that this is a bijection and an isomorphism of bimodules both of which are now straightforward. $\square$

We summarize the results of this section in the following theorem.

Theorem 6.5. *With each diagram of partial homomorphisms D we may associate a covering bimodule $\mathsf{B}(D)$ and every covering bimodule is isomorphic to one of this form. Diagrams of partial isomorphisms correspond to bifree covering bimodules.*

7. Presentations of skeletal left Rees categories

In this section, we shall prove that every skeletal left Rees category has a presentation of a particular form. This presentation is then the final link in showing that the theory of graphs of groups is a special case of the theory of skeletal left Rees categories.

Let C be a skeletal left Rees category. Then for each atom $x \in eCf$, we have as before the following definitions:

$$(G_e)_x^+ = \{g \in G_e \colon gx = xh \text{ for some } h \in G_f\},$$

and

$$(G_f)_x^- = \{h \in G_f \colon gx = xh \text{ for some } g \in G_e\},$$

and

$$\phi_x(g) = h \text{ if } g \in (G_e)_x^+ \text{ and } h \in (G_f)_x^- \text{ and } gx = xh.$$

We have that $(G_e)_x^+ \leq G_e$ and $(G_f)_x^- \leq G_f$ and ϕ_x is a surjective homomorphism.

Lemma 7.1. *Let C be a skeletal left Rees category. Then C is right cancellative if and only if all the homomorphisms ϕ_x defined above are also injective and so in fact isomorphisms.*

Proof. If C is right cancellative, it is immediate that all the homomorphisms ϕ_x are injective. We prove the converse using Section 6. Choose a basis for C so that $C = X^* G$ where X^* is a free category and G the groupoid of isomorphisms. Let $a, b, c \in C$ such that $ab = cb$. We shall prove that $a = c$. We have that $a = xg$, $b = yh$ and $c = zk$ where $x, y, z \in X^*$ and $g, h, k \in G$. Thus

$$x(g \cdot y) g|_y h = z(k \cdot y) k|_y h.$$

From length considerations, $x = z$. We also have that $g|_y = k|_y$. It remains only to prove that $g = k$ and we are done. We know that $gy = ky$. Thus $(g^{-1}k)y = y$. Therefore, to prove our result it is enough to prove that $gy = y$ implies that g is an identity. If y is an atom then the result is immediate. We assume the result holds for all elements y whose length is at most n and prove it for those elements of length $n+1$. Let y have length $n+1$. Let $y = uv$ where v has length 1. Then $gy = guv = (g \cdot u)(g|_u \cdot v)g|_{uv} = y = uv$. Thus $g|_y = g|_{uv} = \mathbf{r}(g)$. Also $u = g \cdot u$ and $v = g|_u \cdot v$. We have that $g|_u v = v g|_{uv}$. It follows that $g|_u$ is an identity. We then have $gu = u$ and so g is an identity, as required. □

The proof of the following is immediate from the definitions and Lemma 3.1.

Lemma 7.2. *Let C be a skeletal left Rees category. If x and y are atoms then $x \mathscr{J} y$ if and only if $x \mathscr{C} y$.*

The following two results will enable us to refine the way that we choose a basis for a left Rees category. We refer the reader to Section 5 for the structure theory of left Rees categories which we use here.

Lemma 7.3. *Let C be a left Rees category and let X be a transversal of the generators of the submaximal principal right ideals so that $C = X^*G$. Let $x, y \in X$. Then $x \mathscr{J} y$ if and only if $y = g \cdot x$.*

Proof. Suppose that $y = g \cdot x$. Then $CyC = Cg \cdot xC = C(g \cdot x)g|_x C = CgxC = CxC$. Thus $x \mathscr{J} y$. Conversely, suppose that $x \mathscr{J} y$. Then $y = gxh$ for some $g, h \in G$. Thus $y = (g \cdot x)g|_x h$. But by the uniqueness of the factorization, we must have that $y = g \cdot x$. □

Lemma 7.4. *Let C be a skeletal left Rees category and let x be an atom where $e \xrightarrow{x} f$. Choose a coset decomposition $G_e = \bigcup_{i \in I} g_i (G_e)_x^+$. The set $\{g_i x \colon i \in I\}$ consists of pairwise $\mathscr{R}$-inequivalent elements and every atom $\mathscr{J}$-related to x is $\mathscr{R}$-related to one of these elements.*

Proof. Suppose that $g_i x \mathscr{R} g_j x$. Then $g_i x = g_j x g$ for some $g \in G_f$. Then $x = (g_i^{-1} g_j) x g$. But $g_i^{-1} g_j \in (G_e)_x^+$. By the definition of coset representatives, we have that $g_i = g_j$, as required. Let $y \mathscr{J} x$ be an arbitrary atom. Then $y = gxh$ for some $g, h \in G$. Write $g = g_i a$ where $a \in (G_e)_x^+$. Then $y = g_i a x h = g_i x h$. Hence $y \mathscr{R} g_i h$. □

We now define a special type of basis for a left Rees category C. For each hom-set eCf, choose a transversal Y of the $\mathscr{J}$-classes of the atoms. For each $x \in Y$, choose a coset decomposition $G_e = \bigcup_{i \in I} g_i (G_e)_x^+$. Denote by T_x^+ the transversal $\{g_i \colon i \in I\}$. We shall assume that the appropriate identities are always elements of these transversals. For each such x, we have a set of atoms $\{kx \colon k \in T_x^+\}$ and the totality of those atoms as x varies over Y then provides a basis for C. We call a basis constructed in this way a *co-ordinatization.* Observe that it contains two components: a transversal of the $\mathscr{J}$-classes of the set of atoms of C, we call this an *atomic transversal*, and for each atom x in that transversal a set of coset representatives T_x^+.

Theorem 7.5 (Skeletal left Rees categories and their diagrams)**.**

(1) Let D be a diagram of partial homomorphisms. Then we may construct a skeletal left Rees category C equipped with an atomic transversal such that the diagram of partial homomorphisms constructed from the bimodule of atoms of C relative to that transversal is equal to D. If D is a diagram of partial isomorphisms then C is a Rees category.

