Some properties of left (right) negatively orderable semigroups

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Negatively orderable semigroups

Definition

A semigroup is called *negatively orderable* (or *positively orderable*) if it admits a partial order which is:

- *negative*, i.e. $st \leq s$ and $st \leq t$ for every s, t in S;
- *operation-compatible*, i.e. for any r, s, t in S, $s \leq t$ implies $rs \leq rt$ and $sr \leq tr$.

Fact

The class of negatively orderable semigroups is a quasivariety; it is closed with respect to taking

- *subsemigroups and*
- *direct products*

The Straubing-Thérien Theorem

Theorem

Every finite $\mathcal{J}$-trivial semigroup is a homomorphic image of a finite negatively orderable semigroup. (Equivalently, the pseudovariety generated by all finite negatively orderable semigroups equals the class of all finite $\mathcal{J}$-trivial semigroups.)

Definition of left (right) negatively orderable semigroups

Definition

Let us call a semigroup *left (right) negatively orderable* if it admits a partial order which is:

- *left negative,* i.e. $st \leq t$ ($st \leq s$) for every s, t in S;
- *operation-compatible*, i.e. for any r, s, t in S, $s \leq t$ implies $rs \leq rt$ and $sr \leq tr$.

Quasiorders $\leq_{\mathcal{K}}$, $\leq_{\mathcal{M}}$, $\leq_{\mathcal{S}}$

Definition

Denote by $\leq_{\mathcal{K}}, \leq_{\mathcal{M}}$ and $\leq_{\mathcal{S}}$ the smallest operation-compatible quasiorder containing $\leq_{\mathcal{J}}, \leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$, respectively.

Definition

On any semigroup S, define the relations $\leq'_{\mathcal{K}}$, $\leq'_{\mathcal{M}}$ and $\leq'_{\mathcal{S}}$ as follows: for any s, t in S let

- $s \leq'_{\mathcal{K}} t$ iff $s = t_1 s_1 t_2$ and $t = t_1 t_2$ for some $s_1, t_1, t_2 \in S^1$;
- $s \leq'_{\mathcal{M}} t$ iff $s = t_1 s_1 t_2$ and $t = t_1 t_2$ for some $s_1, t_1 \in S^1$, $t_2 \in S$;
- $s \leq'_{\mathcal{S}} t$ iff $s = t_1 s_1 t_2$ and $t = t_1 t_2$ for some $s_1, t_2 \in S^1$, $t_1 \in S$.

Fact

The quasiorder $\leq_{\mathcal{K}}$ ($\leq_{\mathcal{M}}$, $\leq_{\mathcal{S}}$) is the transitive closure of the relation $s \leq'_{\mathcal{K}} t$ ($s \leq'_{\mathcal{M}} t$, $s \leq'_{\mathcal{S}} t$).

Congruences $\mathcal{K}$, $\mathcal{M}$, $\mathcal{S}$

Definition

On any semigroup S define the equivalences $\mathcal{K}$, $\mathcal{M}$ and $\mathcal{S}$ in the following way: for any $s, t \in S$ let

- $s\mathcal{K}t$ iff $s \leq_{\mathcal{K}} t$ and $t \leq_{\mathcal{K}} s$.
- $s\mathcal{M}t$ iff $s \leq_{\mathcal{M}} t$ and $t \leq_{\mathcal{M}} s$.
- $s\mathcal{S}t$ iff $s \leq_{\mathcal{S}} t$ and $t \leq_{\mathcal{S}} s$.

Fact

On any semigroup, $\mathcal{K}$ ($\mathcal{M}$, $\mathcal{S}$) is the smallest $\mathcal{K}$-trivial ($\mathcal{M}$-trivial, $\mathcal{S}$-trivial) congruence.

Definition

Let us call a semigroup S *$\mathcal{K}$-trivial* (*$\mathcal{M}$-trivial*, *$\mathcal{S}$-trivial*) if $\mathcal{K}$ ($\mathcal{M}$, $\mathcal{S}$) is the identity relation on S.

Corollary

A semigroup is

- *negatively orderable if and only if it is $\mathcal{K}$-trivial;*
- *left negatively orderable if and only if it is $\mathcal{M}$-trivial;*
- *right negatively orderable if and only if it is $\mathcal{S}$-trivial.*

Corollary

In a monoid $\leq_{\mathcal{K}}=\leq_{\mathcal{M}}=\leq_{\mathcal{S}}$, hence a monoid is negatively orderable if and only if it is left (right) negatively orderable.

