Classes of Groups with Rational Cross-Section

John Fountain and Jonathan Hillier

26 November 2014

York Semigroup

Languages

A is a nonempty finite set (an alphabet).

Word over A: a finite sequence $a_1 \dots a_n$ of elements a_i of A (including the empty word ε).

Length: $|\varepsilon| = 0$; $|a_1 \dots a_n| = n$.

$A^* :=$ set of all words over A.

A language (over A) is a subset of A^*.

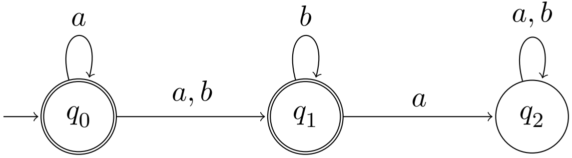
Examples

- $L_1 = \{a^m b^n : m, n \geqslant 0\}$
- $L_2 = \{a^n b^n : n \geqslant 0\}$
- $L_3 = \{a^n b^n c^n : n \geqslant 1\}$

Three classes of languages

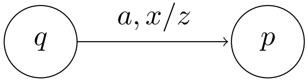
Languages	**Recognizers**
Regular (Reg)	Finite state automata
Context-free (CF)	Pushdown automata (PDA)
Context -Sensitive (CS)	Linear bounded automata (LBA)

Regular languages are also known as rational languages.
L_1 is regular:

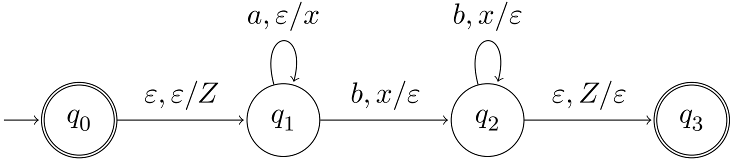


Pushdown automata

Read-only input tape; pushdown stack; finite number of states.
Move: *may* 'pop' top symbol of stack, 'push' a string of symbols onto the top of the stack, move to next symbol on input tape, enter new state.

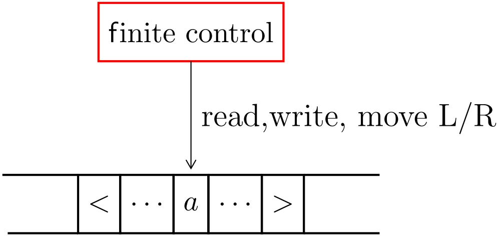


In state q, reading a, pop x, push z and change to state p. Any of a, x, z may be replaced by ε, and we may have $q = p$.
$L_2 = \{a^n b^n : n \geqslant 0\}$ is not regular but is context-free:



Linear bounded automata I

A linear bounded automaton (LBA) is a nondeterministic Turing machine which cannot move beyond the end-markers.



Depending on current state and tape symbol, the machine *may*:

- overwrite the tape symbol with a new symbol;
- move the tape head left or right by one cell;
- jump to a new state.

Word on tape is accepted if machine halts in a final state.
$L_3 = \{a^n b^n c^n : n \geqslant 1\}$ is not CF but is CS.

Closure properties

Closed under	Reg	CF	CS
Intersection	Yes	No	Yes
Intersection with regular set	Yes	Yes	Yes
Union	Yes	Yes	Yes
Concatenation	Yes	Yes	Yes
Complement	Yes	No	Yes
Star	Yes	Yes	Yes

For $L, K \subseteq A^*$,

- $LK := \{vw : v \in L, w \in K\}$;
- $L^* := \{\varepsilon\} \cup L \cup L^2 \cup L^3 \cup \cdots \cup L^n \cup \ldots$.

Groups

A (finite) set A is a generating set for a group G if there is a (monoid) homomorphism $\theta : A^* \to G$ from A^* onto G.

A is symmetric if $A = X \cup X^{-1}$ where $X^{-1} = \{x^{-1} : x \in X\}$ and $x^{-1}\theta = (x\theta)^{-1}$ for all x.