(2) *Let C be a skeletal left Rees category equipped with an atomic transversal. Then we may construct a diagram of partial homomorphisms D using that transversal such that the tensor category of the covering bimodule associated with D is isomorphic to C. If C is a Rees category then D is a diagram of partial isomorphisms.*

Proof. (1). Let D be a diagram of partial homomorphisms. Then by Lemma 6.3, we may construct a covering bimodule $\mathsf{B}(D)$. This bimodule has a transversal with respect to which the associated diagram of partial homomorphisms is D. By Theorem 4.3, we may construct a left cancellative, skeletal equidivisible category equipped with a length functor $\mathsf{T}(\mathsf{B}(D))$. By Proposition 3.10 this is a skeletal left Rees category whose associated bimodule is $\mathsf{B}(D)$. It follows by our results proved earlier that if we start with a diagram of partial isomorphisms we obtain a Rees category.

(2). Let C be a skeletal left Rees category equipped with an atomic transversal. Then by Proposition 3.10 and Lemma 4.1, we may construct a covering bimodule B from the set of atoms acted on by the groupoid of invertible elements. From such a bimodule and an atomic transversal, we may construct a diagram of homomorphisms by Lemma 6.2 D. But then by part (3) of Theorem 4.3, the category C is isomorphic to the tensor category $\mathsf{T}(\mathsf{B}(D))$ where B is essentially $\mathsf{B}(D)$. It follows by our results proved earlier that if we start with a Rees category we obtain a diagram of partial isomorphisms. □

Remark 7.6. We have now seen that diagrams of partial homomorphisms and skeletal left Rees categories are two ways of viewing the same structure. The path from diagram to category taking in both the construction of bimodules from diagrams and then (tensor) categories from bimodules. We shall now describe a *direct* construction of left Rees categories from diagrams of partial homomorphisms.

Let D be a diagram of partial homomorphisms. We shall define a category $\langle D\rangle$ by means of a presentation constructed from D. We first construct a new directed graph D' from D. This contains the directed graph D but at each vertex we adjoin additional loops labelled by the elements of G_e. We construct the free category $(D')^*$. We denote elements of this category thus $x_1 \cdot \ldots \cdot x_m$ where the x_i are edges that match. We now factor out by two kinds of relations: those of the form $g \cdot h = gh$ where $g, h \in G_e$; and those of the form $g \cdot x = x \cdot \phi_x(g)$ where $g \in (G_e)^+_x$. The resulting category is denoted by $\langle D\rangle$.

Theorem 7.7 (Presentation theorem)**.** *Let D be a diagram of partial homomorphisms. Then the category $\langle D\rangle$ is a skeletal left Rees category isomorphic to the category obtained from D by constructing the tensor category of the covering bimodule associated with D. The category $\langle D\rangle$ is a skeletal Rees category if and only if D is a diagram of partial isomorphisms.*

Proof. Denote by C the skeletal left Rees category constructed from the diagram of partial homomorphisms D according to part (1) of Theorem 7.5. This category satisfies all the defining relations of the category $\langle D\rangle$ and so C is a functorial image of $\langle D\rangle$.

It remains therefore only to show that this functor is injective. We work first in the category C. In the category C we choose an atomic transversal whose elements can be identified with the edges of the diagram D. For each such atom x choose a coset decomposition $G_e = \bigcup_{i \in I} g_i (G_e)_x^+$. This leads to a co-ordinatization for C as described earlier in this section. We now appeal to the structure theory of Section 5, and deduce that every element of x can be written as a unique product of atoms in this co-ordinatization followed by an invertible element. Call this a normal form. We now work in the category $\langle D \rangle$. Using the same coset decomposition as above, we may show that every element in $\langle D \rangle$ is equivalent in the presentation, using our two types of relations, to an element in normal forms. However, different normal forms correspond to different elements of C and so these normal forms are unique and we have established our isomorphism. $\square$

We may paraphrase the above theorem as saying that *every skeletal left Rees category may be presented by a diagram of partial homomorphisms.*

The following theorem was originally suggested by [38, 39] and [6] but the proof is a straightforward generalization of Higgins's main result [8] and at the same time this shows how our approach is related to his.

Theorem 7.8. *Every skeletal Rees category may be embedded in its universal groupoid.*

Proof. Let C be a skeletal Rees category. By Theorem 7.7, we may assume that $C = \langle D \rangle$ where D is a diagram of partial isomorphisms. Denote by $\mathcal{G}$ the universal groupoid of C. Our goal is to obtain a normal form for the elements of $\mathcal{G}$. To that end, let $e \xrightarrow{x} f$. Choose a coset decomposition $G_e = \bigcup_{i \in I} g_i (G_e)_x^+$ to obtain the transversal T_x^+ as before. However, now that we want to work in a groupoid, we shall also need a coset decomposition $G_f = \bigcup_{j \in J} h_j (G_f)_x^-$ to obtain the transversal T_x^-. In both transversals, we assume that the identity elements of their respective groups have been choosen. We may now follow the proof of the theorem in Section 3 of [8] by defining suitable normal forms since finiteness plays no role in Higgins's proof. This shows that C is in fact embedded in $\mathcal{G}$. $\square$

Example 7.9. Free monoids on n generators are Rees monoids. The group constructed according to the above theorem is the free group on n generators.

Remark 7.10. Alternative ways of proving the above theorem are suggested by the results of Cohn [6] and von Karger [38, 39] though we do not pursue these here.

Remark 7.11. The connection between our work and that of Higgins will be clarified in the next section, but we anticipate what we do there by outlining the connection phrased in our language. Higgins uses the relations $g \cdot x = x \cdot \phi_x(g)$ in two directions to construct the fundamental groupoid of a diagram of partial isomorphisms. We, on the other hand, use these relations in one direction only to construct a Rees category. It is then evident that the Rees category sits inside the fundamental groupoid. What is perhaps surprising is that these cancellative categories can be abstractly characterized.

8. A categorical approach to Bass-Serre theory

We shall now explain the connection between the theory we have developed and the theory of graphs of groups. Our references for this theory are [36] and [41]. We start with an observation. A graph of groups equipped with a given orientation is essentially the same thing as a diagram of partial isomorphisms where the directed graph underlying it is finite and weakly connected in the sense that as a graph it is connected. Essentially, graphs of groups represent partial isomorphisms by means of relations and so are unoriented. We shall call the diagrams of partial isomorphisms that arise from graphs of groups equipped with an orientation *Serre diagrams of partial isomorphisms.*

Let C be a category. A *zig-zag* joining the identity e to the identity f is determined by a sequence of identities $e = e_1, \ldots, e_n = f$ such that for each consecutive pair of identities e_i and e_{i+1} we have that either e_iCe_{i+1} or $e_{i+1}Ce_i$ is non-empty. We say that a category C is *connected* if any two identities $e, f \in C_o$ are joined by a zig-zag.

