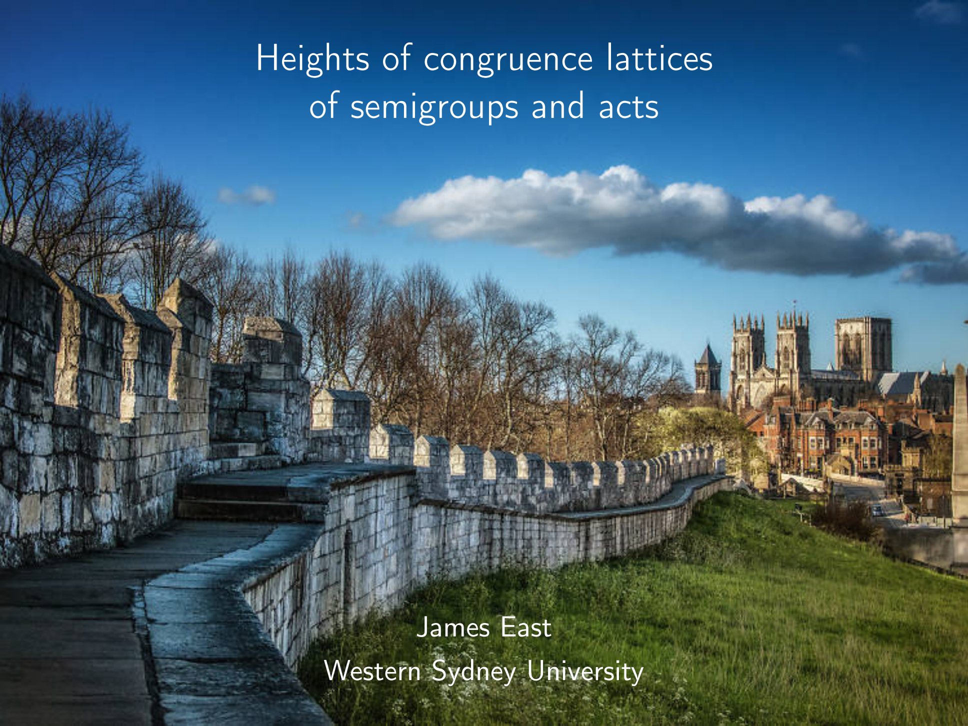
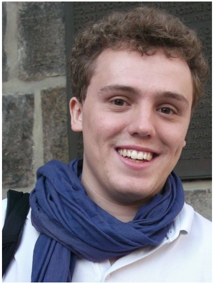
# Heights of congruence lattices of semigroups and acts



James East
Western Sydney University

# Joint work (in progress) with......



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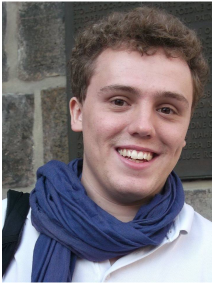

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**Basic question**

What is the height of the (left/right) congruence lattice of a semigroup?

# Congruences

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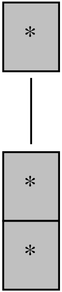
- We'd like to understand these!

# Cong($\mathcal{T}_n$) – Mal'cev (1952) — always a chain!



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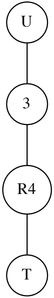


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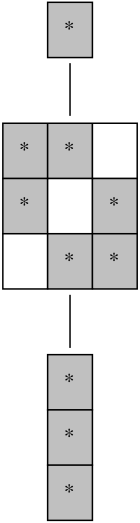

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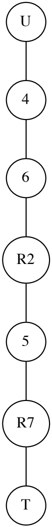


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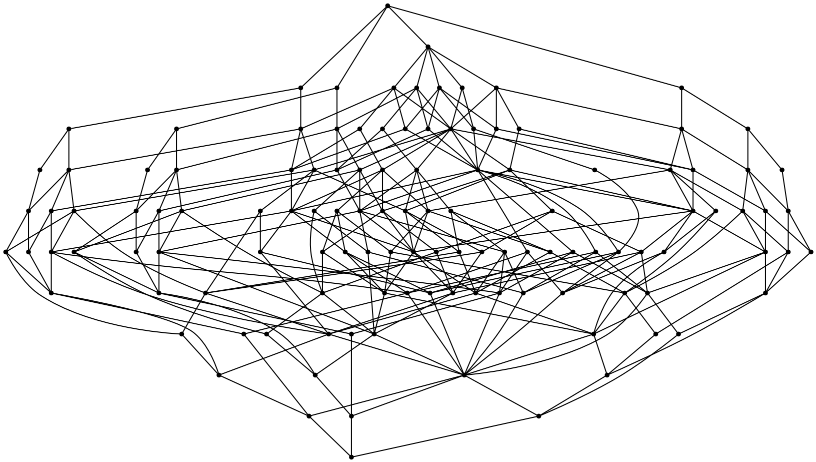

