Clones determined by clausal relations

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Outline

Clausal Relations

Clones

C-clones

Clausal Relations

Let $p, q \in \mathbb{N}_+ := \{1, 2, \ldots\}$, $D := \{0, 1, \ldots, n-1\}$ and $(D; \leq)$ chain.

Definition

For given parameters $\mathbf{a} = (a_1, \ldots, a_p) \in D^p$ and $\mathbf{b} = (b_1, \ldots, b_q) \in D^q$, the *clausal relation* $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ *of arity* $p + q$ is the set of all tuples $(x_1, \ldots, x_p, y_1, \ldots, y_q) \in D^{p+q}$ satisfying

$$(x_1 \geq a_1) \vee \cdots \vee (x_p \geq a_p) \vee (y_1 \leq b_1) \vee \cdots \vee (y_q \leq b_q).$$

In this expression $\leq$ denotes the canonical linear order on D and $\geq$ its dual.

Let $D = \{0, 1, 2\}$, then

$$\begin{aligned} \mathrm{R}_1^2 &= \{(x_1, y_1) \in D^2 \mid x_1 \geq 2 \vee y_1 \leq 1\} \\ &= \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 \end{pmatrix} = D^2 \setminus \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}. \end{aligned}$$

$$\begin{aligned} \mathrm{R}_0^{(2,2)} &= \left\{(x_1, x_2, y_1) \in D^3 \mid x_1 \geq 2 \vee x_2 \geq 2 \vee y_1 \leq 0\right\} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 0 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix} \end{aligned}$$

$$CR_D := \bigcup_{(p,q)\in\mathbb{N}_+^2} \{R_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in D^p, \mathbf{b} \in D^q\}$$

the *set of all finitary clausal relations* on D.

Fact

- *If* $(\exists i \in \{1,\dots,p\} : a_i = 0) \implies R_{\mathbf{b}}^{\mathbf{a}} = D^{p+q}$.
- *If* $(\exists j \in \{1,\dots,q\} : b_j = n-1) \implies R_{\mathbf{b}}^{\mathbf{a}} = D^{p+q}$.

Lemma

$$\begin{aligned} CR_D \cap \mathrm{diag}(D) &= \{ D^{p+q} \mid p, q \in \mathbb{N}_+ \} . \\ CR_D^* &:= CR_D \setminus \mathrm{diag}(D) \\ &= \{ \mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \mid \mathbf{a} \in (D \setminus \{0\})^p, \mathbf{b} \in (D \setminus \{n-1\})^q ; p, q \in \mathbb{N}_+ \} . \end{aligned}$$

Non-trivial clausal relations (Essential predicates)

$$(0, \ldots, 0, n-1, \ldots, n-1) \notin \mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \in CR_D^* \qquad \text{but}$$

$$(a_1, 0, \ldots, 0, n-1, \ldots, n-1), \ldots, (0, \ldots, 0, n-1, \ldots, n-1, b_q) \in \mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$$

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For $n \in \mathbb{N}_+$ called arity,

$$f : D^n \longrightarrow D$$
$$(x_1, \ldots, x_n) \longmapsto f(x_1, \ldots, x_n)$$

n-ary operation on D

$$O_D^{(n)} := D^{D^n}$$

set of n-ary operations on D

$$O_D := \bigcup_{k \in \mathbb{N}_+} O_D^{(k)}$$

set of all finitary operations on D.

For $m \in \mathbb{N}_+$ subsets $\varrho \subseteq D^m$ are

m-ary relations on D

$$R_D^{(m)} := \mathcal{P}(D^m)$$

set of m-ary relations on D

$$R_D := \bigcup_{m \in \mathbb{N}_+} R_D^{(m)}$$

set of all finitary relations on D.

Projections

Let $n \in \mathbb{N}_+$ and $j \in \{1, \ldots, n\}$

$$e_j^{(n)} : \begin{array}{ccc} A^n & \longrightarrow & A \\ (a_1, \ldots, a_n) & \longmapsto & a_j \end{array}$$

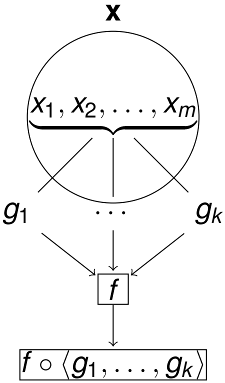
the *j*-th projection of arity *n* on *A*.

$$J_A := \left\{ e_j^{(n)} \mid n \in \mathbb{N}_+, 1 \leq j \leq n \right\}$$

is the set of all projections on *A*.