A skeletal Rees category C is called a *Serre category* if it satisfies the following conditions.

(S1): The number of identities in C is finite and nonzero.
(S2): In each hom-set, the number of $\mathscr{J}$-classes of atoms is finite.
(S3): C is connected.

The following theorem is now immediate from the above definitions and what we proved in the previous section.

Theorem 8.1 (Graphs of groups as categories)**.**

1. *There is a correspondence between graphs of groups with a given orientation and Serre diagrams of partial isomorphisms.*
2. *There is a correspondence between Serre diagrams of partial isomorphisms and Serre categories.*
3. *The fundamental groupoid of a graph of groups with a given orientation is isomorphic to the universal groupoid of the Serre category constructed from the diagram of partial isomorphisms associated with the oriented graph of groups.*
4. *The universal groupoid of a Serre category is connected.*

The following two examples are the basic building blocks of Bass-Serre theory.

Examples 8.2.

1. **HNN extensions**. These are constructed from Rees *monoids*. If we make the additional assumption that the monoid has the property that any two atoms are $\mathscr{J}$-related, then the universal groups are precisely HNN extensions with one stable letter. This case was the subject of our paper [24] and motivated the work of the current paper.
2. **Amalgamated free products.** The building blocks of these are (G, H)-bisets X where G and H are both groups.[3] We say that such a biset is *irreducible* if there exists $x \in X$ such that $GxH = X$. There is a bijective correspondence between conjugacy classes of partial isomorphisms from G to H and isomorphism classes of irreducible, bifree (G, H)-bisets. Consider now any irreducible, bifree biset (G, X, H). Choose and fix $x \in X$. Let $A = G_x^+$, $B = H_x^-$ and $\theta = \phi_x$ be the associated partial isomorphism. We may regard (G, X, H) as a cancellative category with two identities in the following way. We let the identity of G, 1_G say, be one of the identities and the identity of H, 1_H say, the other. Thus we take the disjoint union $G \cup X \cup H$. The products in G and H are the group products. The product

[3]Recall that a biset is a set on which groups act on the left and right and whose actions associate.

gx is the action and the product xh is the action. We denote by C the above biset regarded as a category in this way. Then C is a Rees category and is in fact a Serre category. In this case, the tensor product cosntruction is essentially degenerate. The universal groupoid of C is connected and any vertex group is isomorphic to $G *_\theta H$ the amalgamated free product of G and H via the identifying partial isomorphism θ.

We have not yet explained how the Bass-Serre tree can be constructed within our theory. That can best be understood by translating our categorical approach into an inverse semigroup theoretic one which we do in the next section.

9. An inverse semigroup approach to Bass-Serre theory

In this section, we shall describe how our categorical approach to Bass-Serre theory can be transformed into an approach using inverse semigroups and ordered groupoids.

This section was also motivated by the papers [4, 11, 17, 18, 19, 23].

9.1. **Ordered groupoids.** A groupoid G is said to be *ordered* if it is equipped with a partial order $\leq$ that satisfies the following conditions:

(OG1): If $g \leq h$ then $g^{-1} \leq h^{-1}$.
(OG2): If $g \leq h$ and $g' \leq h'$ and $\exists gg'$ and $\exists hh'$ then $gg' \leq hh'$.
(OG3): If $e \leq gg^{-1}$ then there exists a unique element $(e|g)$ such that $(e|g) \leq g$ and $(e|g)(e|g)^{-1} = e$.
(OG4): If $e \leq g^{-1}g$ then there exists a unique element $(g|e)$ such that $(g|e) \leq g$ and $(g|e)^{-1}(g|e) = e$.

Let G be an ordered groupoid and let $g, h \in G$. Suppose that $e = g^{-1}g \wedge hh^{-1}$ exists in the poset $(G_o, \leq)$. Define $g \bullet h = (g|e)(e|h)$. Then $g \bullet h$ is called the *pseudoproduct* of g and h. This partially defined product is associative when this makes sense.

If S is an inverse semigroup and $s, t \in S$, define the *restricted product* of s and t to be st if $s^{-1}s = tt^{-1}$ and undefined otherwise. Every inverse semigroup can be regarded as an ordered groupoid with respect to its restricted product and natural partial order; if the inverse semigroup has a zero, we shall discard that zero in forming the associated ordered groupoid. In this way, inverse semigroup theory can be viewed as being part of the theory of ordered groupoids. For more on the connections between inverse semigroup theory and ordered groupoid theory, see [16].

9.2. **The maximum enlargement theorem.** The goal of this section is simply to state this theorem. We refer the reader to Chapter 8 of [16] for the details. We first make some definitions. Let G be an ordered subgroupoid of an ordered groupoid H. We say that H is an *enlargement* of G if the following hold:

(E1): G_o is an order ideal of H_o.
(E2): If $x \in H$ and $x^{-1}x, xx^{-1} \in G$ then $x \in G$.
(E3): If $e \in H_o$ then there exists $x \in H$ such that $xx^{-1} = e$ and $x^{-1}x \in G$.

A functor $\theta \colon H \to K$ between groupoids is said to be *star injective* if $\theta(h_1) = \theta(h_2)$ and $h_1h_1^{-1} = h_2h_2^{-1}$ implies that $h_1 = h_2$. Such a functor is said to be a *covering functor* if, in addition, whenever $e \in H$ is an identity such that $\theta(e) = kk^{-1}$ there exists $h \in H$ such that $e = hh^{-1}$ and $\theta(h) = k$.

Theorem 9.1 (Ehresmann's Maximum enlargement theorem). *Let $\theta\colon H \to K$ be an ordered star injective functor. Then there is an ordered groupoid G, an ordered covering functor $\Theta\colon G \to K$, and an ordered embedding $\iota\colon H \to G$ such that $\Theta\iota = \theta$ and G is an enlargement of $\iota(H)$.*

Although it is not immediately obvious from the statement of the theorem, there is a sense in which this construction is universal. An ordered pair (ι', Θ') is called an *extension of θ to a covering* if $\Theta'\colon G \to K$ is an ordered covering functor, $\iota'\colon H \to G'$ is an ordered embedding, and $\Theta'\iota' = \theta$. It is easy to describe a category in which the objects are such extensions. This category has an initial object: namely, (ι, Θ). It is therefore the 'best' way of extending θ to a covering.