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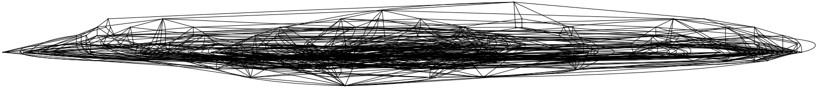

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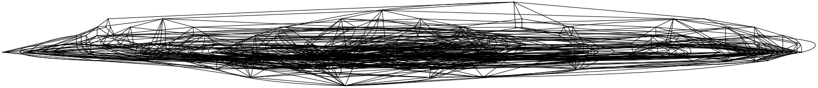

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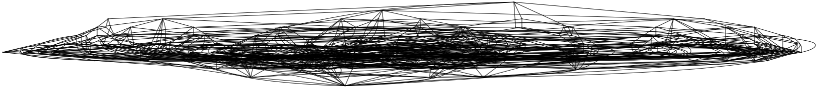

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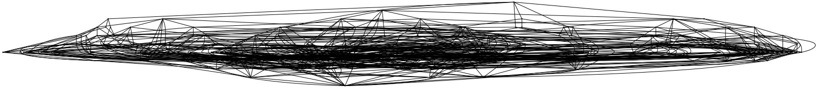

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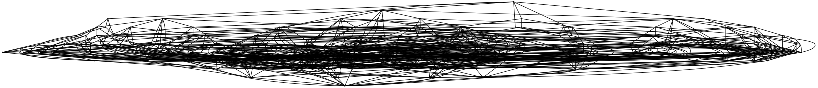

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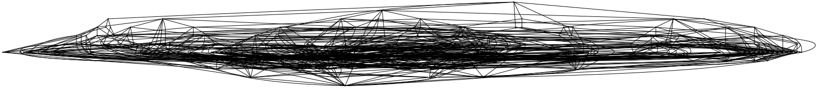

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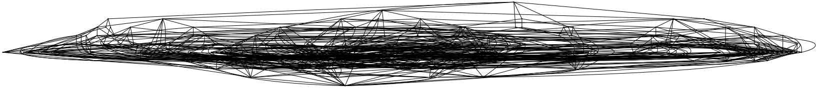

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|---|---|---|---|---|---|---|---|---|---|
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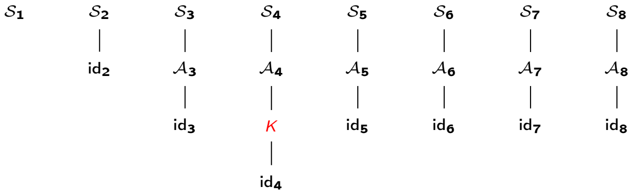

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**Theorem (follows from Mal'cev's classification)**

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- Similar results for $\mathcal{PT}_n$, $\mathcal{I}_n$, $\mathcal{P}_n$, $\mathcal{B}_n$, $\mathcal{TL}_n$.........

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- Another story: $\mathrm{Ht}(\mathrm{Sub}(S))$......
  - Cameron, Gadouleau, Mitchell, Péresse, 2017.

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- Symmetric groups:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
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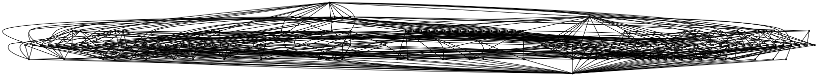

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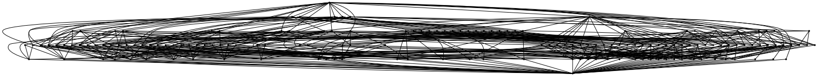

- OEIS: A007238

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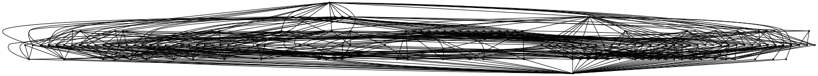
**Theorem (Cameron, Solomon, Turull, 1989)**

- $\mathrm{Ht}(\mathrm{Sub}(\mathcal{S}_n)) = \lceil \frac{3n}{2} \rceil - b(n)$, where
  - $b(n)$ = number of 1s in binary expansion of $n$.

- e.g., $\mathrm{Ht}(\mathrm{Sub}(\mathcal{S}_5)) = \lceil \frac{15}{2} \rceil - b(5) = 6$.



- OEIS: A007238 (+1)
  - Height = number of vertices or edges?

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**Theorem**

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- Eventually generalised to more general classes of regular semigroups via right ideals and acts.