If L is a regular language over A with $L\theta = G$, say (A, L) is a rational structure for G. If the restriction of θ to L is injective, (A, L) is a rational cross-section of G.

Gilman (1987) found various properties of groups with a rational cross-section including:

1. The class of groups with rational cross-section is closed under free product, direct product and extension.
2. If G is a group with a rational cross-section and H is a subgroup of finite index in G, then H has a rational cross-section.

In Epstein et al (1992) it is proved that a finitely presented group with rational cross-section has decidable word problem.

Automatic groups I

Let L be a language over an alphabet A and let $\diamond$ be a symbol not in A.

$A_\diamond := A \cup \{\diamond\}$.

For $v = a_1 \dots a_m, w = b_1 \dots b_n \in A^*$, define $\otimes(v, w)$ to be the word $x_1 \dots x_k$ of length $\max\{|v|, |w|\}$ over $A_\diamond \times A_\diamond$ where

$$x_i = \begin{pmatrix} c_i \\ d_i \end{pmatrix}$$

and

$$c_i = \begin{cases} a_i & \text{if } i \leqslant |v| \\ \diamond & \text{otherwise} \end{cases} \text{ and } d_i = \begin{cases} b_i & \text{if } i \leqslant |w| \\ \diamond & \text{otherwise} \end{cases}$$

For example, if $v = ababa$ and $w = aaa$, then

$$\otimes(v, w) = \begin{pmatrix} a \\ a \end{pmatrix}\begin{pmatrix} b \\ a \end{pmatrix}\begin{pmatrix} a \\ a \end{pmatrix}\begin{pmatrix} b \\ \diamond \end{pmatrix}\begin{pmatrix} a \\ \diamond \end{pmatrix}.$$

Automatic groups II

Let G be a group with finite symmetric generating set A and homomorphism $\theta : A^* \to G$ onto G. (G, A) is automatic if there is a rational structure (A, L) for G such that

(1) the language $L_= = \{\otimes(v, w) : v, w \in L \text{ and } v\theta = w\theta\}$ is regular, and

(2) for each $a \in A$, the language $L_a = \{\otimes(v, w) : v, w \in L \text{ and } v\theta = (wa)\theta\}$ is regular.

We say that (A, L) is an automatic structure for G.

A basic result is that if (G, A) is automatic, then there is a rational cross-section (A, J) of G satisfying (2) (with L replaced by J). Note that (1) is immediate. Thus automatic groups form a subclass of the class of groups with rational cross-section.

Examples

The following groups are automatic:

1. finite groups,
2. finitely generated free groups,
3. finitely generated abelian groups,
4. Artin groups of finite type (e.g. braid groups),
5. finitely generated Coxeter groups,
6. word hyperbolic groups.

The following groups are not automatic:

1. Baumslag-Solitar groups $BS(m,n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$ with $m \neq n$.
2. The Heisenberg group H, i.e., the (multiplicative) group of integer matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$.

Examples II

Although not automatic, H does have a rational cross-section. H has group presentation

$$\langle x, y, z \mid [y, z] = [x, z] = 1, [x, y] = z \rangle.$$

$$A = \{x, y, z, X, Y, Z\}$$

is a set of monoid generators for H with homomorphism $\theta : A^* \to H$ determined by

$$x\theta = x,\ y\theta = y,\ z\theta = z,\ X\theta = x^{-1},\ Y\theta = y^{-1},\ Z\theta = z^{-1}.$$

Put $L = \{x^i y^j z^k : i, j, k \in \mathbb{Z}\}$ where for negative i we understand $X^{|i|}$, etc.

(A, L) is a rational cross-section for H.

Properties of automatic groups

- Let (A, L) be an automatic structure for G. If B is any other finite generating set for G, then there is a regular language J over B such that (B, J) is an automatic structure for G.

- If G is automatic, then G is finitely presented.

- If G is automatic, then the word problem is solvable in quadratic time.

