Binary Relations, Algebras, Games

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March 11, 2015

Binary Relations

Special cases:

- unary functions (partial or total), linear transformations,
- injections,
- surjections,
- permutations.

Constants and Operations

For functions

$$0, 1', \cdot, ;, D, R$$

For relations, also

$$1, +, -, \smile, *$$

E.g. Permutations

$$(Perms, 1', {}^{\smile}, ;) \rightsquigarrow \text{groups}$$

Every group is isomorphic to a set of permutations with identity, converse, composition.
Every set of permutations with identity, closed under converse and composition forms a group.

Classical Representations

Algebra $\mathcal{A} = (A, ops)$. Let X be a class of relations, e.g. total functions.
A *representation of type* X is injection $h : A \to \wp(D \times D) \cap X$ respecting operations
E.g.

$$\begin{aligned}(x, y) \in h(a; b) &\iff \exists z((x, z) \in h(a) \wedge (z, y) \in h(b)) \\ (x, y) \in h(1') &\iff x = y\end{aligned}$$

$\mathbf{R}_X(ops) = \{\mathcal{A} : \exists$representation of type X of $\mathcal{A}\}$.

Problems

- $\exists$ finite set of axioms $\mathcal{A} \models \Sigma \iff \mathcal{A} \in \mathbf{R}_X(ops)$?

- Is it decidable whether a finite $\mathcal{A}$ is in $\mathbf{R}_X(ops)$?

- If $\mathcal{A} \in \mathbf{R}_X(ops)$ is finite, does it have a representation on a finite base?

Relation Algebra [Tarski 1940s]

$$\mathcal{A} = (A, 0, 1, +, -, 1', ^{\smile}, ;)$$

- $(A, 0, 1, +, -)$ is a boolean algebra
- $(A, 1', ^{\smile}, ;)$ is an involuted monoid
- additive operators
- triangle law $a;b \cdot c = 0 \iff a^{\smile};c \cdot b = 0$

Examples

Type of rep.	Operators	Axioms	FRP	Decidable
Perms	$\{1', \smile, ;\}$	Group	Yes	Yes
Funcs/Rels	$\{;\}$	Assoc.	Yes	Yes
Funcs/Rels	$\{1', ;\}$	Monoid	Yes	Yes
Relations	$\{0, 1, +, -\}$	BA	Yes	Yes
Injections	$\{D, R, ;\} \subseteq S \subseteq \{D, R, 0, 1', \cdot, ;\}$	∞	No	No
Relations	$\{+, \cdot, 1', ;\} \subseteq S \subseteq RA$	∞	No	No
	$\{\cdot, \smile, ;\} \subseteq S \subseteq RA$			
	$\{+, \cdot, ;\} \subseteq S \subseteq RA \setminus \{\smile\}$			
	$\{\leq, -, ;\} \subseteq S \subseteq RA \setminus \{\smile\}$			
Relations	$\{1', \cdot, ;\}$	∞	No	?
Relations	$\{-, ;\}$	?	?	?

Atom Structure

If boolean part is atomic (e.g. if $\mathcal{A}$ is finite)

- which atoms are below identity?
- converse of each atom?
- composition of each pair of atoms?

determines the operators.
For composition, list the *forbidden triples* $(a, b, c) : a; b \cdot c = 0$.

Representation of a Relation Algebra

$$\mathcal{A} = (A, 0, 1, +, -, 1', \smile, ;)$$

$$h : \mathcal{A} \to \wp(X \times X)$$

such that

$$\begin{array}{rcl} a \neq 0 & \Rightarrow & h(a) \neq \emptyset \;\; (h \text{ is 1-1}) \\ h(0) & = & \emptyset \\ h(a+b) & = & h(a) \cup h(b) \\ h(-a) & = & h(1) \setminus h(a) \\ h(1') & = & \{(x,x) : x \in X\} \\ (x,y) \in h(a^\smile) & \iff & (y,x) \in h(a) \\ (x,y) \in h(a;b) & \iff & \exists z \, [(x,z) \in h(a) \wedge (z,y) \in h(b)] \end{array}$$

In a square representation $h(1) = X \times X$.

Point Algebra (temporal reasoning)

3 *atoms* $1', L, G$ (so 8 elements)

$;$	$1'$	L	G
$1'$	$1'$	L	G
L	L	L	1
G	G	1	G

where $1 = 1' + L + G,\ (1')^{\smile} = 1',\ L^{\smile} = G,\ G^{\smile} = L$.

Representation over $\mathbb{Q}$.

$$h(L) = \{(q, r) : q < r\}$$

Outline of rest of talk

- How can you tell if a relation algebra is representable?
- Two player games to test representability.
- Obtaining first-order axioms from the games.
- Constructing relation algebras with required properties.

Characterising representability

Can consider various types of representations: classical, relativized, complete, etc. One approach: find first-order theory (or better, an equational theory) Δ such that

$$\mathcal{A} \models \Delta \iff \mathcal{A} \text{ has approp. rep.}$$

This may or may not be possible, and it is almost always fearsomely difficult.

Characterising representability by games

Our approach: devize two player game G such that

$$\exists \text{ has a w.s. in } G(\mathcal{A}) \iff \mathcal{A} \text{ has an approp. rep.}$$

Actually, in many cases we can use these games to obtain first-order theories as above.

Abelarde and Héloïse

Representation — Finite Algebra Case

$(x, y) \in h(1) \;\Rightarrow\; \exists!$ atom $\alpha(x, y) \in h(\alpha)$.
If h is a square, we can define a labelled graph (X, λ) by

$$\begin{aligned} \lambda &: X \times X \to At(\mathcal{A}) \\ \lambda(x, y) &= \bigwedge\{a \in \mathcal{A} : (x, y) \in h(a)\} \end{aligned}$$

Conversely, if $\lambda : X \times X \to \mathsf{At}(\mathcal{A})$ satisfies

$$\begin{aligned} \lambda(x, y) \leq 1' &\iff x = y \\ \lambda(x, y)^{\smile} &= \lambda(y, x) \\ \lambda(x, z); \lambda(z, y) &\geq \lambda(x, y) \end{aligned}$$

and for all atoms $\alpha, \beta \in \mathsf{At}(\mathcal{A})$,

$$\lambda(x, y) \leq \alpha; \beta \Rightarrow \exists z\, [\lambda(x, z) = \alpha \wedge \lambda(z, y) = \beta]$$

then λ defines a square representation h, by

$$h(a) = \{(x, y) : a \geq \lambda(x, y)\}$$

Atomic $\mathcal{A}$-network: $N = (X, \lambda)$

$$\lambda : X \times X \to At(\mathcal{A})$$

satisfies

$$\begin{array}{rcl} \lambda(x,y) \leq 1' & \iff & x = y \\ \lambda(x,y)^{\smile} & = & \lambda(y,x) \\ \lambda(x,z);\lambda(z,y) & \geq & \lambda(x,y) \end{array}$$

But maybe there are nodes x, y and atoms a, b such that

$$\lambda(x,y) \leq a;b \text{ yet } \not\exists z\ [\lambda(x,z) = a \wedge \lambda(z,y) = b]$$

Then (x, y, a, b) is a *defect* of the atomic network.

Write N instead of X or λ.

Games on atomic $\mathcal{A}$-networks

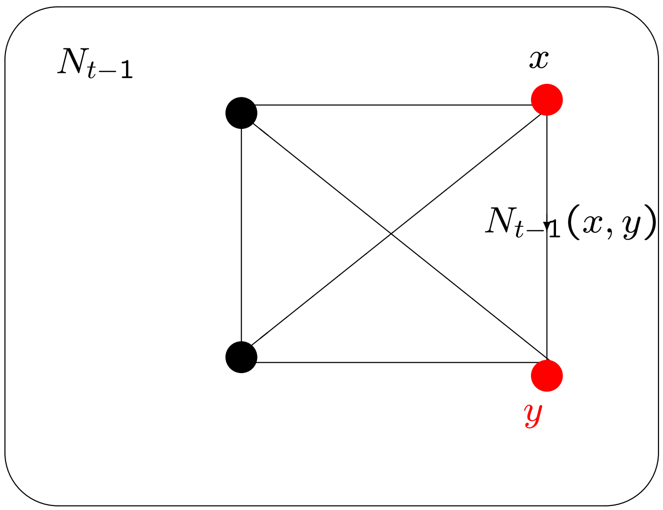
Two players: $\forall$ and $\exists$. The game $G_n(\mathcal{A})$ has n rounds (where $n \leq \omega$). A play of the game will be

$$N_0 \subseteq N_1 \subseteq \ldots \subseteq N_{t-1} \subseteq N_t \subseteq \ldots \quad (t < n)$$

Round 0:

- $\forall$ picks $a_0 \in \mathsf{At}\,\mathcal{A}$.
- $\exists$ plays an atomic network N_0 with a_0 occurring as a label in it.