Composition

Let $f \in \mathrm{O}_A^{(k)}, g_1, \ldots, g_k \in \mathrm{O}_A^{(m)}$,



$$
\begin{array}{rcl}
f \circ \langle g_1, \ldots, g_k \rangle : A^m & \to & A \\
\mathbf{x} := (x_1, \ldots, x_m) & \mapsto & (f[g_1, \ldots, g_k])(\mathbf{x}) \\
 & := & f(g_1(\mathbf{x}), \ldots, g_k(\mathbf{x}))
\end{array}
$$

called *composition* or *superposition*.

What is a clone?

Definition

$F \subseteq O_A$ is *clone* (of operations) on A iff

1. $J_A \subseteq F$
2. F is closed w.r.t. composition.

Examples

1. J_A Clone of all projections.
2. O_A Clone of all operations.
3. $O^c_{(A,\tau)}$ continuous functions of a topological space (A, τ).

Importance of clones (for universal algebra)

Proposition

For every algebra $\mathbf{A} = \langle A; F \rangle$ *its set of term operations* $\mathrm{Term}(\mathbf{A})$ *forms a clone.*
Every clone $F \subseteq \mathrm{O}_A$ *is the set of term operations of some algebra, namely that of* $\mathbf{A} = \langle A; F \rangle$*.*

Example

Let $\mathbf{A} = \langle \mathbb{R}, + \rangle$

$$t_1 := (+, x_1, x_2) \qquad t_1^{\mathbf{A}}(a_1, a_2) = a_1 + a_2$$
$$t_2 := (+, x_1, (+, x_1, x_2)) \qquad t_2^{\mathbf{A}}(a_1, a_2) = 2a_1 + a_2$$
$$t_3 := (+, (\ldots (+, (+, (+, x_1, x_1), x_1), x_1) \ldots), x_1) \qquad t_3^{\mathbf{A}}(a_1) = k_1 a_1$$

Example

Let $\mathbf{A} = \langle \mathbb{R}, + \rangle$

$$t_4 := (+, (\ldots(+, (+, (+, (+, x_1, x_2), x_1), x_2), x_1)\ldots), x_2)$$

$$t_4^{\mathbf{A}}(a_1, a_2) = k_1 a_1 + k_2 a_2, \text{ where } k_1, k_2 \in \mathbb{N} \setminus \{0\}, a_1, a_2 \in \mathbb{R}.$$

$$\mathrm{Term}(\mathbf{A}) = \left\{ \sum_{i=1}^{n} k_i a_i \mid k_i \in \mathbb{N} \setminus \{0\}, a_i \in \mathbb{R} \right\} = \langle + \rangle_{\mathrm{O}_{\mathbb{R}}}$$

Let $F \subseteq \mathrm{O}_A$. The Clone generated by F is

$$\langle F \rangle_{\mathrm{O}_A} := \bigcap \{ C \text{ is clone } \mid F \subseteq C \}$$

and it is the smallest clone containing F.

Lattice of Clones

- $\mathcal{L}_A := \{F \subseteq O_A \mid F \text{ is a clone } \}$
- $(\mathcal{L}_A, \subseteq)$ lattice of all clones on A.
- a complete and dually (for finite A) lattice.

Preservation condition

For $f \in \mathrm{O}_D^{(k)}$, and $\varrho \in \mathrm{R}_D^{(m)}$, we say f preserves ϱ, denoted by $f \rhd \varrho$ if one of the following equivalent conditions is fulfilled:

- $f : D^k \to D$ induces a homomorphism between the k-th direct power $\langle D; \varrho\rangle^k$ and the relational structure $\langle D; \varrho\rangle$. f is a polymorphism of ϱ.
- ϱ is a subuniverse of the m-th direct power $\langle A; f\rangle^m$. This motivates the alternative names subpower or invariant relation for ϱ.
- For every tuples $\mathbf{r}_1, \ldots, \mathbf{r}_k \in \varrho$, the composition of f with these tuples belongs again to the relation ϱ.