Example 9.2. We give a simple example of such an enlargement. Let H be an ordered groupoid whose underlying groupoid is totally disconnected and let $\theta\colon H \to K$ be an ordered star injective functor to the *group* K. We let G be the enlargement as above. To ease notation, we assume that ι is the identity function. We shall describe in more concrete terms the poset of identities G_o. Observe first that since θ is star injective, each local group G_e is embedded in K as a subgroup. We denote its image by G'_e. Put

$$X = \bigcup_{e \in H_o} \{e\} \times K/G'_e.$$

Define a relation $\leq$ on this set by

$$(e, g_1G'_e) \leq (f, g_2G'_f) \Leftrightarrow e \leq f \text{ and } g_1^{-1}g_2 \in G'_e.$$

It is easy to check that this is a partial order on X.

Let $\mathbf{e}$ be an arbitrary identity in G. Since H is totally disconnected, there is a unique identity $e \in H_o$ and some element $x \in G$ such that $xx^{-1} = \mathbf{e}$ and $x^{-1}x = e$. We map $\mathbf{e}$ to the pair $(e, \Theta(x)G'_e)$. Suppose that $y \in G$ such that $yy^{-1} = \mathbf{e}$ and $y^{-1}y = e$. Then $x^{-1}y \in G_e$ and so $\Theta(x^{-1}y) \in G'_e$. It follows that $(e, \Theta(x)G'_e) = (e, \Theta(y)G'_e)$. We have therefore defined a function from G_o to X.

Suppose that $\mathbf{e} \leq \mathbf{f}$ in G_o. Let $x \in G$ be such that $xx^{-1} = \mathbf{f}$ and $x^{-1}x = f \in H_o$. We may form the corestriction $y = \mathbf{e}|y \leq x$. Observe that $yy^{-1} = \mathbf{e}$ and $y^{-1}y = e \leq f$. By the order ideal property, we have that $e \in H_o$. We therefore have that

$$\mathbf{f} \mapsto (f, \Theta(x)G'_f) \text{ and } \mathbf{e} \mapsto (e, \Theta(y)G'_f.$$

By construction $e \leq f$. Now $y \leq x$ and so $\Theta(y)^{-1}\Theta(x) = \Theta(y)^{-1}\Theta(y) \in G'_e$. Thus our map is order-preserving and so injective.

Consider now the element (e, gG'_e) of X. There is a unique element $x \in G$ such that $x^{-1}x = x$ and $\Theta(x) = g$. Put $\mathbf{e} = xx^{-1}$. Then under our mapping define above we have that $\mathbf{e} \mapsto (e, gG'_e)$. Thus our mapping is surjective.

Finally, suppose that $(e, g_1G'_e) \leq (f, g_2G'_f)$. Then $e \leq f$ and $g_1^{-1}g_2 \in G'_e$. First, there is a unique $x \in G$ such that $x^{-1}x = f$ and $\Theta(x) = g_2$. Put $\mathbf{f} = xx^{-1}$. Put $y = x|e$. Then $y^{-1}y = e$ and $y \leq x$. Define $\mathbf{e} = yy^{-1}$. Clearly $\mathbf{e} \leq \mathbf{f}$. The element $\mathbf{e}$ is mapped to $(e, \Theta(y)G_{e'}) = (e, \Theta(x)G'_e) = (e, g_1G'_e)$, as required. It follows that the mapping we have defined is an order isomorphism.

The construction of the poset X is part of Theorem II.12.18 of [2]. We have shown that its construction is a special case of the maximum enlargement theorem. In fact, the ordered groupoid G is a semidirect product of X under the obvious group action by K. This action is again part of Theorem II.12.18.

9.3. **The inverse semigroup associated with a skeletal Rees category.** The theory on which this section is based is developed in detail in [13], so most of the proofs will simply be sketched. The goal is to show how skeletal left Rees categories can be converted first into ordered groupoids and then, via the pseudoproduct, into inverse semigroups. We begin with the general construction we shall base our work upon.

Lemma 9.3. *With each left cancellative category C, we may associate an ordered groupoid $\mathsf{G}(C)$.*

Proof. This is a standard construction described in [19]. We shall therefore sketch out the proof here. For definitions related to ordered groupoids see [16]. The elements of our ordered groupoid are equivalence classes $[a,b]$ where the class $[a,b]$ consists of all ordered pairs (a,b) where $\mathbf{r}(a) = \mathbf{r}(b)$ where $(a,b) \equiv (a',b')$ if and only if $(a,b) = (a',b')u$ where $u \in G$ the groupoid of invertible elements of C. Define the domain of $[a,b]$ to be $[a,a]$ and the range of $[a,b]$ to be $[b,b]$. Suppose that $[a,b][c,d]$ is defined. Then $b = cu$ for an isomorphism u. Define $[a,b][c,d] = [a,du]$. This turns $\mathsf{G}(C)$ into a groupoid where $[a,b]^{-1} = [b,a]$. The identities are the elements of the form $[a,a]$ and so are in bijective correspondence with the non-empty principal right ideals aC. The elements are therefore arrows

$$[a,a] \xrightarrow{[a,b]} [b,b].$$

Define $[a,b] \leq [c,d]$ if and only if $(a,b) = (c,d)p$. This turns $\mathsf{G}(C)$ into a poset. Observe that $[c,c] \leq [a,a],[b,b]$ if and only if $cC \subseteq aC \cap bC$. It follows that $[a,a] \wedge [b,b]$ exists if and only if $aC \cap bC = cC$ for some $c \in C$ in which case $[a,a] \wedge [b,b] = [c,c]$.

The groupoid $\mathsf{G}(C)$ equipped with the above order becomes an ordered groupoid as follows. If $[c,c] \leq [b,b]$, where $c = bp$, define

$$[a,b]|[c,c] = [ap,c]$$

and if $[c,c] \leq [a,a]$, where $c = bp$, define

$$[c,c]|[a,b] = [c,bp].$$

The pseudoproduct $\bullet$ of $[a,b]$ and $[c,d]$ is defined if and only if $bC \cap cC \neq \emptyset$ and $bC \cap cC = wC$ for some $w \in C$. If $w = bx = cy$ then $[a,b] \bullet [c,d] = [ax,dy]$. $\square$

In the light of the above characterization of the pseudoproduct, we make the following definition. A *Leech category* is a left cancellative category C such that if $aC \cap bC \neq \emptyset$ then $aC \cap bC = cC$ for some $c \in C$. In this case, the ordered groupoid $\mathsf{G}(C)$ has the property that if a pair of identities has a lower bound then it has meet.

An inverse semigroup with zero is said to be E^*-*unitary* if the elements above non-zero idempotents are themselves idempotents.