Round t *(*$1 \leq t < n$*):* Suppose that the current atomic network at the start of the round is N_{t-1}. Play goes as follows:



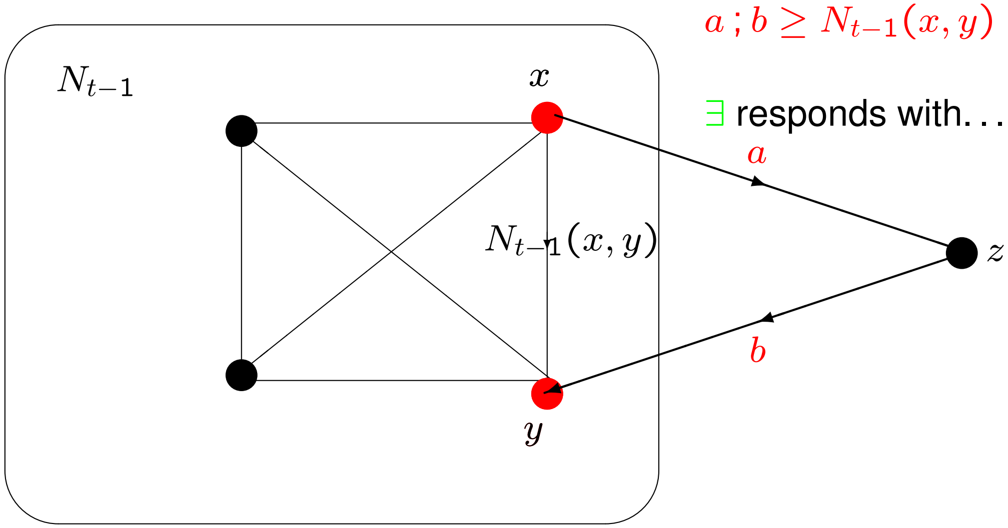
Round t of $G_n(\mathcal{A})$

$\forall$ picks $x, y \in N_{t-1}$ and $a, b \in \mathsf{At}(\mathcal{A})$ with $a ; b \geq N_{t-1}(x, y)$

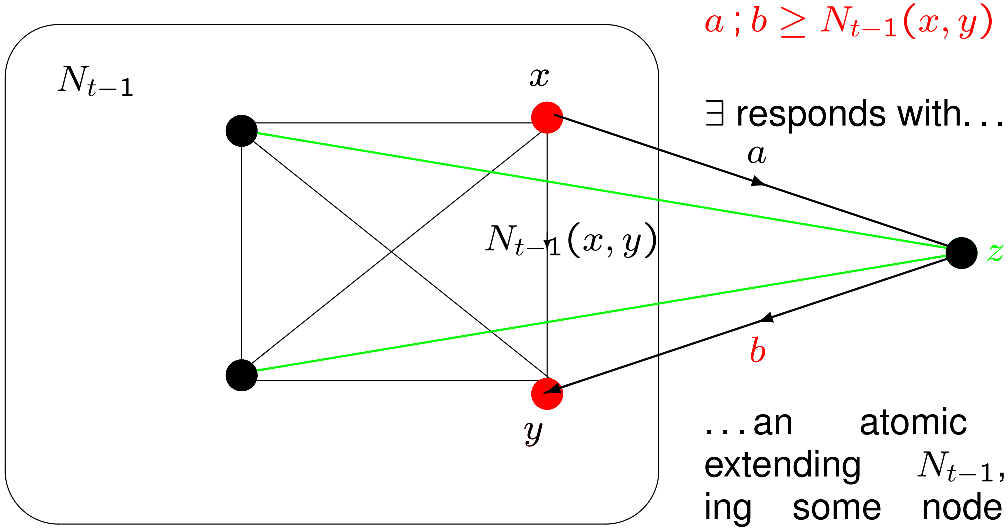
Round t of $G_n(\mathcal{A})$

$\forall$ picks $x, y \in N_{t-1}$ and $a, b \in \mathsf{At}(\mathcal{A})$ with $a \,;\, b \geq N_{t-1}(x, y)$

$\exists$ responds with. . .



Round t of $G_n(\mathcal{A})$



$\forall$ picks $x, y \in N_{t-1}$ and $a, b \in \mathsf{At}(\mathcal{A})$ with $a \,;\, b \geq N_{t-1}(x, y)$

$\exists$ responds with. . .

. . . an atomic network N_t, extending N_{t-1}, & containing some node z such that $N_t(x, z) = a,\ N_t(z, y) = b$

Who wins?

In any round, if $\exists$ cannot play, or if she plays a labelled graph that fails to be an atomic network, then $\forall$ wins.

If $\exists$ plays a legitimate atomic network in each round then she wins.

Characterising representability for finite RAs, by games

Theorem 1 *Let $\mathcal{A}$ be a finite relation algebra.*

1. *$\mathcal{A} \in$ **RRA** iff $\exists$ has a winning strategy in $G_\omega(\mathcal{A})$.*

2. *$\exists$ has a winning strategy in $G_\omega(\mathcal{A})$ iff she has one in $G_n(\mathcal{A})$ for all finite n.*

3. *One can construct first-order sentences σ_n for $n < \omega$ (independently of $\mathcal{A}$) such that $\mathcal{A} \models \sigma_n$ iff $\exists$ has a winning strategy in $G_n(\mathcal{A})$.*

Conclude that for a finite relation algebra $\mathcal{A}$,

$$\mathcal{A} \in \textbf{RRA} \iff \mathcal{A} \models \{\sigma_n : n < \omega\}.$$

The axioms σ_n (sketch)

Given an atomic network N, and $k < \omega$, we write an axiom $\tau_k(N)$ saying that $\exists$ *can win* $G_k(\mathcal{A})$ *starting from* N*.* We go by induction on k. All quantifiers are implicitly relativised to atoms.

$$\begin{aligned}\tau_0(N) \;=\; & \bigwedge_{x \in N} \Big(N(x,x) \leq 1' \\ & \wedge \textstyle\bigwedge_{y \in N \setminus \{x\}} N(x,y) \not\leq 1' \Big) \\ & \wedge \bigwedge_{x,y \in N} N(x,y) = N(y,x)^{\smile} \\ & \wedge \bigwedge_{x,y,z \in N} N(x,y) \leq N(x,z) \,;\, N(z,y).\end{aligned}$$

$$\begin{aligned}\tau_{k+1}(N) \;=\; & \bigwedge_{x,y \in N} \forall a,b \Big(N(x,y) \leq a \,;\, b \to \exists N' \supseteq N \\ & \Big(\tau_k(N') \wedge \bigvee_{z \in N'} (N'(x,z) = a \\ & \wedge N'(z,y) = b) \Big) \Big).\end{aligned}$$

$$\sigma_k \;=\; \forall a_0 \exists N (\tau_{k-1}(N) \wedge \textstyle\bigvee_{x,y \in N} N(x,y) = a_0).$$

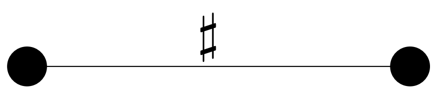
McKenzie's algebra

4 atoms: $1', <, >, \sharp$.

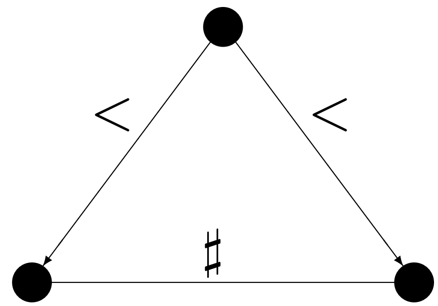
$$1'^{\smile} = 1', \quad <^{\smile} = >, \quad >^{\smile} = <, \quad \sharp^{\smile} = \sharp.$$

;	$<$	$>$	$\sharp$
$<$	$<$	1	$(< +\sharp)$
$>$	1	$>$	$(> +\sharp)$
$\sharp$	$(< +\sharp)$	$(> +\sharp)$	$-\sharp$