Proposition

The class of all left (right) negatively orderable semigroups is a quasivariety.

A description of right negatively orderable semigroups as semigroups of transformations

Right negatively orderable semigroups as semigroups of transformations

Theorem

A semigroup is negatively orderable if and only if it is isomorphic to a semigroup T of mappings on a partially ordered set $(P, \leq)$, such that for every $\alpha \in T$:

- *α is decreasing on P, that is $p\alpha \leq p$ for every $p \in P$;*
- *α is order-preserving on P, that is $p_1 \leq p_2$ implies $p_1\alpha \leq p_2\alpha$, for every $p_1, p_2 \in P$.*

Right negatively orderable semigroups as semigroups of transformations

Theorem

A semigroup is right negatively orderable if and only if it is isomorphic to a semigroup T of mappings on a partially ordered set $(P, \leq)$ with a distinguished element $p_0 \in P$, such that for every $\alpha \in T$:

1. $Im(\alpha) \subseteq P \setminus \{p_0\}$,
2. *α is decreasing on P, that is, $p\alpha \leq p$ for every p in P,*
3. *the restriction $\alpha|_{P\setminus\{p_0\}}$ of α to $P \setminus \{p_0\}$ is order-preserving, that is, for every p_1, p_2 in $P \setminus \{p_0\}$, $p_1 \leq p_2$ implies $p_1\alpha \leq p_2\alpha$.*

Outline of proof

$\Rightarrow$ Suppose S is right negatively orderable. If S a mondoid then $\leq_{\mathcal{K}} = \leq_{\mathcal{S}}$, hence S is also negatively orderable, so the statement holds. Otherwise

- let $P \setminus \{p_0\} = S$ and $p_0 = 1 \in S^1$
- let
  - $\leq|_{P \setminus \{p_0\}} = \leq|_S = \leq_{\mathcal{S}}$;
  - for every $p \in P$ let $p \leq p_0$ and for every $p \in P \setminus \{p_0\}$ let $p_0 \not\leq p$;
- let T be the semigroup of extended right actions of S.

$\Leftarrow$ Let T be a semigroup of mappings on a partially ordered set $(P, \leq)$, satisfying the conditions of the Theorem. Define the relation $\preceq$ on T as follows:

- for any α_1, α_2 in T let $\alpha_1 \preceq \alpha_2$ iff $p\alpha_1 \leq p\alpha_2$ for every p in P
- $\preceq$ is a partial order on T;
- $\preceq$ is right negative and operation-compatible.

Some question about left (right) negatively orderable semigroups and $\mathcal{K}$, $\mathcal{M}$ and $\mathcal{S}$ - an overview

Questions

1. Is $\mathcal{M} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{M}$ in every semigroup?
2. Is $\mathcal{M} \vee \mathcal{S} = \mathcal{K}$ in every semigroup?
3. Is it true that if a semigroup is both left and right negatively orderable then it is also negatively orderable?
4. What is the pseudovariety generated by all finite left (right) negatively orderable semigroups?

Note: (2) $\Rightarrow$ (3)

1. Is $\mathcal{M} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{M}$ in every semigroup?
   - True in every semigroup in which every element has both a left and a right identity.
   - In general? No counter-example has been found so far.
2. Is $\mathcal{M} \vee \mathcal{S} = \mathcal{K}$ in every semigroup?
   - True in every semigroup in which every element has both a left and a right identity.
   - In general? No counter-example has been found so far.
3. Is it true that if a semigroup is both left and right negatively orderable then it is also negatively orderable?
   - True.
4. What is the pseudovariety generated by all finite left (right) negatively orderable semigroups?
   - A one-sided analogue of the Straubing-Thérien Theorem? False.

The congruences $\mathcal{K}$, $\mathcal{M}$ and $\mathcal{S}$ in some semigroups