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  - $\mathrm{Cong}^S(S) = \mathrm{RCong}(S)$!
  - So maybe now we want to compute $\mathrm{Ht}(\mathrm{Cong}^S(A))$?

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## Key Proposition

If $B$ is a sub-act of $A$, then

$$\mathrm{Ht}(\mathrm{Cong}^S(A)) = \mathrm{Ht}(\mathrm{Cong}^S(B)) + \mathrm{Ht}(\mathrm{Cong}^S(A/B)) - 1.$$

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- Proved by separately showing:
  - LHS $\geq$ RHS and LHS $\leq$ RHS.

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- Join to give a chain in $\mathrm{Cong}^S(A)$:
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## Magic

Let $B \leq A$ and $\sigma, \sigma' \in \mathrm{Cong}^S(A)$. If

- $\sigma \subseteq \sigma'$,
- $\sigma \cap \rho_B = \sigma' \cap \rho_B$,
- $\sigma \vee \rho_B = \sigma' \vee \rho_B$,

then $\sigma = \sigma'$.

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## Theorem

- If $S$ is a finite semigroup with:
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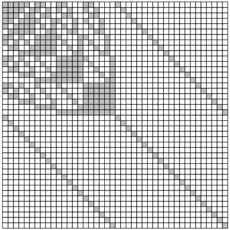
- Say $D$ is column-faithful if the columns of $M(D)$ are distinct.



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## Theorem

- If $S$ is a finite regular semigroup with:
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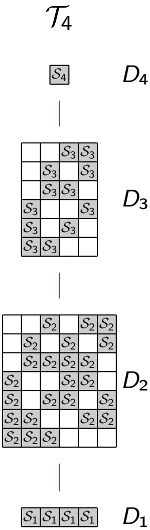
# Full transformation semigroups



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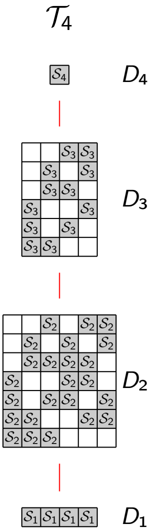


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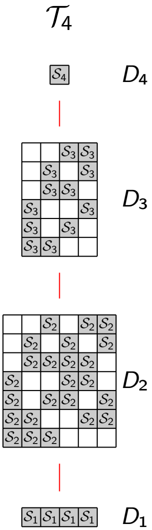


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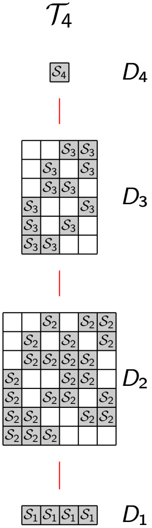


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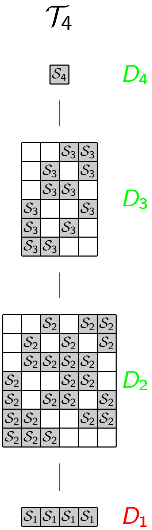


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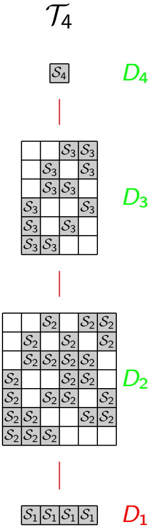


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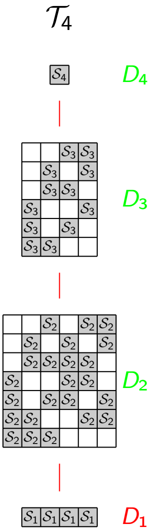


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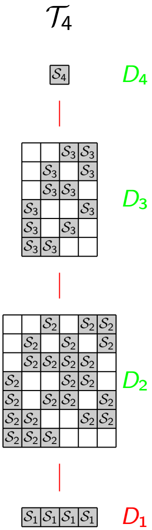


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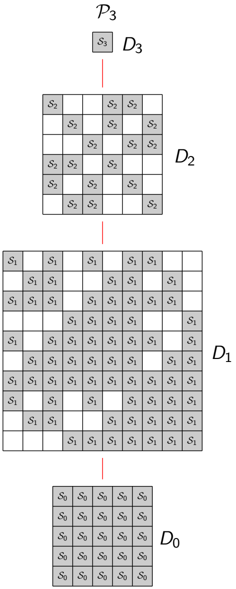
- Similarly:

  $\mathrm{Ht}(\mathrm{LCong}(\mathcal{T}_n)) = \sum_{k=1}^{n} \binom{n}{k} \mathrm{Ht}(\mathcal{S}_k)$.
  - Every $D_k$ is row-faithful!

