SEMIGROUP NEAR-RINGS

by

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A near-ring is a generalization of a ring where two axioms are omitted: addition is not necessarily abelian and only one distributive law holds.

Definition 1.1. Let $(R, +, .)$ be a set with two binary operations + and . satisfying

(i) $(R, +)$ is a group, not necessarily abelian;

(ii) $(R, .)$ is a semigroup;

(iii) $(x + y)z = xz + yz$ for all $x, y, z \in R$.

A similar definition with (iii) replaced by the left distributive law gives rise to a left near-ring. We will be using only right near-rings. The standard example of a near-ring is given by the following.

Example 1.2. Let $(G, +)$ be a group. Define $M(G)$ to be the set of all functions from G to G with pointwise addition ($(\alpha + \beta)(g) = \alpha(g) + \beta(g)$ for all $\alpha, \beta \in M(G), g \in G$) and composition of maps. Then $(M(G), +, .)$ is a right near-ring and all near-rings can be embedded in a suitable $M(G)$.

For more information on near-rings the following books are useful: Pilz [**6**], Meldrum [**4**], Clay [**1**] and Ferrero-Cotti [**2**]. In this account we wish to define and begin an investigation of semigroup near-rings. Group near-rings have been investigated starting with Le Riche, Meldrum and van der Walt [**3**] and several subsequent papers. By replacing the group involved by a semigroup we get a semigroup near-ring. Many of the early results about semigroup near-rings are proved in a very similar way to the group near-rings case and for those we will not give proofs in this account. First we set the scene with a series of definitions.

Definition 1.3. Let $(R, +, .)$ be a near-ring. A subset A of R such that A is closed under addition and composition is called a subnear-ring.
A subnear-ring B such that $(B, +)$ is normal in $(R, +)$ and

(i) $BR \subseteq B$ is a right ideal;

(ii) $r(x + b) - rx \in B$ for all $b \in B$, $r, x \in R$ is a left ideal;

(iii) if (i) and (ii) are satisfied then B is an ideal.

Definition 1.4. Let $(R, +, .)$ be a near-ring. If $(H, +)$ is a group and R can be embedded in $M(H)$ then H is an R-module where the action of R on H is defined using the embedding.

If $K \subseteq H$ is a subgroup and $RK \subseteq K$ then K is a (R)-submodule. If $r(k + h) - rh \in K$ for all $k \in K$, $h \in H$, $r \in R$ and K is normal in H then K is an R-ideal of H.

The various definitions of ideals and modules parallel the use of these ideas in ring theory. In particular the ideas of R-homomorphisms and R-module homomorphisms parallel the ring case very closely.

We now turn to defining group near-rings and semigroup near-rings. The definitions may seem strange compared with the ring case but they have proved to be necessary in order for progress to be made.

Let R be a right near-ring with identity 1 and which is zero-symmetric (i. e. $r0 = 0 = 0r$ for all $r \in R$). Let G be a multiplicative group with identity e. Denote by R^G the cartesian direct sum of $|G|$ copies of $(R, +)$ indexed by the elements of G. Define a set of functions in $M(R^G)$ by

$$[r, g]\mu(h) = r\mu(hg) \text{ for all } \mu \in R^G,\ h, g \in G, r \in R \tag{1.5}$$

Definition 1.6. The subnear-ring of $M(R^G)$, denoted by $R[G]$ is the subnear-ring generated by

$$\{[r, g]; r \in R, g \in G\}.$$

Definition 1.7. Let S be a multiplicative semigroup with identity e. Define $[r, s]$ as a function on R^S by (1.5) with suitable change in notation. Then the subnear-ring of $M(R^S)$ generated by

$$\{[r, s]; r \in R, s \in S\}.$$

is denoted by $R[S]$.

Definition 1.8. With the notation of the two previous definitions we call $R[G]$ the group near-ring of R over G and $R[S]$ the semigroup near-ring of R over S.