Lemma 9.4. *With each Leech category, we may associate an inverse semigroup with zero $\mathsf{S}(C)$.*

1. *This inverse semigroup is E^*-unitary if and only if C is cancellative.*
2. *This semigroup has the additional property that every non-zero idempotent is beneath a unique maximal idempotent.*
3. *If the Leech category is skeletal, then each $\mathscr{D}$-class contains a unique maximal idempotent.*

Proof. The set $\mathsf{S}(C)$ is the set $\mathsf{G}(C)$ with a zero adjoined. The non-zero product is the pseudoproduct.

The proof of (1) is straightforward.

(2). Consider the idempotents of the form $[e,e]$ where e is an identity of C. If $[e,e] \leq [a,a]$ then $e = ap$ for some p. But C is left cancellative and so a is invertible with inverse p. It follows that $[e,e] = [a,a]$. Thus the idempotents $[e,e]$ where $e \in C_o$ are maximal idempotents. Observe that $[a,a] \leq [\mathbf{d}(a), \mathbf{d}(a)]$. Thus every non-zero idempotent lies beneath a maximal idempotent; in fact, a unique one.

(3). It is easy to check that $[a,a] \mathscr{D} [\mathbf{r}(a), \mathbf{r}(a)]$. Thus each $\mathscr{D}$-class contains a maximal idempotent. Observe that $[e,e] \mathscr{D} [f,f]$ if and only if e and f are isomorphic. It follows that if C is skeletal then each $\mathscr{D}$-class contains a unique maximal idempotent. □

We say that an inverse semigroup with zero is a *reduced Leech semigroup* if it satisfies the following conditions:

- (RLS1): Each non-zero idempotent is beneath a unique maximal idempotent.
- (RLS2): Each $\mathscr{D}$-class contains a unique maximal idempotent.

A reduced Leech semigroup is called a *reduced Perrot semigroup* if it satisfies two further conditions:

- (PS1): If e and f are idempotents such that $ef \neq 0$ then either $e \leq f$ or $f \leq e$.
- (PS2): If e is a non-zero idempotent then the set of idempotents f such that $e \leq f$ is finite.

We have therefore proved the following.

Proposition 9.5. *Let C be a skeletal Leech category. Then $\mathsf{S}(C)$ is a reduced Leech semigroup. If, in addition, C is a left Rees category, then $\mathsf{S}(C)$ is a reduced Perrot semigroup.*

In [13], it is proved that there is a correspondence between skeletal left Rees categories and reduced Perrot semigroups so that in passing between the language of categories and the language of inverse semigroups there is no loss of information.

Remark 9.6. If we start with a left Rees monoid C then it is automatically skeletal. The associated inverse semigroup $\mathscr{S}(C)$ has one non-zero $\mathcal{D}$-class and so is 0-bisimple. The maximal idempotent is just the identity element and so $\mathsf{S}(C)$ is a 0-bisimple inverse monoid. In other words, left Rees monoids are associated with the 0-*bisimple Perrot monoids*.

Let S be an inverse semigroup with zero. We write $S^* = S \setminus \{0\}$. Let $\mathcal{G}$ be a groupoid. A function $\psi \colon S^* \to \mathcal{G}$ is called a *prehomomorphism* if $st \neq 0$ implies that $\psi(s)\psi(t)$ is defined and that $\psi(st) = \psi(s)\psi(t)$. Such a prehomomorphism is said to be *idempotent pure* if $\psi(s)$ an identity implies that s is an idempotent. An inverse semigroup is said to be *strongly E^*-unitary* if it admits an idempotent pure prehomomorphism to a groupoid.

Remark 9.7. In the case of inverse semigroups not having a zero, it is usual to refer to E-unitary rather than E^*-unitary semigroups.

Lemma 9.8. *Let C be a skeletal Leech category and let S be its associated reduced Leech semigroup. Let $\mathcal{G}$ be a groupoid.*

(1) Functors $\theta\colon C \to \mathcal{G}$ correspond to prehomomorphisms $\bar{\theta}\colon \mathsf{S}(C)^* \to \mathcal{G}$.
(2) Those functors θ, satisfying the additional condition $\theta(a) = \theta(b)$ and $\mathbf{r}(a) = \mathbf{r}(b)$ imply that $a = b$, correspond to idempotent pure prehomomorphisms $\bar{\theta}$.
(3) θ is an injective functor if and only if $\bar{\theta}$ is star injective and maximal idempotent separating.

Proof. (1). Let $\theta\colon C \to \mathcal{G}$ be a functor to a groupoid. Define $\bar{\theta}\colon \mathsf{S}(C)^* \to \mathcal{G}$ by $\bar{\theta}[a,b] = \theta(a)\theta(b)^{-1}$. This is a well-defined function. It is easy to check that if $[a,b]\bullet[c,d] \neq 0$ then $\bar{\theta}([a,b]\bullet[c,d]) = \bar{\theta}([a,b])\bar{\theta}([c,d])$. Thus $\bar{\theta}$ is a prehomomorphism to the groupoid.

Let $\psi\colon S^* \to \mathcal{G}$ be a prehomomorphism. Define $\psi'\colon \mathsf{C}(S) \to \mathcal{G}$ by $\psi'(e,a) = \psi(a)$. Then it is easy to check that ψ' is a functor.

(2). Let $\theta\colon C \to \mathcal{G}$ be such a functor. Suppose that $\bar{\theta}[a,b] = \theta(a)\theta(b)^{-1} = e$, an identity. Then $a = b$. But $\mathbf{r}(a) = \mathbf{r}(b)$ and so by assumption $a = b$ and so $[a,b]$ is an idempotent.

Let $\psi\colon S^* \to \mathcal{G}$ be idempotent pure. Let (e,a) and (f,b) be such that $a^{-1}a = b^{-1}b$ and $\psi'(e,a) = \psi'(f,b)$. Then $\psi(a) = \psi(b)$. Now $ab^{-1} \neq 0$ and so $\psi(ab^{-1}) = \psi(a)\psi(a)^{-1}$. By assumption ab^{-1} is an idempotent. Similarly $a^{-1}b$ is an idempotent. Thus a and b are compatible and $\mathscr{L}$-related and so equal.