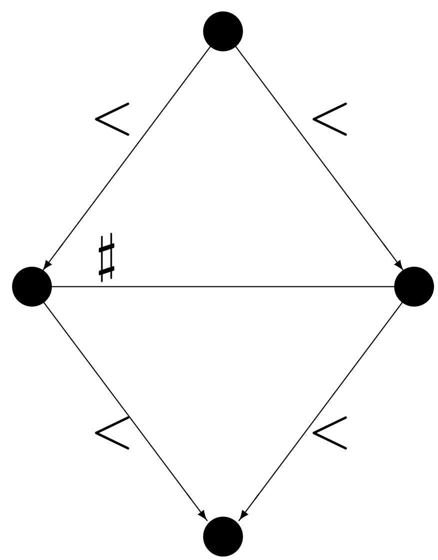
McKenzie's algebra



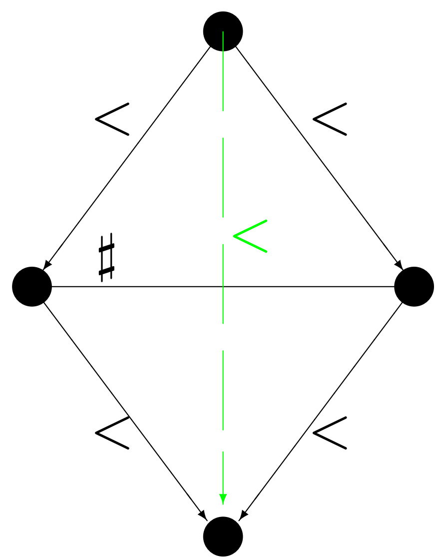
McKenzie's algebra



McKenzie's algebra



McKenzie's algebra



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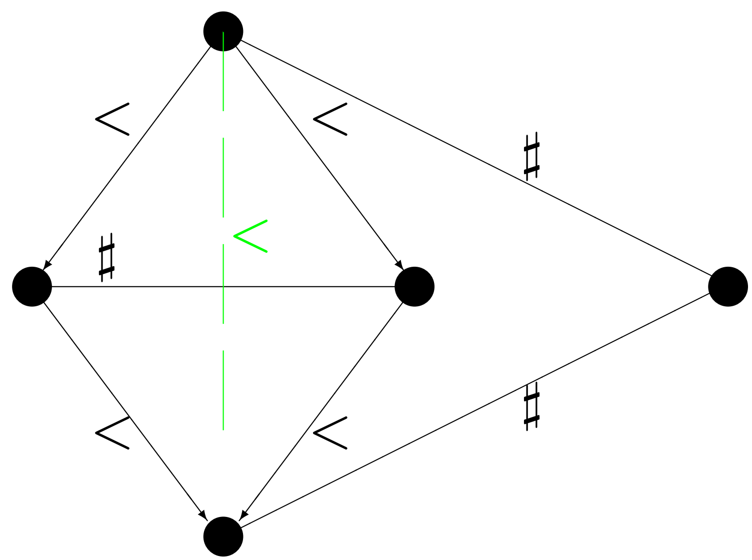

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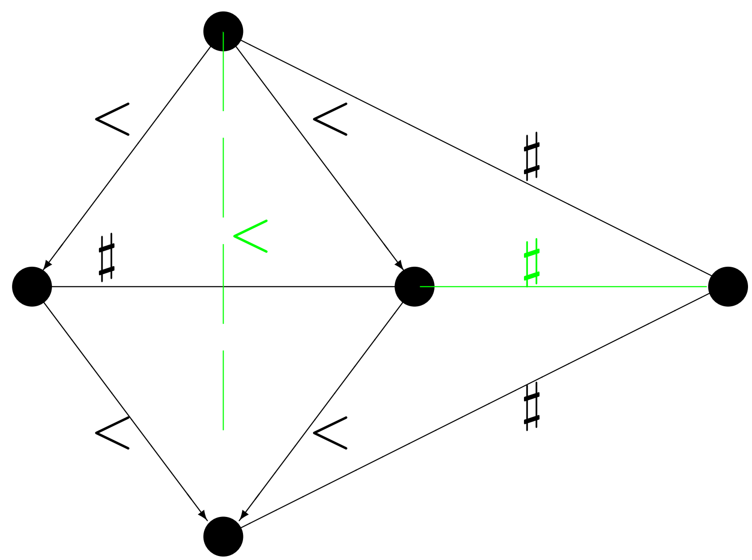
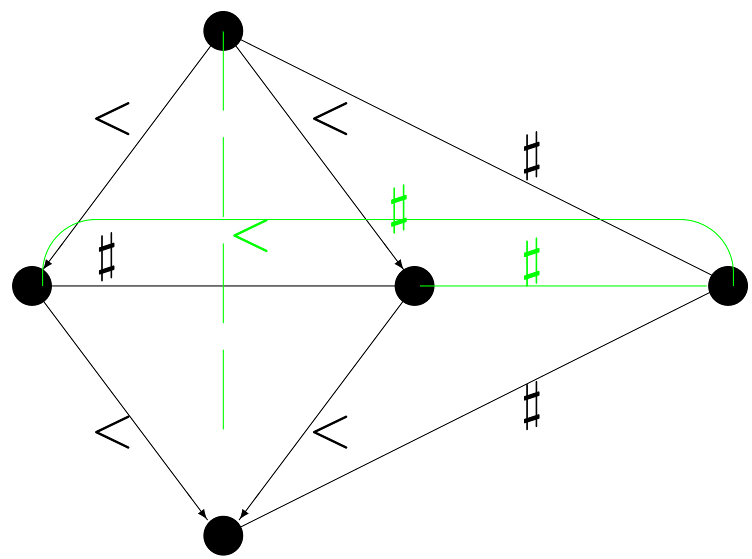


McKenzie's algebra



$\forall$ wins.

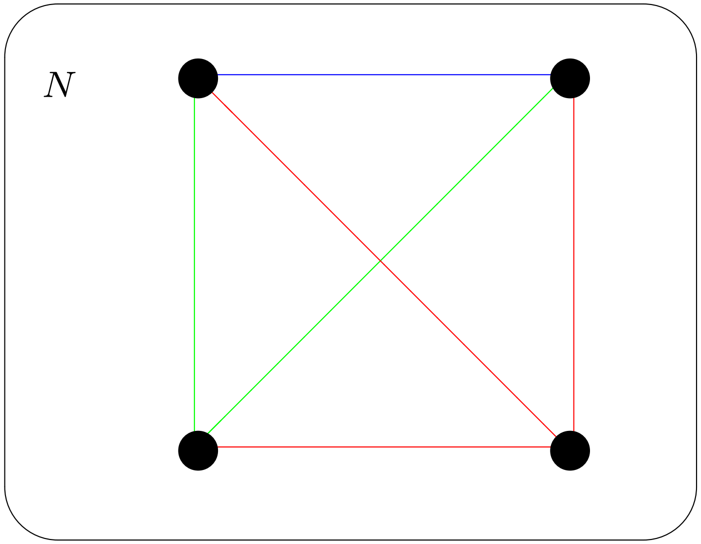
Maddux algebra

4 atoms: $1', r, b, g$.

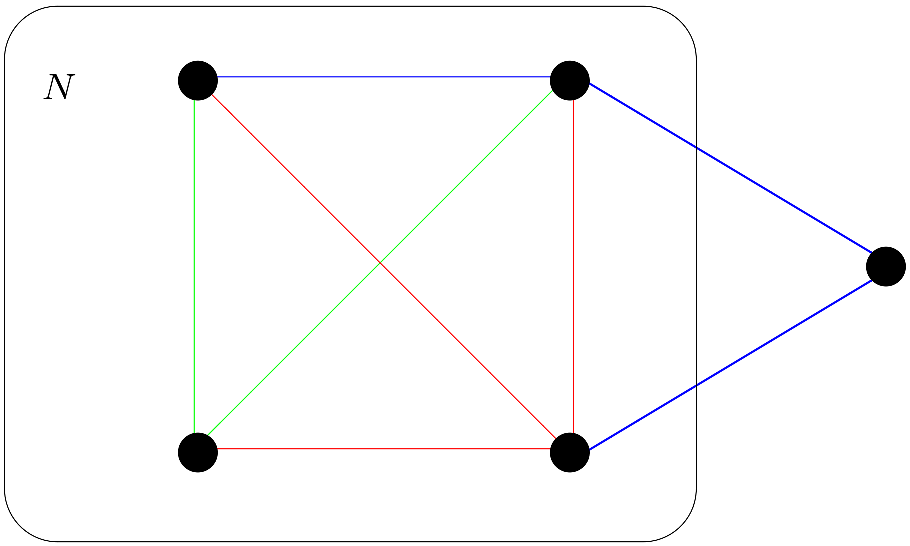
$x^{\smile} = x$ for all atoms x ('symmetric algebra')

All triples are consistent except Peircean transforms of:
$(1', a, a')$ for $a \neq a'$, and (r, b, g).