- If H is a subgroup of finite index in a group G, then G is automatic if and only if H is automatic.

- For groups G,H, the free product $G * H$ is automatic if and only if G and H are both automatic.

- The direct product of two automatic groups is automatic.

$\mathcal{C}$-structures for groups I

Let $\mathcal{C}$ be a class of languages with Reg $\subseteq \mathcal{C}$. Suppose that $\mathcal{C}$ is closed under (finite) union, concatenation and intersection with a regular language, i.e, if $L, K \in \mathcal{C}$ and $J \in$ Reg, then $L \cup K, LK, L \cap J \in \mathcal{C}$.

Definition A rational structure (A, L) for a group G is a $\mathcal{C}$-structure for G if

(1) $L_=$ is regular;

(2) $L_a \in \mathcal{C}$ for all $a \in A$;

(3) there is a positive integer k such that for all $a \in A$ and $v, w \in L$ with $\otimes(v, w) \in L_a$ we have $k(|v| + 1) \geqslant |w|$.

$\mathcal{C}$-structures for groups II

Obviously, any group with a Reg-structure is automatic, and

Fact

If G is an automatic group, then G has a Reg-*structure.*

However, a group may have an automatic structure which is not a Reg-structure. For example, let $G = \{1, g\}$ be the cyclic group of order 2, $A = \{a\}$ and $\theta : A^* \to G$ be determined by $a\theta = g$. Let $L = \{a, a^2, a^3, \dots\}$. It is easy to verify that (A, L) is an automatic structure for G but (A, L) does not satisfy (3) so it is not a Reg-structure.

The Heisenberg group has a CF-structure.

Properties of groups with a $\mathcal{C}$-structure

- If (A, L) is a $\mathcal{C}$-structure for a group G, then there is a subset J of L such that (A, J) is both $\mathcal{C}$-structure for G and a rational cross-section of G.

  For the proof, order $A = \{a_1, \ldots, a_n\}$ by $a_1 < \cdots < a_n$. Extend the order to A^* first by length ($v < w$ if $|v| < |w|$) and then lexicographically on words of equal length. Then

  $$J = \{v : v \in L \text{ and for all } w \in L \text{ if } \otimes(v, w) \in L_=, \text{ then } v \leqslant w\}$$

  is regular. Then (A, J) is a rational cross-section of G. $J_a = L_a \cap \otimes(J, J) \in \mathcal{C}$ since $\otimes(J, J)$ is regular and $\mathcal{C}$ is closed under intersection with regular sets. Condition (3) is immediate since $J_a \subseteq L_a$.

- If (A, L), (B, K) are $\mathcal{C}$-structures with uniqueness for groups G, H respectively with ε representing the identity, then $(A \cup B, L((K \setminus \{\varepsilon\})(L \setminus \{\varepsilon\}))^* K)$ is a $\mathcal{C}$-structure with uniqueness for the free product $G * H$.

CS-groups

We now consider $\mathcal{C} = CS$.

- Let A and B be finite sets of monoid generators for a group G. Then there exists $L \subseteq A^*$ such that (A, L) is a CS-structure for G if and only if there exists $K \subseteq B^*$ such that (B, K) is a CS-structure for G

  This justifies speaking of CS-groups.

- If G, H are CS-groups, then the direct product $G \times H$ is a CS-group.

- If H is a subgroup of finite index in a group G, then G is a CS-group if and only if H is a CS-group.

- The word problem for a CS-group is solvable by a Turing machine operating within an exponential space bound.

Further directions

1. Improve the bound for the word problem for CS-groups, and get bounds for groups with a $\mathcal{C}$-structure for other language classes $\mathcal{C}$.

2. Find properties of groups with a $\mathcal{C}$-structure when $\mathcal{C}$ is CF, the class of indexed languages etc.

3. Investigate the use of transducers rather than acceptors in defining the groups.

4. Find more examples and non-examples.

5. Free products with amalgamation?

6. Investigate connections with work of Elder and Taback.