(3). Suppose that $\psi\colon S^* \to \mathcal{G}$ is idempotent pure and maximal idempotent separating. Let $\psi'(e,a) = \psi'(f,b)$. Then $\psi(a) = \psi(b)$. Observe that $aa^{-1} \leq e$ and so $\psi(aa^{-1}) = \psi(e)$. Similarly, $\psi(bb^{-1}) = \psi(f)$. It follows that $\psi(e) = \psi(f)$ and so since ψ separates maximal identities we have that $e = f$. By assumption $a^{-1}a$ and $b^{-1}b$ are maximal idempotents. It follows once again that $a^{-1}a = b^{-1}b$. We now use the fact that ψ is idempotent pure to deduce that $a = b$. We have therefore shown that $(e,a) = (f,b)$, as required. $\square$

It is a consequence of this section, that if we start with a skeletal Rees category C embedded in its universal groupoid $\mathcal{G}$, then we may construct a strongly E^*-unitary inverse semigroup $\mathsf{S}(C)$ equipped with an idempotent pure prehomomorphism $\gamma\colon \mathsf{S}(C)^* \to \mathcal{G}$ which is maximal idempotent separating.

9.4. **Bass-Serre theory and the maximum enlargement theorem.** All the Perrot semigroups in this section will be reduced.

Let C be a Serre category. We call the associated inverse semigroup $\mathsf{S}(C)$ the *Serre semigroup*. From Section 7, the category C is embedded in its universal groupoid $\mathcal{G}$ in such a way that $C_o = \mathcal{G}_o$. We shall now see how this embedding yields a natural interpretation for the Serre tree that can be constructed from the original graph of groups. This will be obtained by using the Serre inverse semigroup.

Let C be a skeletal Rees category embedded in its universal groupoid $\mathcal{G}$. We are mainly interested in the case where C is a Serre category but we can prove our results more generally. Define

$$X = \{gC \colon g \in \mathcal{G}\} \text{ and } Y = \{aC \colon a \in C\}.$$

Clearly, $Y \subseteq X$. We regard X as a poset under subset inclusion. The set X is equipped with a map $X \to \mathcal{G}_o$ given by $gC \mapsto \mathbf{d}(g)$. This enables us to define a groupoid action $\mathcal{G} * X \to X$ given by $g \cdot hC = ghC$ if $\mathbf{r}(g) = \mathbf{d}(h)$.

Lemma 9.9.
(1) $\mathcal{G} \cdot Y = X$.
(2) Y is an order ideal of X.

Proof. (1). Let $gC \in X$. Let $e = \mathbf{r}(g)$. Then $eC \in Y$ and $g \cdot eC$ is defined and equals gC.

(2). Let $gC \subseteq aC$ where $aC \in Y$. Then $g = ad$ where $d \in C$ and so $g \in C$ giving $gC \in Y$, as required. □

We may use this data to build an inverse semigroup with zero S equipped with an idempotent pure prehomomorphism $\gamma \colon \mathsf{S}^* \to \mathcal{G}$. The non-zero elements of this semigroup will be the ordered pairs (aC, g) where $g^{-1}aC \in Y$. The product $(aC, g)(bC, h)$ is defined as follows: if $aC \cap gbC = \emptyset$ then the product is defined to be zero; if $aC \cap gbC \neq \emptyset$ then $(aC, g)(bC, h) = (aC \cap gbC, gh)$.

Proposition 9.10. *With the above definitions,* S *is an* E^**-unitary Perrot semigroup equipped with an idempotent pure prehomomorphism* γ *to the groupoid* $\mathcal{G}$*. In fact,* S *is isomorphic to the Serre inverse semigroup* $\mathsf{S}(C)$*.*

Proof. Let $(aC, g), (bC, h) \in \mathsf{S}$ and suppose that $aC \cap gbC \neq \emptyset$. By definition, $(aC, g)(bC, h) = (aC \cap gbC, gh)$. We need to prove that the righthand side is an element of S. By assumption, $g^{-1}a = c$ and $h^{-1}b = d$ for some $c, d \in C$. Thus $(aC, g) = (aC, ac^{-1})$ and $(bC, h) = (bC, bd^{-1})$. By assumption, $aC \cap gbC$ is non-empty. It follows that $ax = gby$ for some x and y in C. But $a = gc$ and so $gcx = gby$. By left cancellation, we have that $cx = by$. Hence $cC \cap bC \neq \emptyset$. By assumption, $cC \cap bC = pC$ for some p. Thus $p = cw = bz$ for some $w, z \in C$. We now claim that $aC \cap gbC = awC$. Let $g_1 = aa_1 = gbb_1$. Then $gca_1 = gbb_1$ and so $ca_1 = bb_1$ by left cancellation. It follows that $ca_1 = bb_1 = pd_1$. Thus $a_1 = wd_1$ and $b_1 = zd_1$. Hence $g_1 = awd_1 \in awC$. To prove the reverse inclusion, observe that $aw = ac^{-1}cw = gcw = gbz$. We have therefore proved that

$$(aC, g)(bC, h) = (awC, gh).$$

We now calculate $(gh)^{-1}awC$. Observe that

$$(gh)^{-1}aw = (ac^{-1}bd^{-1})^{-1}aw = db^{-1}ca^{-1}aw = db^{-1}cw = db^{-1}bz = dz$$

and the result is proved.

We now prove that $(aC, ac^{-1})(bC, bd^{-1})$ is non-zero if and only if $cC \cap bC \neq \emptyset$. Only one direction remains to be proved. We are given elements (aC, ac^{-1}) and (bC, bd^{-1}). Suppose that $cC \cap bC \neq \emptyset$. We shall prove that $aC \cap ac^{-1}bC \neq \emptyset$. by assumption, $cC \cap bC = pC$ and so $p = cw = bz$. Thus $ac^{-1}cw = ac^{-1}bz$. It follows that $aw = (ac^{-1}b)z$ and so $aC \cap ac^{-1}bC \neq \emptyset$, as required.

We shall assume for the time being that S is a semigroup.

Observe that a necessary condition for (aC, ab^{-1}) to be an idempotent is that ab^{-1} be an identity but it is easy to check that this is also sufficient. It follows that the idempotents in S are the elements of the form $(aC, \mathbf{d}(a))$. We calculate the product $(aC, \mathbf{d}(a))(bC, \mathbf{d}(b))$. By definition, this is non-zero if and only if $aC \cap bC \neq \emptyset$. In particular, we need that $\mathbf{d}(a) = \mathbf{d}(b) = e$, say. Suppose that $aC \cap bC = pC$ and that $p = aw = bz$. Then $(aC, \mathbf{d}(a))(bC, \mathbf{d}(b)) = (aC \cap bC, e) = (pC, e)$. Thus the semilattice of idempotents of S is isomorphic to the poset of principal right ideals of C together with the emptyset set. A (von Neumann) inverse of (aC, ab^{-1}) is (bC, ba^{-1}). Hence S is inverse.