Finitary operations with Finitary relations.

$f \in \mathrm{O}_D^{(k)}$ preserves $\varrho \in \mathrm{R}_D^{(m)}$ denoted by $f \rhd \varrho$

$$f \circ \left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{pmatrix} \right) = \begin{pmatrix} f(a_{11} & \ldots & a_{1k}) \\ & \ddots & \\ f(a_{m1} & \ldots & a_{mk}) \end{pmatrix}$$

$$\begin{matrix} \Cap & & \Cap & & \Cap \\ \varrho & & \varrho & & \varrho \end{matrix}$$

Polymorphisms and Invariant relations

Definition

$F \subseteq \mathrm{O}_D$, $Q \subseteq \mathrm{R}_D$

$$\begin{aligned} \mathrm{Pol}_D\, Q &:= \{f \in \mathrm{O}_D \mid \forall \varrho \in Q : f \rhd \varrho\} \\ \mathrm{Inv}_D\, F &:= \{\varrho \in \mathrm{R}_D \mid \forall f \in F : f \rhd \varrho\} \end{aligned}$$

The mappings

$$\mathrm{Pol}\, Q \hookleftarrow Q$$
$$F \mapsto \mathrm{Inv}\, F$$

define a GALOIS *connection* Pol – Inv induced by $\rhd$.

Example

Let $D = \{0, 1, 2\}$, then

$$\begin{aligned} R_1^2 &= \{(x_1, y_1) \in D^2 \mid x_1 \geq 2 \vee y_1 \leq 1\} \\ &= \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 \end{pmatrix} = D^2 \setminus \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}. \end{aligned}$$

$c_0^{(3)}, c_2^{(2)}, e_1^{(2)} \in \mathrm{Pol}_D\, R_1^2 = \left\{ f \in O_D^{(n)} \mid f \rhd R_1^2 \right\}$ because for every tuple in R_1^2 we have

$$c_0^{(3)} \circ \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad c_2^{(2)} \circ \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad e_1^{(2)} \circ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

But $f_6 \not\rhd R_1^2$, where $f_6(0) = 0$, $f_6(1) = 2$, $f_6(2) = 0$ (Explain)

On finite D, clones(of operations) are

Theorem (Bodnarčuk, Kalužnin, Kotov, Romov 69)

For D finite, F is a clone $\implies F = \mathrm{Pol}_D\, Q$ for $Q = \mathrm{Inv}_D\, F$.

Every clone can be described by relations.

Idea

To confine the allowed relations to be clausal relations.

$$C\mathrm{Inv}\, F := \mathrm{Inv}_D\, F \cap C\mathrm{R}_D$$

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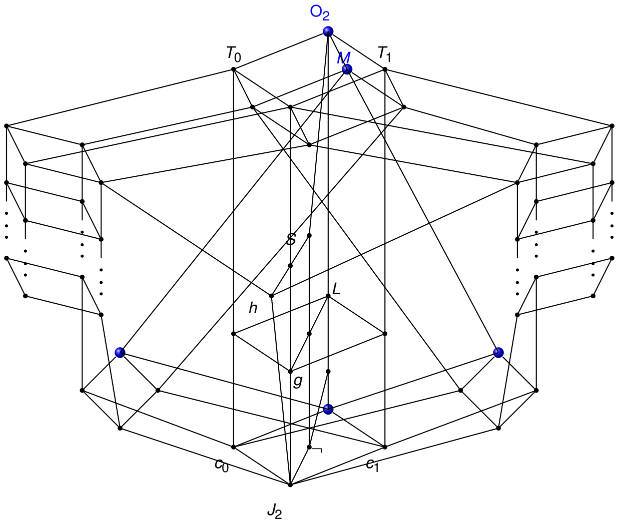
Definition

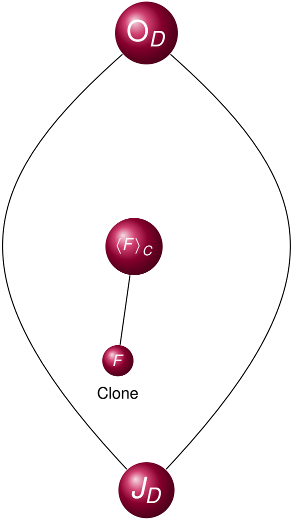
$F \subseteq \mathrm{O}_D$ is C-clone $:\iff F = \mathrm{Pol}_D\, Q$, where Q is a set of clausal relations.