We will now list a number of elementary results about semigroup near-rings which parallel very closely the corresponding results about group near-rings. They will be given without proof but with a reference to the corresponding result in Le Riche, Meldrum and van der Walt [**5**].

Theorem 1.9. *If R happens to be a ring, then $R[S]$ is isomorphic to the standard semigroup ring constructed from R and S.* (Theorem 2.4.)

We denote the restricted direct power of $(R, +)$ by $R^{(S)}$.

Theorem 1.10. *$R^{(S)}$ is a faithful $R[S]$-module.* (Theorem 2.5.)

A key lemma is important in other contexts.

Lemma 1.11. *Let $t \in S$, $\mu \in R^S$, $A \in R[S]$. Then there exists a finite set X, independent of μ, such that for all $X' \supseteq X$ we have $(A\mu)(t) = (A\mu|_{X'})(t)$, where $\mu|_{X'}$ is defined by $\mu|_{X'}(t) = \mu(t)$ if $t \in X'$, $\mu|_{X'}(t) = 0$ if $t \in S \setminus X'$.* (Lemma 2.6.)

Note that $\mu|_{X'} \in R^{(S)}$.

Lemma 1.12. *In $R[S]$ the following rules apply for all values of the parameters.*

(1) $[r_1, s_1][r_2, s_2] = [r_1 r_2, s_1 s_2]$;
(2) $[r_1, s] + [r_2, s] = [r_1 + r_2, s]$;
(3) $[r, s]$ is zerosymmtric if and only if r is;
(4) $[r, s]$ is distributive if and only if r is. (Lemma 2.7.)

An element $r \in R$ is distributive if and only if

$$r(a + b) = ra + rb \text{ for all } a, b \in R.$$

Corollary 1.13. *The map $r \mapsto [r, e]$ is an embedding of R into $R[S]$ and the map $s \mapsto [1, s]$ is an embedding of S into the semigroup $(R[S], .)$.* (Corollary 2.8.)

A near-ring R is dg (distributively generated) by X is $(R, +)$ is generated by X and every element of X is distributive. (Definition 3.1.)

We now turn to homomorphisms of the near-ring and the semigroup. We start with homomorphisms of the near-ring as it is much less complicated than the case of homomorphisms of the semigroup.

Let $\phi : R \to N$ be an epimorphism of near-rings. We wish to show that this leads to an epimorphism $\phi^* : R[S] \to N[S]$. Define $\overline{\phi}$ as an epimorphism from $R^S \to N^S$ given by

$\overline{\phi}(\mu)(s) = \phi(\mu(s))$ for all $s \in S$.

Then we have the following result.

Corollary 1.14. *If $\overline{\phi}(\mu) = \overline{\phi}(\nu)$ for $\mu, \nu \in R^S$, then $\overline{\phi}(A\mu) = \overline{\phi}(A\nu)$ for all $A \in R[S]$.* (Lemma 4.1)

Let $\tilde{\phi}$ be any right inverse of $\overline{\phi}$, so $\tilde{\phi} : N^S \to R^S$ and $\overline{\phi}\tilde{\phi}(\sigma) = \sigma$ for all $\sigma \in N^S$. We define $\phi^* : R[S] \to N[S]$ by

$$\phi^*(A)(\sigma) = \overline{\phi}(A\tilde{\phi}(\sigma)) \text{ for all } \sigma \in N^S. \tag{1.15}$$

Note that ϕ^* does not depend on the particular right inverse $\tilde{\phi}$ of $\overline{\phi}$ chosen.

Theorem 1.16. *If $\phi : R \to N$ is an epimorphism, then $\phi^* : R[S] \to N[S]$ is an epimorphism.* (Theorem 4.2).