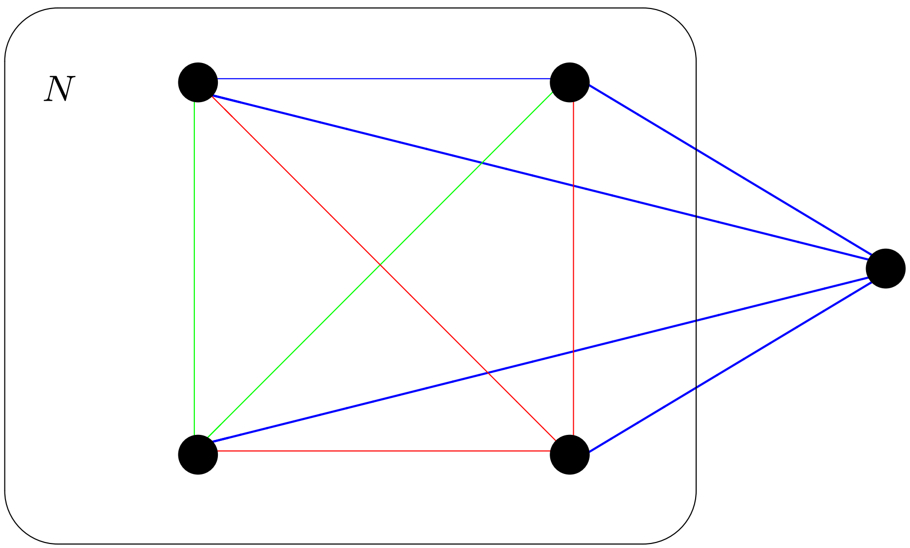
Maddux algebra ($\forall$'s first kind of move)



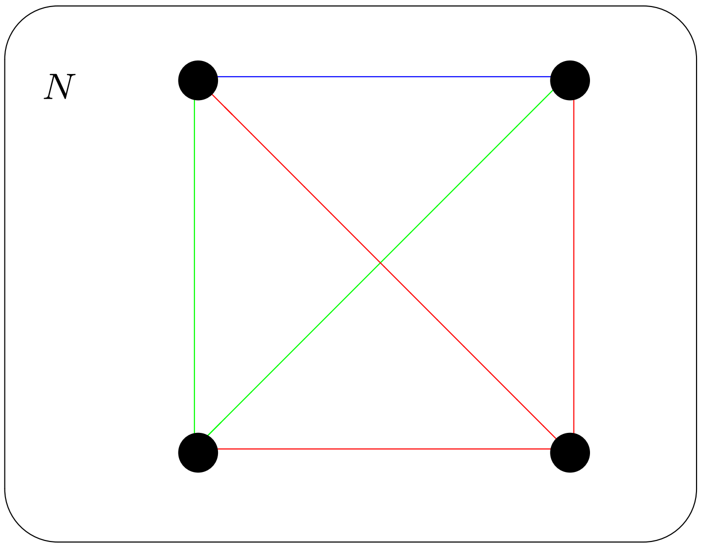
Maddux algebra (∀'s first kind of move)



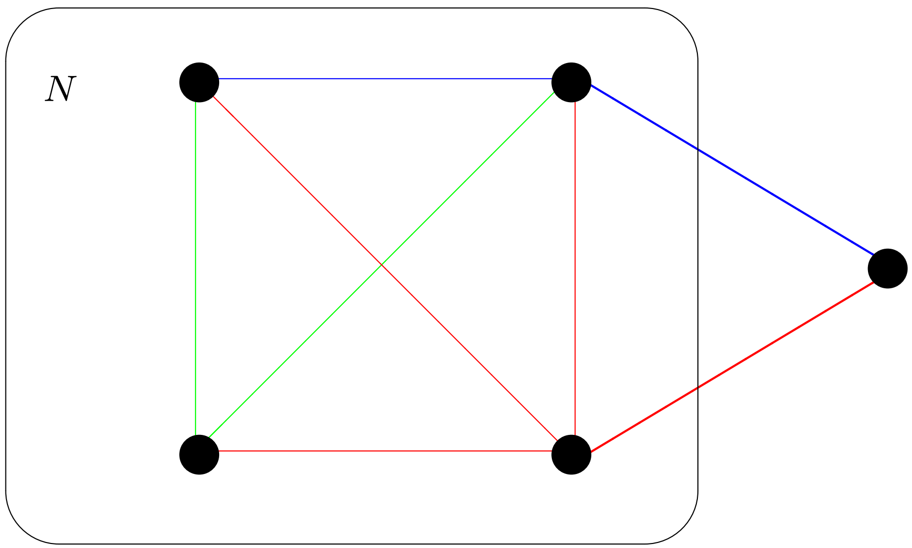
Maddux algebra (∀'s first kind of move)



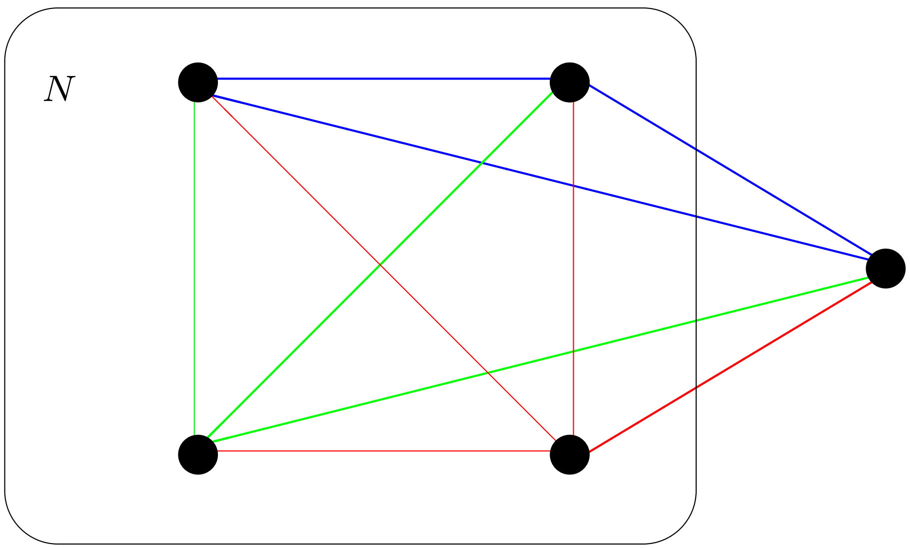
Maddux algebra ($\forall$'s second kind of move)



Maddux algebra ($\forall$'s second kind of move)



Maddux algebra ($\forall$'s second kind of move)



Hence

1. McKenzie's algebra $\mathcal{K} \notin$ **RRA**.

   So **RRA** $\subset$ **RA**, as Lyndon (1950) showed.

   In fact, $\mathcal{K}$ is one of the smallest non-representable relation algebras. All relation algebras with ≤ 3 atoms are representable.

2. The Maddux algebra $\mathcal{M} \in$ **RRA**.

   Exercise: show that if (X, λ) is any representation of $\mathcal{M}$, then X is infinite.

   This is perhaps surprising, given that $\mathcal{M}$ is symmetric.

Infinite Case

For infinite relation algebras there may not be atoms.
For atomic $\mathcal{A}$ with countably many atoms:

$$\exists \text{ has winning strategyin } G_\omega(\mathcal{A}) \iff \mathcal{A} \in \mathbf{CRA}.$$

Could define a slightly different game and get axiomatisation of **RRA**. Alternatively,

$$\mathcal{A} \in \textbf{RRA} \iff \mathcal{A}^+ \in \mathbf{CRA}$$

so to determine if $\mathcal{A}$ is representable, play the atomic game over the *canonical extension* $\mathcal{A}^+$.

Constructing Relation Algebras

We want to construct algebras $\mathcal{A}$ and we want to control who will win $G_n(\mathcal{A})$.

Ehrenfeucht–Fraïssé Game

Let A, B be structures in a binary signature (e.g. graphs). We can easily test whether positive existential properties of A hold in B or not — much easier than checking if an **RA** is representable.

$$\mathbf{EF}_r(A, B)$$

Game with r rounds ($r \leq \omega$).

Rules of $\mathsf{EF}_r(A, B)$

- $\forall$ has pebbles $\alpha_0, \alpha_1, \ldots$
- $\exists$ has corresponding pebbles $\beta_0, \beta_1, \ldots$.
- Initially $\forall$ places α_0 at some $a \in A$, $\exists$ must respond by picking $b \in B$ and placing β_0 at b.
- In each subsequent round $\forall$ can place a new pebble α_i on some $a_i \in A$, $\exists$ must choose $b_i \in B$ and place β_i at b_i.

- $\forall$ wins if $\alpha_i, \alpha_j, \beta_i, \beta_j$ are at a_i, a_j, b_i, b_j resp., $(a_i, a_j) \in r^A$ but $(b_i, b_j) \notin r^B$ (some binary predicate r).

- After r rounds, if $\forall$ hasn't won so far then $\exists$ is the winner.

- Can assume $\forall$ never puts two pebbles on same spot.

Rules of $\mathsf{EF}_r^p(A, B)$

- Similar, but each player has only p pebbles.

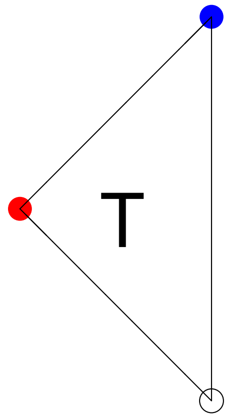
- After p rounds, $\forall$ must pick up a pebble in play and can re-use it ($\exists$ does the same).