# Partition monoids



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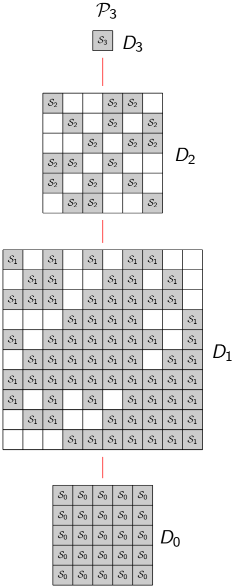

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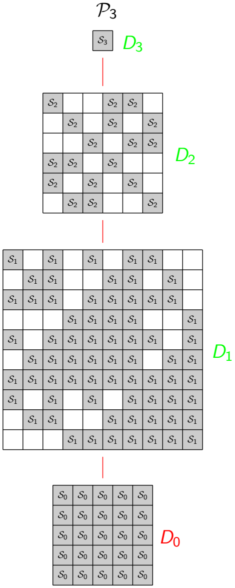


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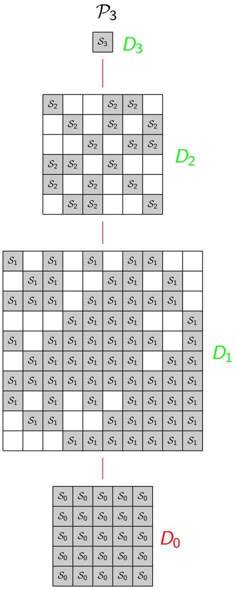


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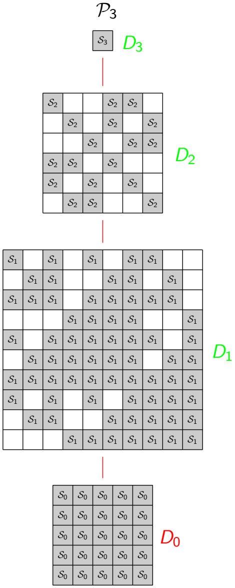


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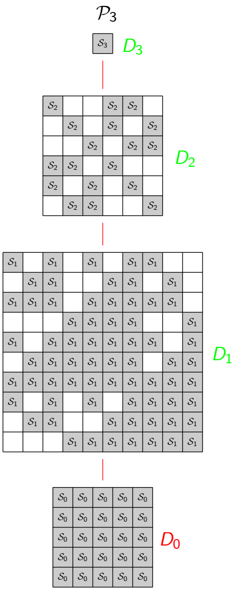


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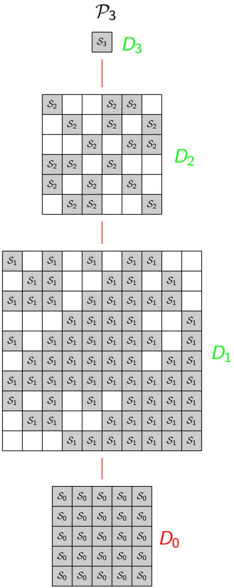


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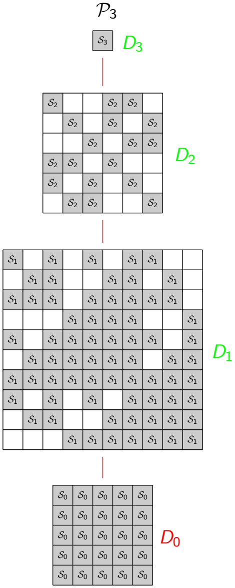


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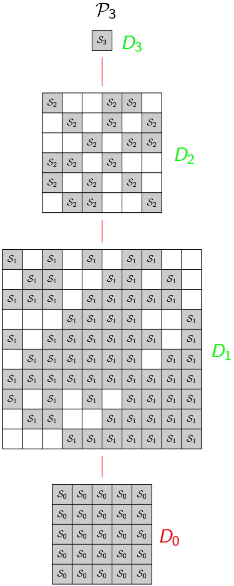


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- Analogous results for $\mathcal{B}_n$, $\mathcal{TL}_n$, etc.

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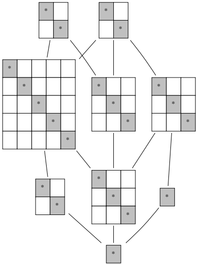
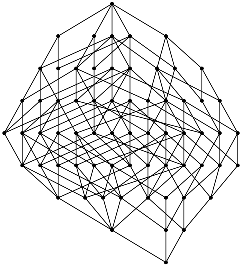
    - Previous holds when $S$ is inverse.
- $\mathrm{Ht}(\mathrm{Cong}(S)) = |S/\mathscr{D}|$ when $S$ is regular + faithful + $\mathscr{H}$-trivial.





# Thanks for listening :-)