	$\mathcal{K}$	$\mathcal{M}$	$\mathcal{S}$
T_n	$T_n \setminus S_n$, S_n	$T_n \setminus S_n$, S_n	$T_n \setminus S_n$, S_n
$T_n \setminus S_n$, $n \geq 3$	$\mathcal{K}$-simple	$\mathcal{M}$-simple	$\mathcal{S}$-simple
$T_2 \setminus S_2$	$\mathcal{K}$-simple	$\mathcal{M}$-trivial	$\mathcal{S}$-simple
T_X, NST_X	$\mathcal{K}$-simple	$\mathcal{M}$-simple	$\mathcal{S}$-simple
O_n	$O_n \setminus \{\iota\}$, $\{\iota\}$	$O_n \setminus \{\iota\}$, $\{\iota\}$	$O_n \setminus \{\iota\}$, $\{\iota\}$
$O_n \setminus \{\iota\}$, $n \geq 3$	$\mathcal{K}$-simple	$\mathcal{M}$-simple	$\mathcal{S}$-simple
$O_2 \setminus \{\iota\}$	$\mathcal{K}$-simple	$\mathcal{M}$-trivial	$\mathcal{S}$-simple
OE_n	$\mathcal{K}$-trivial	$\mathcal{M}$-trivial	S-trivial
$OE_n \setminus \{\iota\}$, $n \geq 2$	$\mathcal{K}$-trivial	$\mathcal{M}$-trivial	$\mathcal{S}$-trivial
Free semigroups	$\mathcal{K}$-trivial	$\mathcal{M}$-trivial	S-trivial
$O_{\mathbb{N}}$	$O_C^{[]}$, $O_C^{[)}$	$O_C^{[]}$, $O_C^{[)}$	$O_C^{[]}$, $O_C^{[)}$
$O_{\mathbb{Z}}, O_{\mathbb{Q}}, O_{\mathbb{R}}$	$O_C^{[]}$, $O_C^{[)}$, $O_C^{(]}$, $O_C^{()}$	$O_C^{[]}$, $O_C^{[)}$, $O_C^{(]}$, $O_C^{()}$	$O_C^{[]}$, $O_C^{[)}$, $O_C^{(]}$, $O_C^{()}$

Semigroups which are both left and right negatively orderable are also negatively orderable

Some lemmas

Definition

On any semigroup S, define the relations $\leq'$ and $\preceq$ as follows:

- for any s, t in S let $s \leq' t$ if and only if $s = t_1 s_1 t_2$ and $t = t_1 t_2$ for some s_1, t_1, t_2 in S;
- denote by $\preceq$ the reflexive and transitive closure of $\leq'$.

Lemma 4.1

In every semigroup we have

- $\leq_{\mathcal{K}} = \preceq \circ \leq_{\mathcal{J}}$
- $\leq_{\mathcal{M}} = \preceq \circ \leq_{\mathcal{L}}$
- $\leq_{\mathcal{S}} = \preceq \circ \leq_{\mathcal{R}}$

Lemma 4.2

Let S be a semigroup and θ be a congruence on S such that $\theta \subseteq \mathcal{K}$. For any $s \in S$ denote by θ_s the congruence class of s. Then if $\theta_s \leq_{\mathcal{K}} \theta_t$ in S/θ for some $s, t \in S$ then $s \leq_{\mathcal{K}} t$ in S.

Lemma 4.3

Let S be a semigroup. If $s \leq_{\mathcal{K}} t$ for some s, t in S then for any r in S we have $rs \leq_{\mathcal{S}} rt$ and $sr \leq_{\mathcal{M}} tr$.

Lemma 4.4 (Main Lemma)

If a semigroup S is $\mathcal{M}$-trivial then $S/\mathcal{S}$ is $\mathcal{K}$-trivial.

Proof.

Let S be an $\mathcal{M}$-trivial semigroup. For any s in S denote by $\bar{s}$ the image of s under the natural homomorphism $S \to S/\mathcal{S}$. Let $s, t \in S$ be such that $\bar{s}\mathcal{K}\bar{t}$ in $S/\mathcal{S}$. We shall show that $\bar{s} = \bar{t}$. By Lemma 4.2 we have $s \leq_{\mathcal{K}} t$ and $t \leq_{\mathcal{K}} s$. Hence, by Lemma 4.1 $s \preceq a \leq_{\mathcal{J}} t$ and $t \preceq b \leq_{\mathcal{J}} s$ for some $a, b \in S$. We have the following four cases.

Case 1: $a \leq_{\mathcal{L}} t$ and $b \leq_{\mathcal{L}} s$. Then $s\mathcal{M}t$, hence – by $\mathcal{M}$-triviality of S – $s = t$ and so $\bar{s} = \bar{t}$.