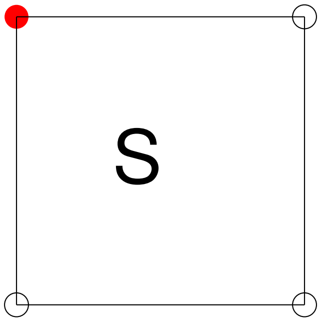
Example game

Example game

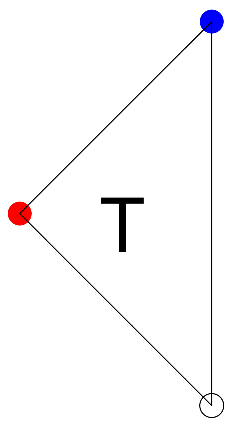
Example game

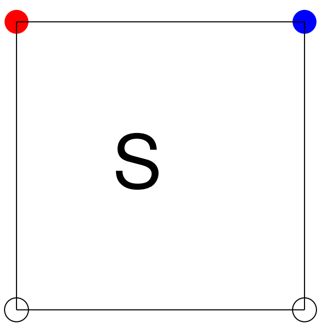
Example game





Example game





Example game

Example game

∀ wins.

But $\forall$ needs 3 turns with 3 different pebbles to win.

- $\forall$ has winning strategy in $\mathbf{EF}_3^3(T, S)$.
- $\exists$ has winning strategy in $\mathbf{EF}_r^2(T, S)$.

$\mathbf{EF}_\omega(A, B)$

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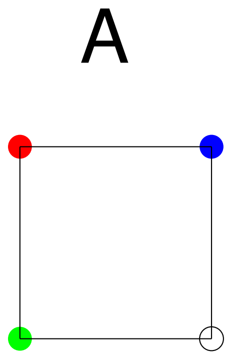
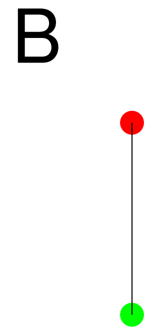


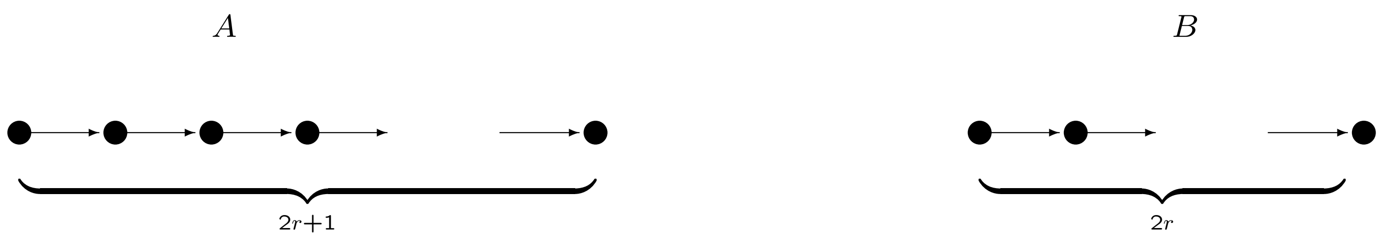


$\mathbf{EF}_\omega(A, B)$

$\mathbf{EF}_\omega(A, B)$

$\exists$ wins.

Third Example Game



- Successor relation.

- $\forall$ has winning strategy in $\mathsf{EF}^2_{r+1}(A, B)$.

- $\exists$ has winning strategy in $\mathsf{EF}^2_r(A, B)$.

- With three pebbles on (transitive) linear orders can do binary search — $\forall$ can win on linear orders of different lengths, $< 2^r$.

Fourth example game

$$A = \mathsf{K}_\omega, \quad B = \dot{\bigcup}_{n<\omega} \mathsf{K}_n$$

- $\forall$ wins $\mathbf{EF}_\omega(A, B)$, but
- $\exists$ wins $\mathbf{EF}_\omega^p(A, B)$ for any $p < \omega$ ($\exists$'s places all her pebbles in K_p).

Extra rule

Initial round changed. $\forall$ picks distinct $a_0, a_1 \in A$ and places α_0, α_1 at these points. $\exists$ responds by picking $b_0, b_1 \in B$ and placing β_0, β_1 there. This counts as two rounds (combined).

At any point, $\forall$ may remove pebbles as before, but he must always leave at least two distinct points of A covered.

Converting to RA

Idea: given binary structures A, B make **RA** $\mathcal{A}_{A,B}$ such that

$$\exists \text{ has w.s. in } \mathbf{EF}_r^p(A, B) \iff \exists \text{ has w.s. in } G_{1+r}^{2+p}(\mathcal{A}_{A,B})$$

Atoms

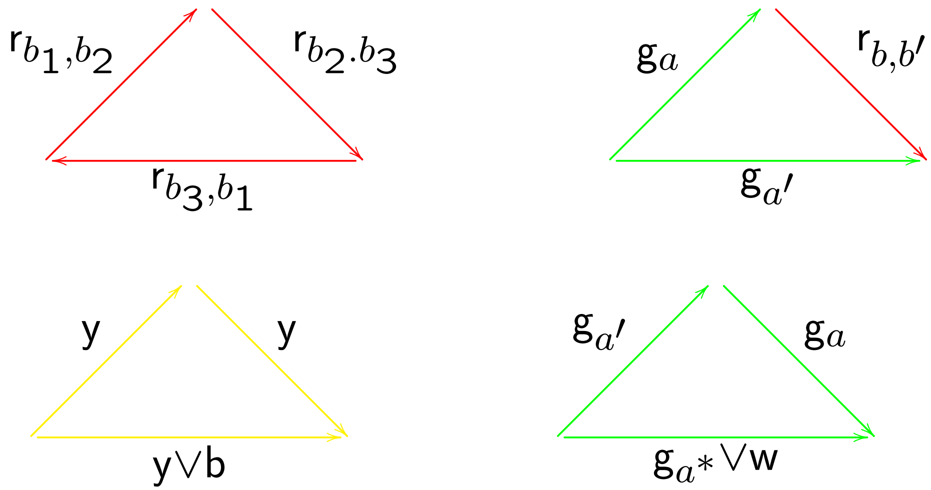
- $1', \mathsf{g}_a \ (a \in A), \ \mathsf{r}_{bb'} \ (b, b' \in B), \ \mathsf{y}, \mathsf{b}, \mathsf{w}$.

- All atoms self-converse, except $\breve{\mathsf{r}_{bb'}} = \mathsf{r}_{b'b}$.

Forbidden triangles

Forbid $(1', x, y)$ unless $x = y$.

indices match $\Leftarrow$

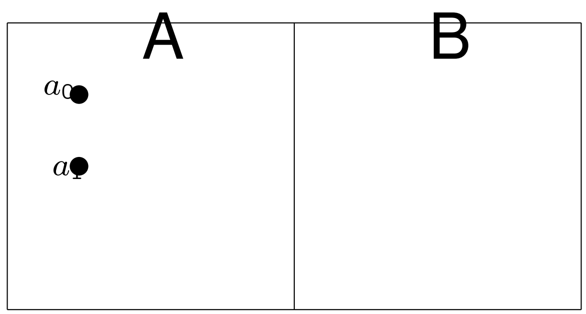


$\Rightarrow$ $\{(a, b), (a', b')\}$ is well-def. partial hom.

At this point we have

$$\forall \text{ has w.s. in } \mathbf{EF}_r^p(A, B) \Rightarrow \forall \text{ has w.s. in } G_{1+r}^{2+p}(\mathcal{A}_{A,B})$$

Correspondence between games.
$\exists$ wins $G_{1+r}^{2+p}(\mathcal{A}_{A,B}) \Rightarrow \exists$ wins $\mathsf{EF}_r^p(A, B)$



Correspondence between games.