Let (aC, ab^{-1}) be a non-zero element of S. We map it to ab^{-1} and denote the map by γ. It is immediate that the only elements mapping to identities are idempotents. Suppose that $(aC, g)(bC, h)$ is defined where $g = ac^{-1}$ and $h = bd^{-1}$. Then we have seen that $cC \cap bC \neq \emptyset$. Hence $\mathbf{d}(c) = \mathbf{d}(b)$. Thus the product gh is defined in the groupoid.

It remains to prove that S is a semigroup. We could prove the associativity of the multiplication directly. Instead, we shall prove it by a different route. We prove that the partial binary operation we have constructed is in fact the pseudoproduct of the ordered groupoid $\mathsf{G}(C)$. It follows that the inverse semigroup S is nothing other than the Serre inverse semigroup. Recall that the elements of our ordered groupoid are equivalence classes $[a,b]$ where the class $[a,b]$ consists of all ordered pairs (a,b) where $\mathbf{r}(a) = \mathbf{r}(b)$ where $(a,b) \equiv (a',b')$ if and only if $(a,b) = (a',b')u$ where $u \in G$ the groupoid of invertible elements of C. We next establish a bijection between the non-zero elements of S and $\mathsf{G}(C)$. Let $(aC, g) \in \mathsf{S}$. Then $g^{-1}a = b$ for some $b \in C$. Consider the element $[a,b]$. Then $ab^{-1} = gbb^{-1} = g$. We therefore map

$$(aC, g) \mapsto [a,b]$$

where $g = ab^{-1}$. Suppose that (aC, ab^{-1}) and (cC, cd^{-1}) are such that $[a,b] = [c,d]$. then $(a,b) = (c,d)u$ where u is an isomorphism. It follows that $aC = cC$ and that $ab^{-1} = cu(du)^{-1} = cd^{-1}$. Thus $(aC, ab^{-1}) = (cC, cd^{-1})$. The map is therefore injective and it is immediate that it is surjective.

It is now easy to check that the pseudoproduct $[a,b] \bullet [c,d]$ exists if and only if $aC \cap ab^{-1}cC \neq \emptyset$ and that the partial product in S maps to the pseudoproduct in G. The pseudoproduct is known to be associative whenever it makes sense. Thus S is an inverse semigroup.

The maximal idempotents are those of the form $(\mathbf{d}(a)C, \mathbf{d}(a))$. It is immediate that each idempotent lies beneath a unique maximal idempotent. Let $(aC, \mathbf{d}(a))$ be an arbitrary non-zero idempotent. Then $(aC, \mathbf{d}(a)) \mathscr{D} (\mathbf{r}(a)C, \mathbf{r}(a))$ via the element (aC, a). Observe that $(eC, e) \mathscr{D} (fC, f)$ if and only if $e = f$. There is therefore a bijection between the maximal idempotents of S and the identities in the category C which in turn correspond to the maximal principal right ideals of C.

The fact that the inverse semigroup S is E^*-unitary follows from the fact that C is right cancellative as well as cancellative. $\square$

Remark 9.11. The above result gives a structural description of the Serre inverse semigroup in terms of a groupoid acting on a partially ordered set. It is therefore a description that generalizes the familiar P-theorem in inverse semigroup theory.

The inverse semigroup S only contains an implicit use of the poset X. However, there is a procedure for revealing it that we now describe which is well-known in inverse semigroup theory. To do this, we work in the category of ordered groupoids and ordered functors.

Consider the set of all ordered pairs, $X * \mathcal{G}$, of the form (hC, g) where $h, g \in \mathcal{G}$ where $\mathbf{d}(g) = \mathbf{d}(h)$. Observe that $\mathsf{S} \subseteq X * \mathcal{G}$. Define $(gC, h)^{-1} = (h^{-1}gC, h^{-1})$, $\mathbf{d}(gC, h) = (gC, \mathbf{d}(h))$, and $\mathbf{r}(gC, h) = (h^{-1}gC, \mathbf{r}(h))$. Define a partial product

$$(g_1C, g_2)(h_1C, h_2) = (g_1C, g_2h_2)$$

when $\mathbf{r}(g_1C, g_2) = \mathbf{d}(h_1C, h_2)$ and a partial order by $(g_1C, g_2) \leq (h_1C, h_2)$ if and only if $g_2 = h_2$ and $g_1C \subseteq h_1C$. Then $X * \mathcal{G}$ is an ordered groupoid. Define $\Gamma \colon X * \mathcal{G} \to \mathcal{G}$ by $\Gamma(gC, h) = h$. Then Γ is an ordered covering functor. We have the following commutative diagram

$$\begin{tikzcd} X * \mathcal{G} \arrow[dr, "\Gamma"] & \\ \mathsf{S} \arrow[u, "\iota"] \arrow[r, "\gamma"'] & \mathcal{G} \end{tikzcd}$$

where $\iota\colon \mathsf{S} \to X * \mathcal{G}$ is an embedding. By Lemma 9.9, it follows that $X * \mathcal{G}$ is an enlargement of S. We therefore may make the following deduction from [16].

Theorem 9.12. *The covering functor Γ arises from the idempotent pure functor γ by an application of the maximum enlargement theorem.*

If we restrict our attention to the case where C is a Serre category then we may say more about the structure of the poset X. Define $\pi\colon X \to \mathcal{G}_o$ by $\pi(gC) = \mathbf{d}(g)$. Observe that if $gC \cap hC \neq \emptyset$ then $\mathbf{d}(g) = \mathbf{d}(h)$. We write $X_e = \pi^{-1}(e)$. Clearly $X = \bigsqcup_{e \in \mathcal{G}_o} X_e$. Observe that if e and f are identities and $k\colon f \to e$ is invertible, then there is an order isomorphism between X_e and X_f given by $gC \mapsto kgC$. Our reference for the construction of Serre trees is [41]. We choose a co-ordinatization for C as in Section 7 and so a basis for C consisting of the atoms that arise. If $f \overset{a=kx}{\to} i$ is such an atom and $e \overset{g}{\to} f$ then we draw an edge labelled by x from gC to gaC.