Case 2: $a \not\leq_{\mathcal{L}} t$ and $b \not\leq_{\mathcal{L}} s$. Then $a = utv$ and $b = xsy$ for some $u, x \in S^1$ and $v, y \in S$. Clearly, $b\mathcal{K}s$, hence $s \leq_{\mathcal{K}} b = xsy \leq_{\mathcal{S}} xs \leq_{\mathcal{L}} s$, thus $s \leq_{\mathcal{K}} xs \leq_{\mathcal{K}} s$, and so $s\mathcal{K}xs$. Hence by Lemma 4.3 $sy\mathcal{M}xsy$, and by $\mathcal{M}$-triviality of S, $sy = xsy = b$. Hence $b \leq_{\mathcal{R}} s$, and by $t \preceq b$, $t \leq_{\mathcal{S}} s$ follows. Similarly, we can show $s \leq_{\mathcal{S}} t$. Therefore $s\mathcal{S}t$, and so $\bar{s} = \bar{t}$. □

Proof continued

Proof.

Case 3: $a \leq_{\mathcal{L}} t$ and $b \not\leq_{\mathcal{L}} s$. Then $a = ut$ and $b = xsy$ for some $u, x \in S^1$ and $y \in S$. Since $s\mathcal{K}xs\mathcal{K}t$, by Lemma 4.3 $sy\mathcal{M}xsy = b\mathcal{M}ty$, and by $\mathcal{M}$-triviality of S, $sy = b = ty$. Hence $t \leq b = sy \leq_{\mathcal{R}} s$ and so $t \leq_{\mathcal{S}} s$. We show that $s \leq_{\mathcal{S}} t$ also holds. Since $b = ty \leq_{\mathcal{R}} t$ and $t \leq b = ty$, we have $ty\mathcal{S}t$ and so – by operation-compatibility of $\mathcal{S}$ – $uty\mathcal{S}ut$. We also have $ut = a\mathcal{K}t$, which – by Lemma 4.3 – implies $uty\mathcal{M}ty$, hence – by $\mathcal{M}$-triviality of S – $uty = ty$, and thus $ty = uty\mathcal{S}ut$. Therefore $s \leq a = ut \leq_{\mathcal{S}} ty \leq_{\mathcal{R}} t$, so $s \leq_{\mathcal{S}} t$ and thus $s\mathcal{S}t$ follows, implying $\bar{s} = \bar{t}$.
Case 4: $a \not\leq_{\mathcal{L}} t$ and $b \leq_{\mathcal{L}} s$. This case is similar to *Case 3*. $\square$

Theorem

A semigroup is negatively orderable if and only if it is both left and right negatively orderable.

Proof.

Since any negative partial order is also left and right negative, clearly, every negatively orderable semigroup is both left and right negatively orderable. Suppose S is a semigroup which is both left and right negatively orderable. Then S is both $\mathcal{M}$-trivial and $\mathcal{S}$-trivial, hence $S/\mathcal{M} \cong S$ and $S/\mathcal{S} \cong S$ and so $(S/\mathcal{M})/\mathcal{S} \cong S/\mathcal{S} \cong S$. Since $S/\mathcal{M} \cong S$ is $\mathcal{M}$-trivial, so by the Main Lemma $S \cong (S/\mathcal{M})/\mathcal{S}$ is $\mathcal{K}$-trivial, hence negatively orderable. $\square$

Lemma 4.5

Let θ be a congruence on a semigroup S. Then for any s, t in S, $s \leq_{\mathcal{K}} t$ ($s \leq_{\mathcal{M}} t$, $s \leq_{\mathcal{S}} t$) implies $\theta_s \leq_{\mathcal{K}} \theta_t$ ($\theta_s \leq_{\mathcal{M}} \theta_t$, $\theta_s \leq_{\mathcal{S}} \theta_t$) in S/θ.

Proposition

Let S be a semigroup and denote by θ the kernel of the natural homomorphism $S \to (S/\mathcal{M})/\mathcal{S}$ ($S \to (S/\mathcal{S})/\mathcal{M}$). Then we have $\mathcal{M} \vee \mathcal{S} \subseteq \theta = \mathcal{K}$.

Proof.