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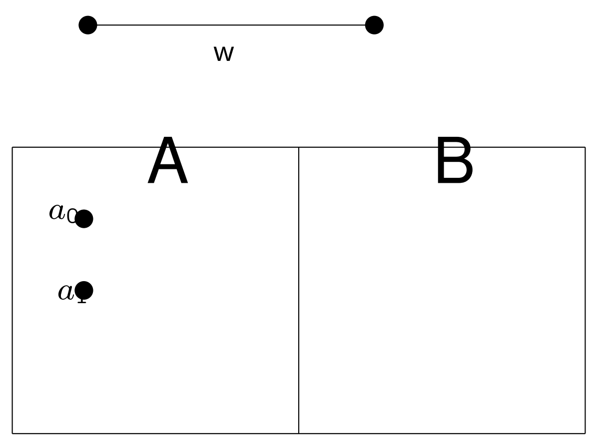
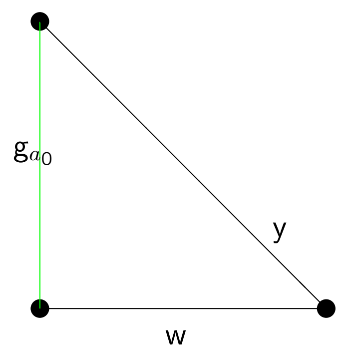
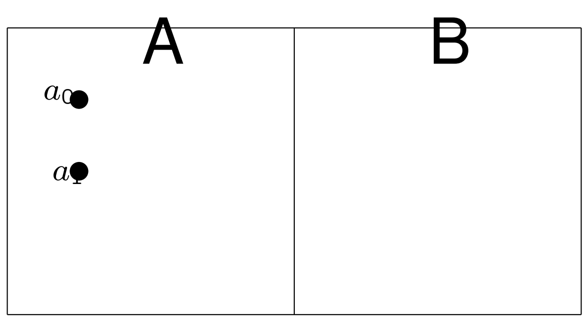


Correspondence between games. $\exists$ wins $G^{2+p}_{1+r}(\mathcal{A}_{A,B}) \Rightarrow \exists$ wins $\mathsf{EF}^p_r(A, B)$





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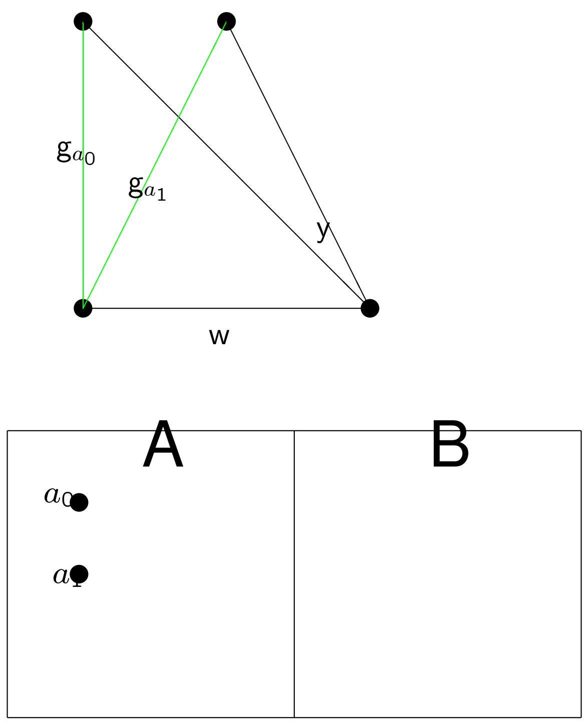
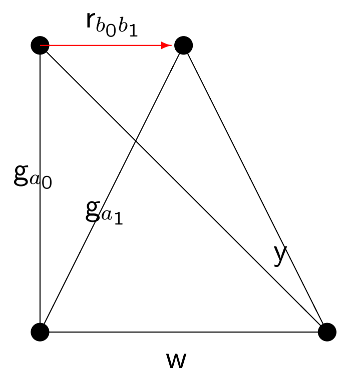
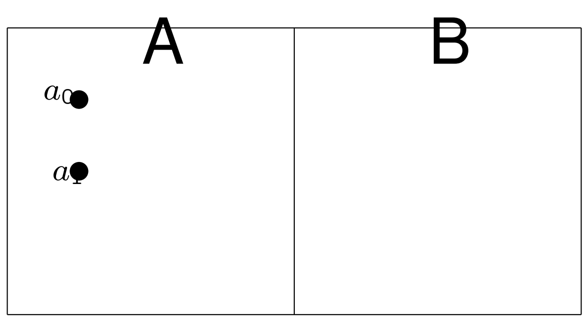
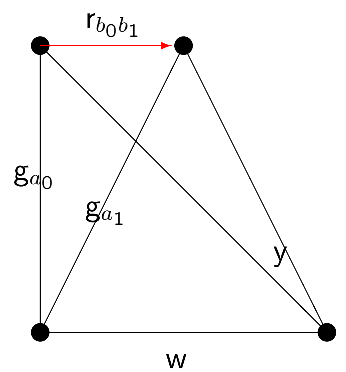


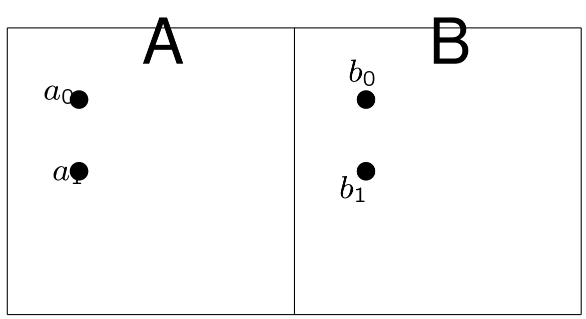
Correspondence between games. $\exists$ **wins** $G^{2+p}_{1+r}(\mathcal{A}_{A,B}) \Rightarrow \exists$ **wins** $\mathsf{EF}^p_r(A,B)$





Correspondence between games. $\exists$ wins $G_{1+r}^{2+p}(\mathcal{A}_{A,B}) \Rightarrow \exists$ wins $\mathsf{EF}_r^p(A,B)$





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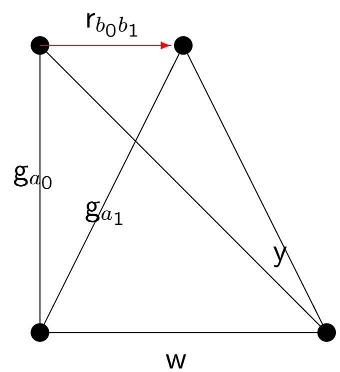

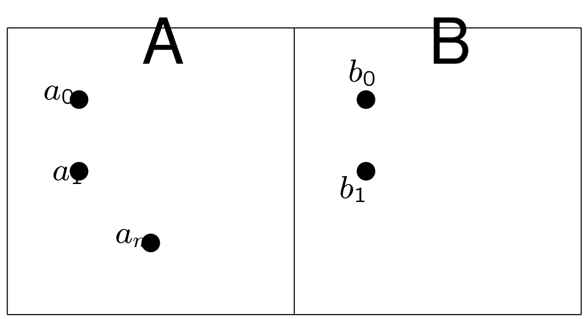

Correspondence between games.
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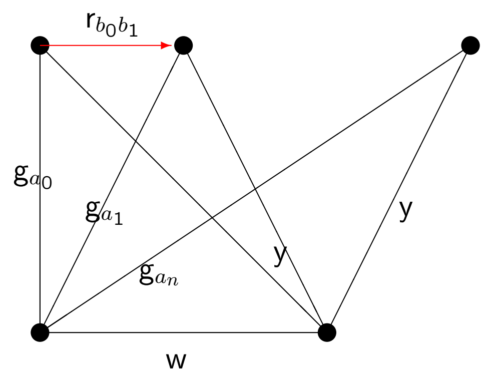

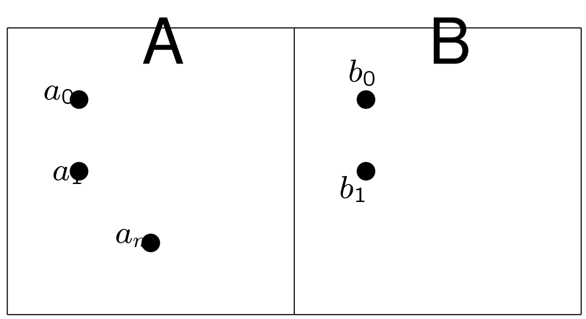

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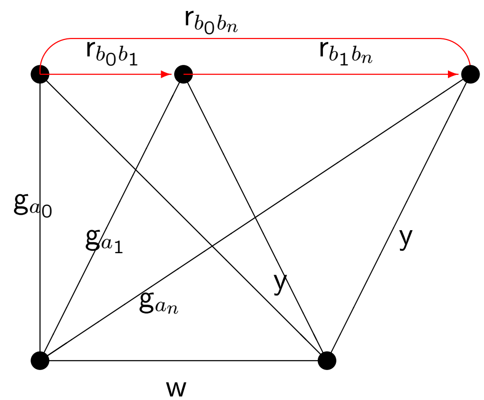