Motivation

Reduction of complexity by confining the allowed relations.


O$_2$
T_0
M
T_1
S
L
h
g
c_0
c_1
J_2

Where do the C-clones live?



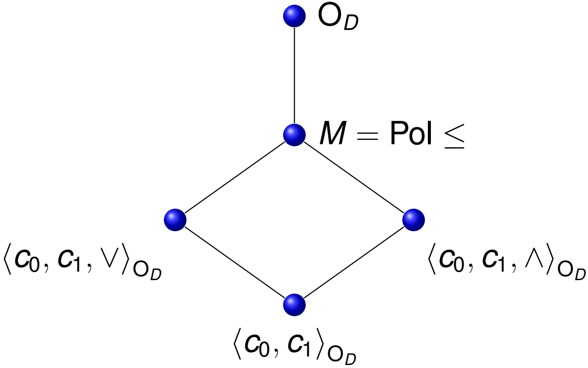
For $F \subseteq \mathrm{O}_D$

$$\mathrm{Inv}\, F \supseteq C\mathrm{R}_D \cap \mathrm{Inv}\, F = C\mathrm{Inv}\, F$$

$$\Rightarrow \langle F \rangle_{\mathrm{O}_D} = \mathrm{Pol}\,\mathrm{Inv}\, F \subseteq \mathrm{Pol}\, C\mathrm{Inv}\, F := \langle F \rangle_C$$

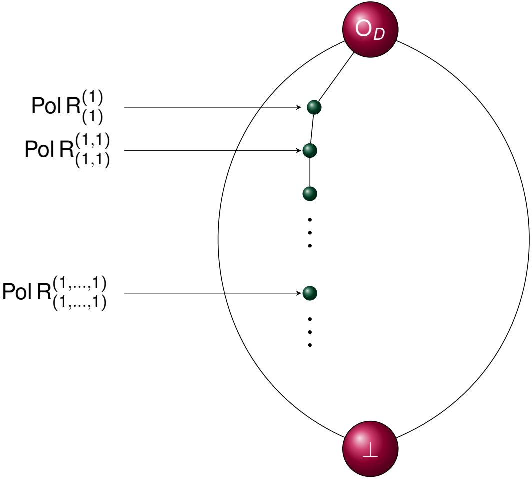
How many C-clones do exist for an arbitrary finite set D?

- For $D = \{0, 1\}$, there are five different C-clones.



Lattice of C-clones for $D = \{0, \ldots, n-1\}$, $n \geq 3$

- Contains countably infinite descending chains.



Definition

$Q \subseteq \mathrm{R}_D$

$$\mathrm{End}\, Q := \mathrm{Pol}\, Q \cap \mathrm{O}_D^{(1)}$$
$$\mathrm{Aut}\, Q := \mathrm{Pol}\, Q \cap \mathrm{Sym}\,(D)\,.$$

Definition

A monoid $\mathcal{M} = \langle M, \circ, id_D \rangle$ where $M \subseteq D^D$ is C-monoid

$$:\iff \exists Q \subseteq C\mathrm{R}_D : M = \mathrm{End}\, Q$$

$G \subseteq \mathrm{Sym}\,(D)$ is C-automorphism group $:\iff \exists Q \subseteq C\mathrm{R}_D : G = \mathrm{Aut}\, Q$.