$\mathcal{M} \vee \mathcal{S} \subseteq \theta$: Let $s, t \in S$ be such that $s(\mathcal{M} \vee \mathcal{S})t$. Then there exist $s = s_0, s_1, \ldots, s_n = t \in S$ such that for every $0 \leq i \leq n-1$ we have $s_i \mathcal{M} s_{i+1}$ if i is even and $s_i \mathcal{S} s_{i+1}$ if i is odd (we can assume this, since $s_i = s_{i+1}$ is possible). Then, by Lemma 4.5 for every $0 \leq i \leq n-1$ we have $\mathcal{M}_{s_i} \mathcal{S} \mathcal{M}_{s_{i+1}}$ in $S/\mathcal{M}$ if i is odd and clearly, we have $\mathcal{M}_{s_i} = \mathcal{M}_{s_{i+1}}$ if i is even. Therefore $\mathcal{M}_s \mathcal{S} \mathcal{M}_t$ in $\mathcal{S}/\mathcal{M}$ and so $s\theta t$. $\square$

Proof continued

Proof.

$\theta \subseteq \mathcal{K}$: Suppose we have $s\theta t$ for some $s, t \in S$. Then – by definition of θ – $\mathcal{M}_s \mathcal{S} \mathcal{M}_t$, hence $\mathcal{M}_s \leq_{\mathcal{S}} \mathcal{M}_t$ and $\mathcal{M}_t \leq_{\mathcal{S}} \mathcal{M}_s$ in $S/\mathcal{M}$. Since $\mathcal{M} \subseteq \mathcal{K}$, by Lemma 4.2 $s \leq_{\mathcal{K}} t$ and $t \leq_{\mathcal{K}} s$, and so $s\mathcal{K}t$.

$\mathcal{K} \subseteq \theta$: Since $S/\mathcal{M}$ is an $\mathcal{M}$-trivial semigroup, by Lemma 4.4 $(S/\mathcal{M})/\mathcal{S} \cong S/\theta$ is a $\mathcal{K}$-trivial semigroup, and so θ is a is a $\mathcal{K}$-trivial congruence. Since on any semigroup $\mathcal{K}$ is the smallest $\mathcal{K}$-trivial congruence, $\mathcal{K} \subseteq \theta$. Therefore $\theta = \mathcal{K}$. □

The pseudovariety generated by all finite left (right) negatively orderable semigroups?

A one-sided version of the Straubig-Thérien Theorem?

Question:

Is the pseudovariety generated by all finite left (right) negatively orderable semigroups equal to the pseudovariety of all $\mathcal{L}$-trivial ($\mathcal{R}$-trivial) semigroups?

Answer:

No!

A counter-example

Example

Let $B = \{e, f, 1\}$ be a two-element band of left zeros with an adjoint identity element 1. Then B is $\mathcal{R}$-trivial, but cannot be obtained as a homomorphic image of a finite right negatively orderable semigroup.

Proof.

Suppose that B is a homomorphic image of a right negatively orderable semigroup S under the homomorphism $\varphi : S \to B$. Let t, u, v in S be such that $\varphi(t) = 1$, $\varphi(u) = e$ and $\varphi(v) = f$. Consider the sequence such that $a_{2i} = t(vu)^i$ and $a_{2i+1} = tu(vu)^i$ for every $i = 0, 1, 2, \ldots$. Then
$\varphi(a_{2i}) = \varphi(t)(\varphi(vu))^i = 1(fe)^i = f$ and
$\varphi(a_{2i+1}) = \varphi(t)\varphi(u)(\varphi(vu))^i = 1e(fe)^i = ef = e$. Hence $\varphi(a_k) \neq \varphi(a_{k+1})$ and so $a_k \neq a_{k+1}$ for every $k = 0, 1, 2, \ldots$. Clearly, $a_{k+1} \leq_{\mathcal{S}} a_k$ and thus $a_{k+1} <_{\mathcal{S}} a_k$ for every $k = 0, 1, 2, \ldots$. Since S is $\mathcal{S}$-trivial, it means that for any natural numbers $m \neq n$ we have $a_m \neq a_n$, which is impossible, since S is a finite semigroup. $\square$

A property of semigroups in $\mathcal{PS}(\mathcal{S})$ $(\mathcal{PS}(\mathcal{M}))$

Proposition

Let $S \in \mathcal{PS}(\mathcal{S})$. Then:

1. for any $r, s, t \in S$, $s\mathcal{L}t$ implies $rs = rt$
2. $\mathcal{L} = \mathcal{J} = \mathcal{J}^\sharp$ is the smallest $\mathcal{J}$-trivial congruence on S.

Proof

Proof.