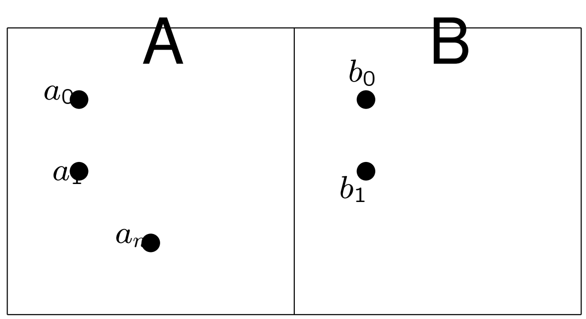

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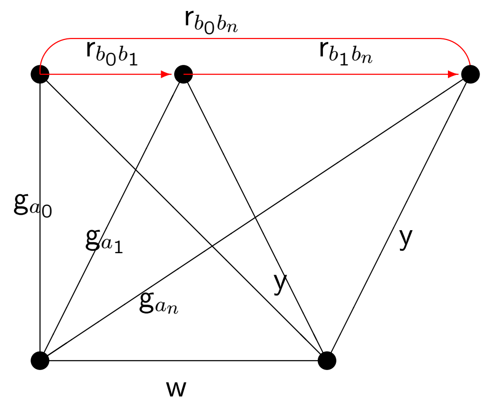

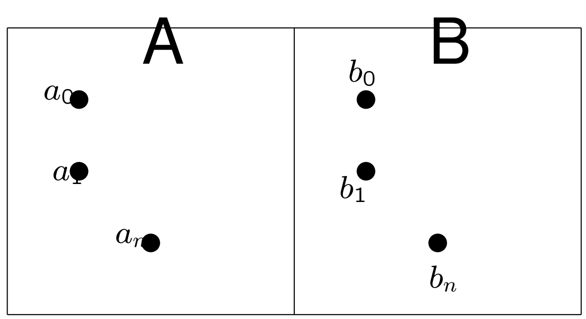

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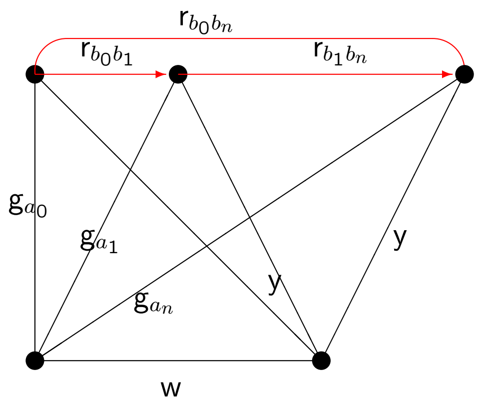

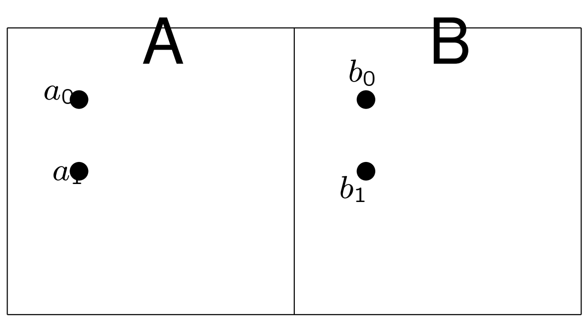

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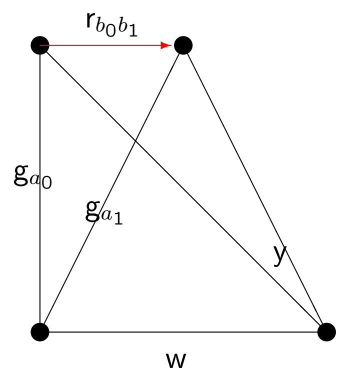

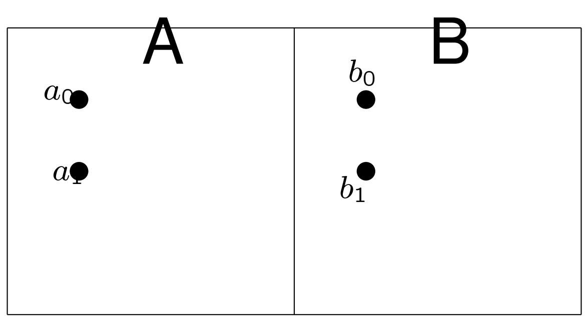

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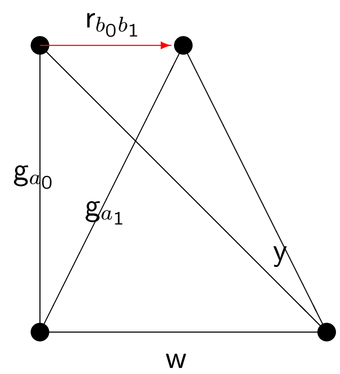
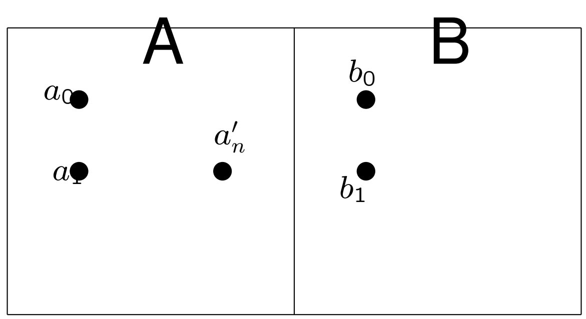
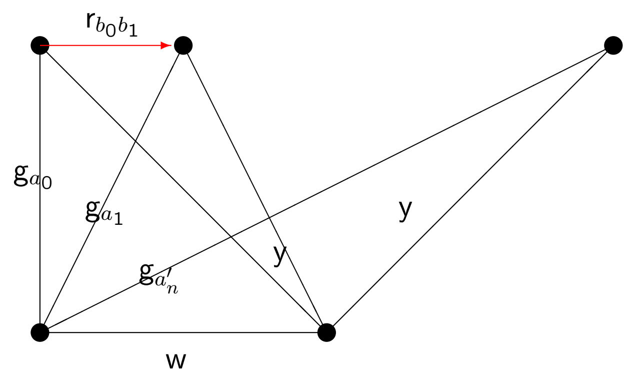
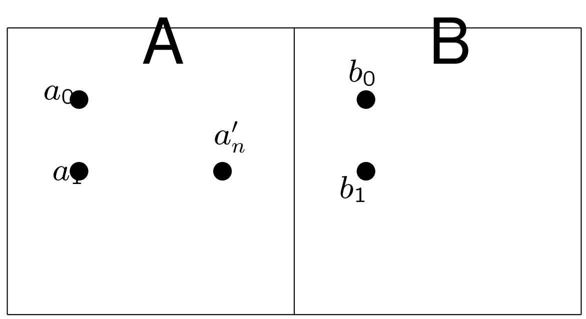




Correspondence between games. $\exists$ wins $G_{1+r}^{2+p}(\mathcal{A}_{A,B}) \Rightarrow \exists$ wins $\mathsf{EF}_r^p(A,B)$





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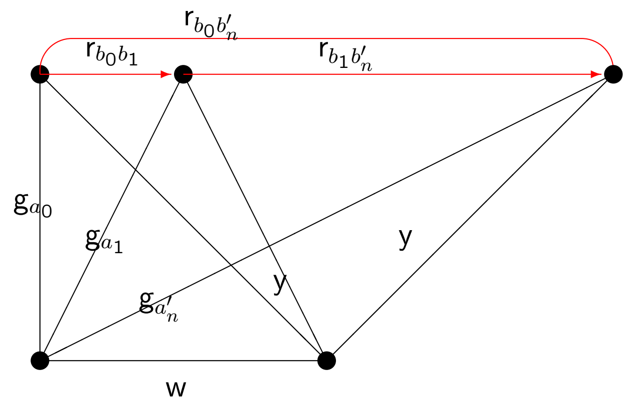

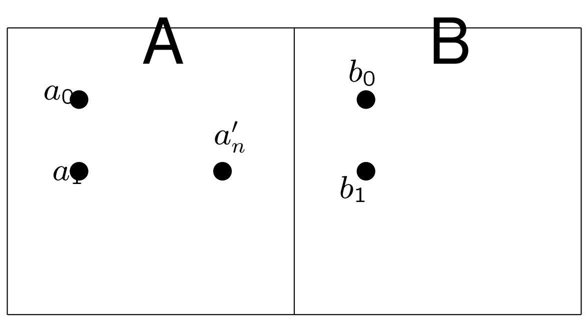

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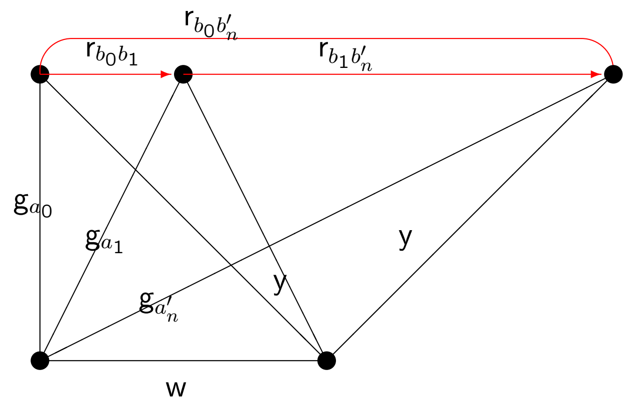
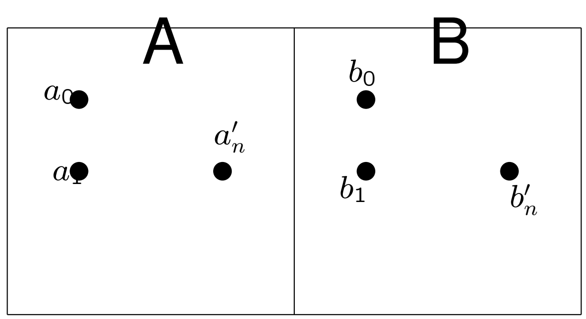
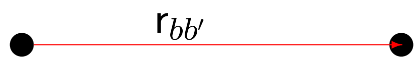