Lemma 9.13. *For each $e \in \mathcal{G}_o$, the set X_e with the edges as defined is an orientation of the Serre tree rooted at e.*

Proof. Suppose that $g, h \in \mathcal{G}$ are such that $gC = hC$. We need to show that $g = hg'$ where $g' \in C$ and a is invertible. By assumption, $g = ha$ and $h = gb$ where $a, b \in C$. But $a = h^{-1}g$. It follows that a is an invertible element of C. It is particualrly easy in this formulation to show that the graph we have constructed is a tree. $\square$

Remark 9.14. We therefore see that the Serre tree arises as part of the structure obtained by applying the maximum enlargement theorem to the Serre inverse semigroup of the graph of groups and its associated idempotent pure prehomomorphism. The connection we have made with the maximum enlargement theorem is intriguing. It suggests that the perspectives of geometric group theory might prove useful in inverse semigroup theory particularly in connection with the theory of E^*-unitary and strongly E^*-unitary inverse semigroups.

The classical theory of E-unitary inverse semigroups [29] can be viewed from the perspective of group actions. The starting point is a group G acting by order automorphisms on a poset X. We are given a subset $Y \subseteq X$ such that $GY = X$. Thus Y is a *fundamental domain*, not necessarily *strict*. The set Y is assumed to be an order ideal and a meet semilattice under the induced order. Finally, for each $g \in G$ we have that $gY \cap Y \neq \emptyset$. Under these conditions (G, X, Y) is called a *McAlister triple*. From this data, one can construct an inverse semigroup $P(G, X, Y)$. We can view this as *local data* about the action. The original group action (G, X) can be constructed via the group S/σ, where σ is the minumum group congruence on S; this can be seen as a generalization of a colimit. The poset X can be recaptured using the maximum enlargement theorem applied to the natural map $S \to S/\sigma$ and working in the category of ordered groupoids. In the monoid case, McAlister's original proof of these results is described using the theory of cancellative categories and groupoids of fractions [11].

When the inverse semigroup has a zero, the situation is much more complicated and interesting. All strongly E^*-unitary inverse semigroups are E^*-unitary but the converse is not true in general. In categorical terms, being E^*-unitary is equivalent to a category being cancellative, whereas being strongly E^*-unitary is equivalent to that category being embeddable in a groupoid. In the case of inverse semigroups without zero, being E-unitary and being strongly E-unitary are equivalent. The distinction between E^*-unitary and strongly E^*-unitary is a theme in [2], though expressed in different language, and similarly in [5].

9.5. **From inverse semigroups to graphs of groups.** In this section, we shall show how to pass from a reduced Perrot semigroup to a diagram of partial homomorphisms.

We shall start by working in an arbitrary inverse semigroup S. Let e and f be non-zero idempotents. Recall that in an inverse semigroup, we have that $f \leq_J e$ if $f \mathscr{D} e' \leq e$ for some idempotent e'. Let $a \in S$ such that $a^{-1}a = f$ and $aa^{-1} \leq e$. We shall define a partial homomorphism ϕ_a from H_e to H_f. Define

$$(H_e)^+_a = \{g \in H_e \colon gaa^{-1} = aa^{-1}g\}$$

and define

$$\phi_a \colon (H_e)^+_a \to H_f \text{ by } \phi_a(g) = a^{-1}ga.$$

Lemma 9.15. *With the above definitions, we have that $(H_e)^+_a$ is a subgroup and ϕ_a is a homomorphism. If S is E^*-unitary then ϕ_a is injective.*

Proof. We prove first that $(H_e)^+_a \leq H_e$. Since $aa^{-1} \leq e$ we have that $e \in (H_e)^+_a$. Suppose that $g, h \in (H_e)^+_a$. Then $ghaa^{-1} = gaa^{-1}h = aa^{-1}gh$. Finally, suppose that $g \in (H_e)^+_a$. Then it is immediate that $g^{-1} \in (H_e)^+_a$.

Let $g \in (H_e)^+_a$. We prove that $\phi_a(g) \in H_f$. We have that

$$(a^{-1}ga)^{-1}a^{-1}ga = f \text{ and } a^{-1}ga(a^{-1}ga)^{-1} = f.$$

Let $g, h \in (H_e)^+_a$. Then

$$\phi_a(gh) = a^{-1}gha = a^{-1}(aa^{-1})gha = a^{-1}g(aa^{-1})ha = \phi_a(g)\phi_b(h),$$

as required.

Suppose that S is now assumed E^*-unitary and that $\phi_a(g) = f$. Then $a^{-1}ga = f$. Thus $afa^{-1} \leq g$. But afa^{-1} is a non-zero idempotent if f is and so g is an idempotent, by assumption. Thus ϕ_a is injective. $\square$

With the above notation we define $(H_f)^-_a$ to be the image of ϕ_a.

In the case of Perrot semigroups, we would choose e and f to be maximal idempotents and the element a would be chosen as an atom such that $a^{-1}a = f$ and $aa^{-1} \leq e$.

We now link these results to the Perrot semigroup $\mathsf{S}(C)$ where C is a skeletal left Rees category. We may identify atoms as those elements (e, x) where $e \xrightarrow{x} f$ and x is an atom in C. We have that

$$H_{[e,e]} = \{[g, e] \colon g \in G_e\}.$$

The element $[x, f]$ is such that $[x, f][x, f]^{-1} = [x, x] \leq [e, e]$ and $[x, f]^{-1}[x, f] = [f, f]$.

Let $g \in (G_e)^+_x$. Then $gx = xh$ for some h. We have that $[g,e][x,x] = [gx,x] = [xh,x]$. Thus

$$[g,e][x,x][g,e]^{-1} = [xh,xh] = [x,x].$$

On the other hand, if $g \in G_e$ is such that

$$[g,e][x,x][g,e]^{-1} = [x,x]$$

then $gx = xh$ for some group element h.

We have therefore constructed the diagram of partial homomorphisms directly from the Perrot semigroup. In the case where the Perrot semigroup is E^*-unitary, we get a diagram of partial isomorphisms. This diagram determines the Perrot semigroup upto isomorphism.

9.6. **Concluding remarks.** Andrew Duncan (Newcastle, UK) has pointed out that there are some interesting parallels between our theory and Stallings theory of pregroups as described in [9]. We do not know, at this point, whether we can derive that theory from ours.

All cancellative equidivisible categories may be embedded in groupoids; see [38] although a direct proof using the ideas of [12] would be useful. Motivated by group theory, one might consider the structure of such categories equipped with other kinds of length functors.

Finally, it may be of interest to investigate the structure of the tight completions of Serre inverse semigroups via the theory developed in [22].

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