Let $\mathcal{A} = \text{Ker}\phi$ then

$\text{Ker}\phi^* = \mathcal{A}^* := (\mathcal{A}^S : R^S) = \{A \in R[S]; A\mu \in \mathcal{A}^S \text{ for all } \mu \in R^S\}$

If $\mathcal{A}$ is a left ideal of R then $\mathcal{A}^S$, the cartesian direct sum of $|S|$ copies of $(\mathcal{A}, +)$ indexed by the elements of S is an $R[S]$-ideal of the $R[S]$ module R^S, so $\mathcal{A}^*$ is an ideal of $R[S]$.

Corollary 1.17. *For any ideal $\mathcal{A}$ of R we have*

$(R/\mathcal{A})[S] \cong R[S]/\mathcal{A}^*$ (Corollary 4.3)

Theorem 1.18. *The mapping $(\ )^*$ is an injection from the set of ideals of R into that of $R[S]$. Moreover*

$(\mathcal{A} \cap \mathcal{B})^* = \mathcal{A}^* \cap \mathcal{B}^*$ *and* $\mathcal{A}^* + \mathcal{B}^* \subseteq (\mathcal{A} + \mathcal{B})^*$. (Theorem 4.4.)

There is another way of associating an ideal in $R[S]$ with an ideal $\mathcal{A}$ in R: $\mathcal{A}^+ = \mathrm{Id}\langle\{[a, e]; a \in \mathcal{A}\rangle$.

Theorem 1.19. *The mapping $(\ )^+$ is an injection from the set of ideals of R into that of $R[S]$. Moreover*

$$\mathcal{A}^+ \subseteq \mathcal{A}^*, (\mathcal{A}+\mathcal{B})^+ = \mathcal{A}^+ + \mathcal{B}^+, (\mathcal{A}\cap\mathcal{B})^+ \subseteq \mathcal{A}^+ \cap \mathcal{B}^+. \text{ (Theorem 4.5.)}$$

The next step is to associate an ideal of R with an ideal in $R[S]$. This part differs from the treatment given in Le Riche, Meldrum and van der Walt [**3**].

Let $\mathcal{A}$ be an ideal in $R[S]$. Define $\mathcal{A}_*$ as an ideal in R by

$$\mathcal{A}_* = \mathrm{Id}\langle\{a \in R; \exists t \in S, \mu \in R^S, A \in \mathcal{A} \text{ such that } (A\mu)(t) = a\}\rangle. \tag{1.20}$$

$\mathcal{A}_*$ is certainly an ideal of R.

Lemma 1.21. *For any ideal $\mathcal{A}$ of $R[S]$, $\mathcal{A}_*$ is an ideal of R.*

Theorem 1.22. *For any ideal $\mathcal{A}$ of $R[S]$, $\mathcal{A} \subseteq (\mathcal{A}_*)^*$.*

Proof. Let $A \in \mathcal{A}$, $\mu \in R[S]$. Then $(A\mu)(t) \in \mathcal{A}_*$ by (1.20). This holds for all $t \in S$. So $(A\mu) \in \mathcal{A}_*^S$ and A maps an arbitrary element of R^S into $\mathcal{A}_*^S$, i. e. $\mathcal{A} \subseteq (\mathcal{A}_*^S : R^S) = (\mathcal{A}_*)^*$.

This proof is quicker than the proof of the corresponding result in Le Riche, Meldrum and van der Walt [**3**], but has a much less satisfactory definition of $\mathcal{A}_*$.

The next situation to consider is that of epimorphisms of the semigroup. This is an area where the results and proofs will differ somewhat from the group near-ring case because the kernel of a semigroup homomorphism is a congruence and not a particular substructure.

So let θ be an epimorphism from S to a semigroup T. Let ρ be the kernel, ρ being a congruence on S. We want to relate $R^T = R^{S/\rho}$ to a semigroup near-ring and find its relation to R^S. It is possible to get a connection using only a right congruence. This parallels the group near-ring case where R^H is considered for a subgroup H of G, not necessarily a normal subgroup.