We shall describe

$$\{\mathrm{Aut}\, Q \mid Q \subseteq CR_D\} = \{\mathrm{Aut}\, Q^* \mid Q^* \subseteq CR^*_D\}\,.$$

$$\mathrm{Aut}\, Q^* = \bigcap_{\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \in Q^*} \mathrm{Aut}\, \mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$$

$(x_1 \geq a_1) \vee \cdots \vee (x_p \geq a_p) \vee (y_1 \leq b_1) \vee \cdots \vee (y_q \leq b_q)$.

$$\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} = \bigcup_{i=1}^{p} \underbrace{D \times \ldots \times D}_{i-1} \times [a_i, n-1] \times \underbrace{D \times \ldots \times D}_{p+q-i} \cup$$

$$\bigcup_{j=1}^{q} \underbrace{D \times \ldots \times D}_{p+j-1} \times [0, b_j] \times \underbrace{D \times \ldots \times D}_{q-j}$$

$$(\mathrm{R}^{\mathbf{a}}_{\mathbf{b}})^c = [0, a_1) \times \ldots \times [0, a_p) \times (b_1, n-1] \times \ldots \times (b_q, n-1]$$

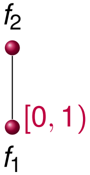
For $f \in \mathrm{Sym}(D)$, $\mathbf{a} \in (D \setminus \{0\})^p$, $\mathbf{b} \in (D \setminus \{n-1\})^q$, it holds:

$$f \rhd \mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \iff f \rhd \left(\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}\right)^c$$
$$\iff f \rhd \prod_{i=1}^{p}[0, a_i) \times \prod_{j=1}^{q}(b_j, n-1]$$
$$\iff f \rhd \{\{[0, a_i) \mid i \in \{1, \ldots, p\}\} \cup \{(b_j, n-1] \mid j \in \{1, \ldots, q\}\}\}$$

$$\mathrm{Aut}\,\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} = \mathrm{Aut}\left(\{[0, a_i) \mid i \in \{1, \ldots, p\}\} \cup \{(b_j, n-1] \mid j \in \{1, \ldots, q\}\}\right)$$
$$= \mathrm{Aut}\left(\{[0, a_i) \mid i \in \{1, \ldots, p\}\} \cup \{[0, b_j + 1) \mid j \in \{1, \ldots, q\}\}\right)$$

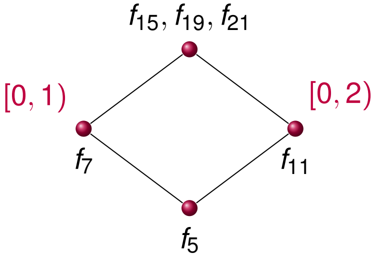
$D = \{0, 1\}$

$\triangleright$	$[0,1)$
$f_1 : id$	$\times$
$f_2 : (01)$	

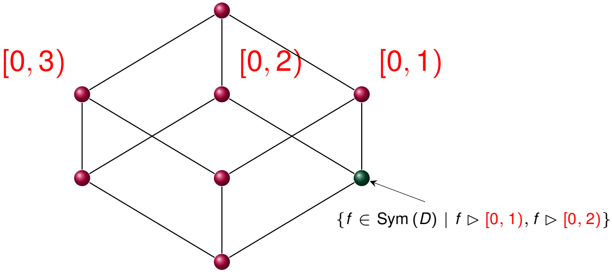


$D = \{0, 1, 2\}$

$\triangleright$	$[0,1)$	$[0,2)$
$f_5 : id$	$\times$	$\times$
$f_7 : (12)$	$\times$	
$f_{11} : (01)$		$\times$
$f_{15} : (012)$		
$f_{19} : (021)$		
$f_{21} : (02)$		



$D = \{0, 1, 2, 3\}$



Theorem (Behrisch-Vargas)

The lattice of all C-automorphism groups is dually isomorphic to $(\mathcal{P}(D \setminus \{0\}), \subseteq)$ *via the following isomorphism*

$$\begin{aligned} \phi : \ (\mathcal{P}(D \setminus \{0\}), \subseteq) &\to (\{\mathrm{Aut}\, Q \mid Q \subseteq CR_D\}, \supseteq) \\ U &\mapsto \mathrm{Aut}\,\{[0, u) \mid u \in U\} \end{aligned}$$

We shall describe

$$\{\mathrm{End}\, Q \mid Q \subseteq CR_D\} = \{\mathrm{End}\, Q^* \mid Q^* \subseteq CR_D^*\}\,.$$

$$\mathrm{End}\, Q^* = \bigcap_{\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \in Q^*} \mathrm{End}\, \mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$$