(1) It is sufficient to show that the statement is true for any homomorphic image of a finite right negatively orderable semigroup. Let S be a semigroup, which is a homomorphic image of a right negatively orderable semigroup T under the homomorphism $\varphi : T \to S$. Assume that there exist r, s, t in S such that $s\mathcal{L}t$ and $rs \neq rt$. Then $s \neq t$ and $s = xt$ and $t = ys$ for some x, y in S. Let r', s', t', x', y' in T be such that $s = \varphi(s')$, $t = \varphi(t')$, $x = \varphi(x')$ and $y = \varphi(y')$. Consider the sequence such that $a'_{2i} = r'(x'y')^i s'$ and $a'_{2i+1} = r'y'(x'y')^i s'$ for every $i = 0, 1, 2, \ldots$. Then $\varphi(a'_{2i}) = r(xy)^i s = rs$ and $\varphi(a'_{2i+1}) = ry(xy)^i s = rt$ for every $i = 0, 1, 2, \ldots$. Therefore $\varphi(a'_k) \neq \varphi(a'_{k+1})$ and so $a'_k \neq a'_{k+1}$ for every $k = 0, 1, 2, \ldots$. Clearly, $a'_{k+1} \leq_{\mathcal{S}} a'_k$ and so $a'_{k+1} <_{\mathcal{S}} a'_k$ for every $k = 0, 1, 2, \ldots$. Since T is $\mathcal{S}$-trivial it means that for any natural numbers $m \neq n$ we have $a'_m \neq a'_n$, which is impossible, since T is finite. $\square$

Proof continued

Proof.

(2) This property is a consequence of Property (1) and $\mathcal{R}$-triviality. Let $S \in PS(\mathcal{S})$. By Property (1), $\mathcal{L}$ is a left congruence on S. It is well-known that $\mathcal{L}$ is a right congruence in every semigroup, hence $\mathcal{L}$ is a congruence in S. Since in a finite semigroup $\mathcal{J} = \mathcal{L} \circ \mathcal{R}$ and S is $\mathcal{R}$-trivial, we have $\mathcal{L} = \mathcal{J}$ and so $\mathcal{L} = \mathcal{J} = \mathcal{J}^\sharp$, which is the smallest $\mathcal{J}$-trivial congruence on S.

$\square$

Proposition

The class of those finite $\mathcal{R}$-trivial semigroups in which:

1. for any $r, s, t \in S$, $s\mathcal{L}t$ implies $rs = rt$ (and hence $\mathcal{L} = \mathcal{J} = \mathcal{J}^\sharp$ is the smallest $\mathcal{J}$-trivial congruence)

is a pseudovariety.

Proof.

Since finite $\mathcal{R}$-trivial semigroups form a pseudovariety, it is sufficient to show that finite semigroups with Property (1) form a pseudovariety. Denote by C the class of all finite semigroups satisfying Property (1).

- C is closed under taking direct-products*:* Let $S, T \in C$ and let $(r_1, r_2), (s_1, s_2), (t_1, t_2) \in S \times T$ be such that $(s_1, s_2)\mathcal{L}(t_1, t_2)$. Then clearly $s_1\mathcal{L}t_1$ and $s_2\mathcal{L}t_2$ and so $r_1 s_1 = r_1 t_1$, $r_2 s_2 = r_2 t_2$. Therefore $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2) = (r_1 t_1, r_2 t_2) = (r_1, r_2)(t_1, t_2)$.

□

Proof continued

Proof.

- C is closed under taking subsemigroups*:* Let $S \in C$ and T be a subsemigroup of S. For any $r, s, t \in S$, if $s\mathcal{L}t$ in T then $s\mathcal{L}t$ in S also holds, so – since $S \in C$ – $rs = rt$.
- C is closed under taking homomorphic images*:* Let $S \in C$ and θ be a congruence on S and let $\theta_1, \theta_2, \theta_3 \in S/\theta$ be such that $\theta_2 \mathcal{L} \theta_3$. Then – since S is finite – it is easy to show that there exist $s \in \theta_2$ and $t \in \theta_3$ such that $s\mathcal{L}t$ in S. Then for any $r \in \theta_1$ we have $rs = rt$, hence $\theta_1\theta_2 = \theta_1\theta_3$.

□

Question

Is the pseudovariety generated by all finite right negatively orderable semigroups equal to the pseudovariety of all finite $\mathcal{R}$-trivial semigroups in which:

- for any $r, s, t \in S$, $s\mathcal{L}t$ implies $rs = rt$ (and hence $\mathcal{L} = \mathcal{J} = \mathcal{J}^\sharp$ is the smallest $\mathcal{J}$-trivial congruence) ?

Thank you!