Let ρ be a right congruence. Let $\mu \in R^S$. Call μ a ρ-function if $(x, y) \in \rho$ implies $\mu(x) = \mu(y)$ for all $x, y \in S$ and denote by R^S_ρ the set of all ρ-functions.

Definition 1.23. The right congruence ρ on S gives rise to ρ-functions and the set of ρ-functions is denoted by R^S_ρ.

The ρ-functions are just functions in R^S which are constant on ρ-classes. To provide the background for the proof of the next result we need to define the complexity of an element of R^S, the basis of a large number of induction arguments.

Definition 1.24. Let $A \in R[S]$. A generating sequence for A is a finite sequence $A_1, A_2, \cdots, A_n$ of elements of $R[S]$ such that $A_n = A$ and for all $i, 1 \leq i \leq n$ one of the following three cases applies:

(i) $A_i = [r, s]$ for some $r \in R, s \in S$;

(ii) $A_i = A_k + A_\ell$ for some $k, \ell, 1 \leq k, \ell < i$;

(iii) $A_i = A_k A_\ell$ for some $k, \ell, 1 \leq k, \ell < i$.

The length of a generating sequence of minimal length for A will be called the complexity of A and be denoted by $c(A)$.

It is obvious that $c(A) \geq 1$ for all $A \in R[S]$ and $c(A) = 1$ if and only if $A = [r, s]$ for some $r \in R$, $s \in S$. If $c(A) \geq 1$ then either $A = B + C$ or $A = BC$ with $c(B), c(C) < c(A)$.

Lemma 1.25. *Using the notation in Definition 1.23, R^S_ρ is an $R[S]$ module, a submodule of the left $R[S]$ module R^S.*

Proof. The elements of R^S_ρ are simply the maps $S \to R$ which are constant on the equivalence classes of ρ. So it is obvious that R^S_ρ is a subgroup of $(R^S, +)$ under addition. We have to show that if $\mu \in R^S_\rho$ and $A \in R[S]$ then $A\mu \in R^S_\rho$ and we can do this by induction on $c(A)$. Let $(x, y) \in \rho$. If $c(A) = 1$, then $A = [r, s]$ for some $r \in R, s \in S$ and so $([r, s]\mu)(x) = r\mu(xs)$ and $([r, s]\mu)(y) = r\mu(ys)$. Since ρ is a right congruence $(x, y) \in \rho$ implies $(xs, ys) \in \rho$ and so $\mu(xs) = \mu(ys)$ and $[r, s]\mu \in R^S_\rho$. Now suppose that the result holds for all $D \in R[S]$ with $c(D) < c(A)$. Then $A = B+C$ or $A = BC$ with $c(B), c(C) < c(A)$. In the former case we have

$$\begin{aligned}(A\mu)(x) &= ((B+C)\mu)(x) = (B\mu + C\mu)(x) = (B\mu)(x) + (C\mu)(x) \\ &= (B\mu)(y) + (C\mu)(y) = ((B+C)(\mu))(y) = (A\mu)(y).\end{aligned}$$

using the induction hypothesis.

In the second case $A\mu = (BC)\mu = B(C\mu)$ and by the induction hypothesis $C\mu \in R^S_\rho$ so $A\mu \in R^S_\rho$ by a second application of the induction hypothesis. The induction is now complete.

So for each right congruence ρ, R^S_ρ is an $R[S]$ submodule and $R[S]$ acts on R^S_ρ but in general not faithfully. So we have the following definition.

Definition 1.26. With the notation given above the kernel of the action of $R[S]$ on R^S_ρ is denoted by $\omega(\rho)$.