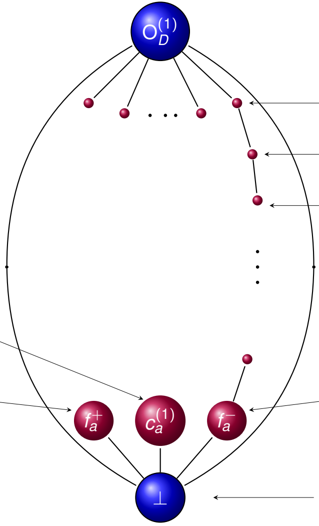
Proposition

Let $b_1, \dots, b_q \in D \setminus \{n-1\}$ and $a_1, \dots, a_p \in D \setminus \{0\}$. Then

$$\mathrm{End}\, \mathrm{R}_{b_1,\dots,b_q,b_q}^{a_1,\dots,a_p} = \mathrm{End}\, \mathrm{R}_{b_1,\dots,b_q}^{a_1,\dots,a_p}$$

$$\mathrm{End}\, \mathrm{R}_{b_1,\dots,b_q}^{a_1,\dots,a_p,a_p} = \mathrm{End}\, \mathrm{R}_{b_1,\dots,b_q}^{a_1,\dots,a_p}$$

$0 < a < n-1$



$\operatorname{End} R_{n-2}^{n-1}$

$\operatorname{End} R_{(n-2,n-3)}^{n-1}$

$\operatorname{End} R_{(n-2,n-3,n-4)}^{n-1}$

$c_a^{(1)}(x) = a$

$$f_a^+(x) := \begin{cases} a+1 & \text{if } x = a \\ x & \text{otherwise.} \end{cases}$$

$$f_a^-(x) := \begin{cases} a-1 & \text{if } x = a \\ x & \text{otherwise.} \end{cases}$$

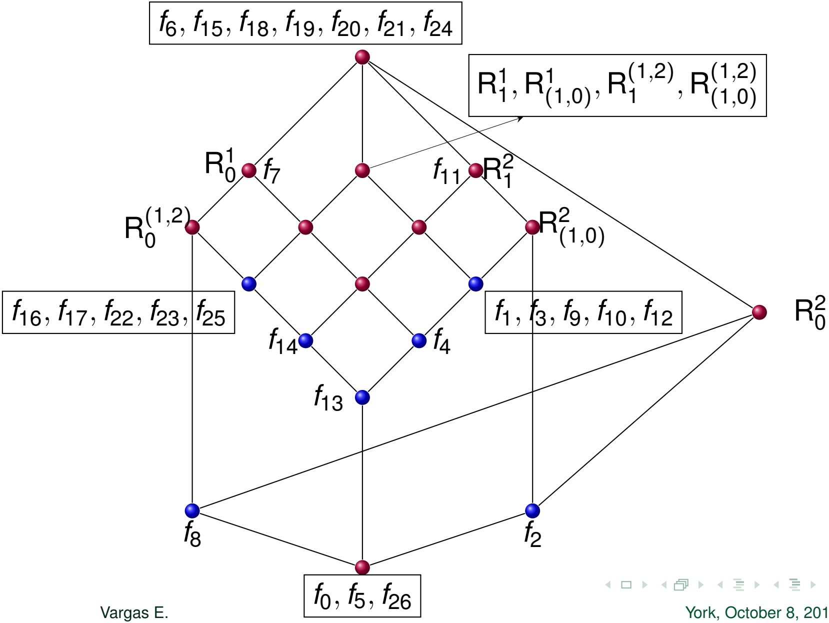
$\operatorname{End} CR_D = \left\{ c_0^{(1)}, c_{n-1}^{(1)}, id \right\}$

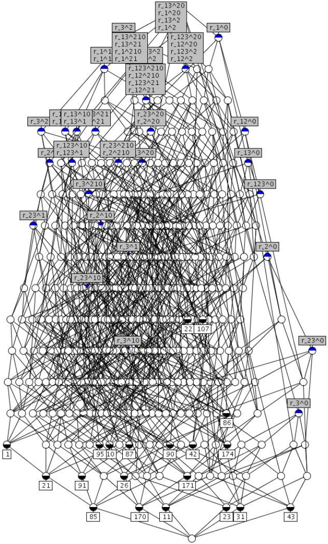
$D = \{0, 1, 2\}$

	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
1	0	0	0	1	1	1	2	2	2	0	0	0	1	1
2	0	1	2	0	1	2	0	1	2	0	1	2	0	1

	f_{14}	f_{15}	f_{16}	f_{17}	f_{18}	f_{19}	f_{20}	f_{21}	f_{22}	f_{23}	f_{24}	f_{25}	f_{26}
0	1	1	1	1	2	2	2	2	2	2	2	2	2
1	1	2	2	2	0	0	0	1	1	1	2	2	2
2	2	0	1	2	0	1	2	0	1	2	0	1	2

▷	R_0^1	R_0^2	R_1^1	R_1^2	$R^2_{(1,0)}$	$R^1_{(1,0)}$	$R_0^{(1,2)}$	$R_1^{(1,2)}$	$R^{(1,2)}_{(1,0)}$
$(0,0,0) = f_0$	×	×	×	×	×	×	×	×	×
$(0,0,1) = f_1$			×	×	×	×		×	×
$(0,0,2) = f_2$		×		×	×				
$(0,1,0) = f_3$			×	×	×	×		×	×
$(0,1,1) = f_4$	×		×	×	×	×		×	×
$(0,1,2) = f_5$	×	×	×	×	×	×	×	×	×
$(0,2,0) = f_6$									
$(0,2,1) = f_7$	×								
$(0,2,2) = f_8$	×	×					×		
$(1,0,0) = f_9$			×	×	×	×		×	×
$(1,0,1) = f_{10}$			×	×	×	×		×	×
$(1,0,2) = f_{11}$				×					
$(1,1,0) = f_{12}$			×	×	×	×		×	×
$(1,1,1) = f_{13}$	×		×	×	×	×	×	×	×
$(1,1,2) = f_{14}$	×		×	×		×	×	×	×
$(1,2,0) = f_{15}$									
$(1,2,1) = f_{16}$	×		×			×	×	×	×
$(1,2,2) = f_{17}$	×		×			×	×	×	×
$(2,0,0) = f_{18}$									
$(2,0,1) = f_{19}$									
$(2,0,2) = f_{20}$									
$(2,1,0) = f_{21}$									
$(2,1,1) = f_{22}$	×		×			×	×	×	×
$(2,1,2) = f_{23}$	×		×			×	×	×	×
$(2,2,0) = f_{24}$									
$(2,2,1) = f_{25}$	×		×			×	×	×	×
$(2,2,2) = f_{26}$	×	×	×	×	×	×	×	×	×


$f_6, f_{15}, f_{18}, f_{19}, f_{20}, f_{21}, f_{24}$
$R^1_1, R^1_{(1,0)}, R^{(1,2)}_1, R^{(1,2)}_{(1,0)}$
R^1_0
f_7
f_{11}
R^2_1
$R^{(1,2)}_0$
$R^2_{(1,0)}$
$f_{16}, f_{17}, f_{22}, f_{23}, f_{25}$
$f_1, f_3, f_9, f_{10}, f_{12}$
R^2_0
f_{14}
f_4
f_{13}
f_8
f_2
f_0, f_5, f_{26}


r_13^20
r_1^20
r_13^2
r_1^2
r_3^2
r_1^0
r_13^210
r_13^21
r_1^210
r_1^21
r_123^20
r_12^20
r_123^2
r_12^2
r_123^210
r_12^210
r_123^21
r_12^21
r_13^10
r_13^1
r_23^20
r_2^20
r_12^0
r_123^10
r_123^1
r_23^210
r_2^210
r_13^0
r_3^210
r_123^0
r_23^1
r_2^10
r_3^1
r_2^0
r_23^10
22
107
r_3^10
r_23^0
r_3^0
86
1
95
10
87
90
42
174
21
91
26
171
85
170
11
23
31
43

Final Clause

Thank $\wedge$ ($\neg$me) $\wedge$ for $\wedge$ your $\wedge$ ($\neg$inattention).

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