Note that $\omega(\rho)$ is a two sided ideal being the kernel of the near-ring homomorphism

$$f : R[S] \to M(R^S_\rho).$$

Lemma 1.27. *If ρ is a congruence on S then there is a natural isomorphism between R^S_ρ and $R^{S/\rho}$.*

Theorem 1.28. *Let ρ be a congruence on S. Then*

$$R[S]/\omega(\rho) \cong R[S/\rho].$$

The kernel of the full congruence on S, generally denoted by Δ, is called the augmentation ideal and plays an important role in group rings and near-rings. So we will write Δ for $\omega(S)$. Note that R^S_S just has one factor R and the natural map $R[S] \to M(R^S_S)$ maps $R[S]$ onto R. This is also brought out in the following argument.

Lemma 1.29. *Let $\mu, \nu \in R[S]$ and let $x \in R$ be such that $\nu(s) = \mu(s)x$ for all $s \in S$. Then*

$$(A\nu)(s) = ((A\mu)(s))x \text{ for all } s \in S.$$

Proof A straightforward induction on $c(A)$ since the near-ring is a right near-ring.

Define $f_\omega : R[S] \to R$ as follows: for any $d \in R$ we let ε_d denote the element of R_S^S such that $\varepsilon_d(s) = d$ for all $s \in S$. Then by lemmas 1.29 and 1.25 we see that for any $A \in R[S]$ there exists a unique (because $1 \in R$) $a \in R$ such that $A\varepsilon_d = \varepsilon_{ad}$ for all $d \in R$. Define $f_\omega(A) = a$. Then f_ω is an epimorphism and $\mathrm{Ker} f_\omega = \Delta$

Corollary 1.30. *The augmentation map $f_\omega : R[S] \to R$ is an epimorphism with kernel Δ, so $R[S]/\Delta \cong R$.*

There is a special set of generators for Δ mirroring the case of group rings and group near-rings.

Theorem 1.31. *The augmentation ideal Δ is generated as an ideal by the set $\{[1,s]-[1,e]; s \in S\}$.*

Proof. Let $\mathcal{J} = \mathrm{Id}\langle\{[1,s]-[1,e]; s \in S\}\rangle$. Then $\mathcal{J} \subseteq \Delta$ because $([1,s]-[1,e]\varepsilon_d = 0$ for all $s \in S, d \in R$. We need to prove the reverse inclusion. For any $r \in R$, $s \in S$ there are $I, I' \in \mathcal{J}$ such that

$$[r,s] = [r,e][1,s] = [r,e]([1,e]+I) = [r,e][1,e] + I' = [r,e] + I'.$$

It is now a straightforward induction that if $A \in R[S]$, then $A = [a,e] + J$ for some $a \in R$, $J \in \mathcal{J}$. So let $A \in \Delta$ and $A = [a,e] + J$, $J \in \mathcal{J}$. Then $A\varepsilon_1 = \varepsilon_a + 0$, i. e. $a = 0$ which means that $A \in \mathcal{J}$.

We now consider more properties of the ideals of the form $\omega(\rho)$.

Theorem 1.32. *The mapping ω from the lattice of right congruences of S into the lattice of ideals of $R[S]$ is an injection, isotone and $\omega(\rho \cap \sigma) \subseteq \omega(\rho) \cap \omega(\sigma)$.*

Proof. Let $\rho \neq \sigma$, say $(t,u) \in \rho \setminus \sigma$. Let $\mu \in R_\rho^S$. Then $\mu(t) = \mu(u)$ and $([1,t]-[1,u])\mu(e) = \mu(t) - \mu(u) = 0$. But $(t,u) \notin \sigma$, so $\mu(t)$ and $\mu(u)$ can take different values. Hence $\mu \in R_\rho^S \setminus R_\sigma^S$ and ω is injective. If $\rho \subseteq \sigma$ then $R_\sigma^S \subseteq R_\rho^S$ so $\omega(\rho) = \mathrm{Ann}_{R[S]} R_\rho^S \subseteq \mathrm{Ann}_{R[S]} R_\sigma^S = \omega(\sigma)$ and ω is isotone and the last part follows immediately.

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