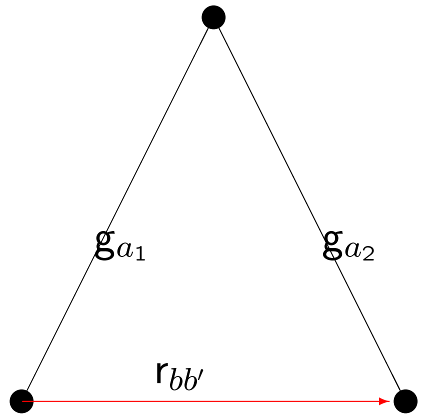
How $\exists$ can win $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$

$\exists$'s strategy will be to play white if possible, else black if possible, else red. But this isn't working.

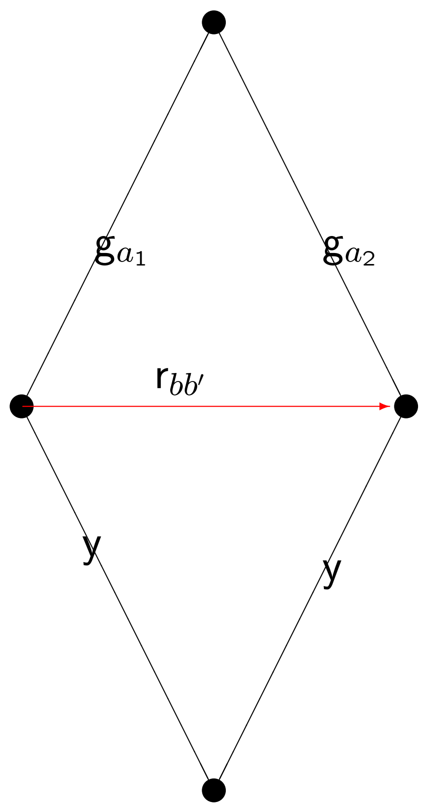
$\forall$ finds loophole



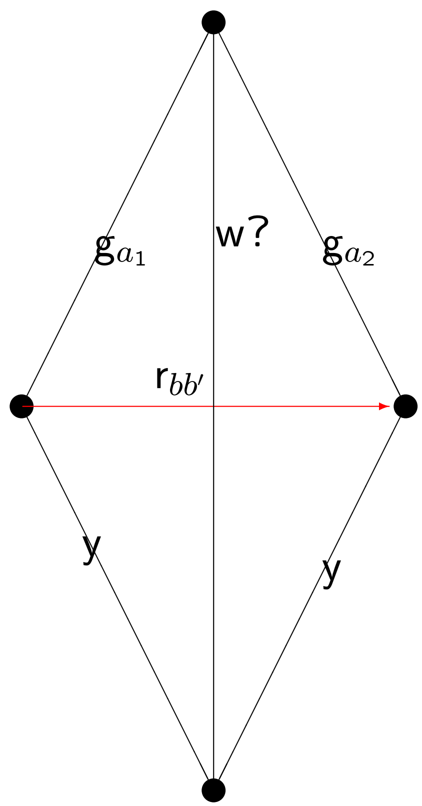
$\forall$ finds loophole



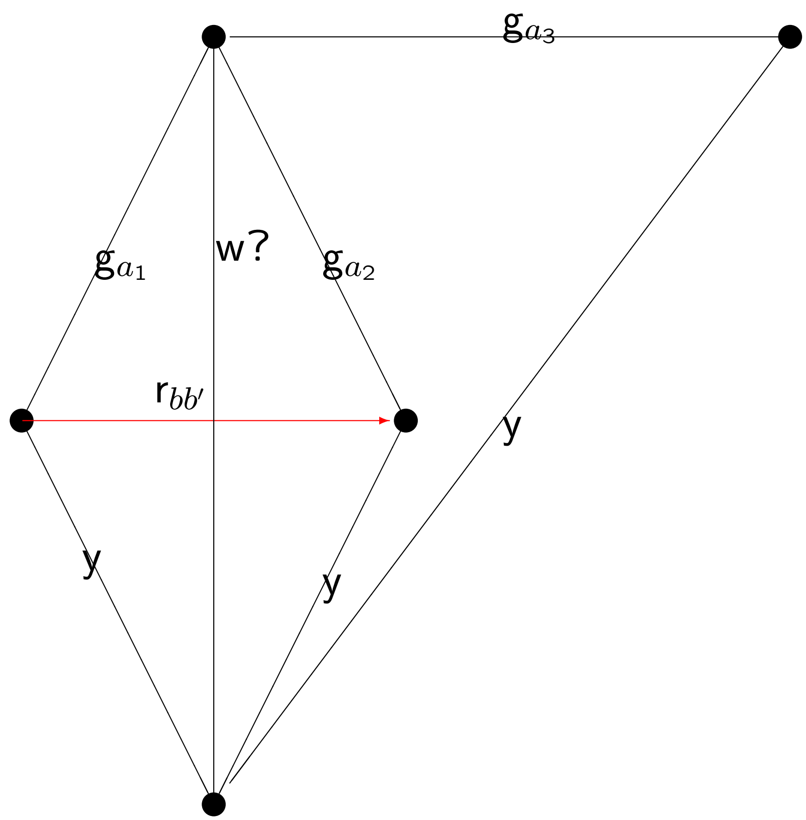
$\forall$ finds loophole



$\forall$ finds loophole



$\forall$ finds loophole



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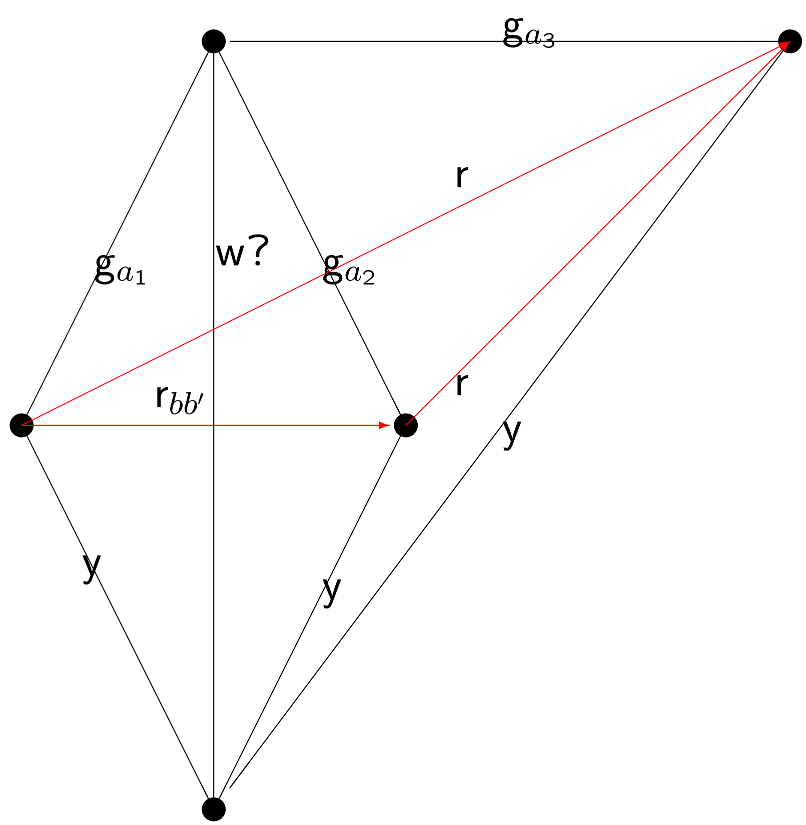


How to fix this

The idea was that $\exists$ could freely choose red atoms.

Don't want $\forall$ to choose red edge and then force a 'red clique' including that edge.

Final atoms to add:-

$$\mathsf{w}_S : S \subseteq A, \ |S| \leq 2$$

all self-converse.

Forbid

$$(\mathsf{w}_S, \mathsf{g}_a, \mathsf{y})$$

unless $a \in S$.

The atom structure in full

Atoms

$$1', \mathsf{g}_a, \mathsf{w}, \mathsf{w}_S, \mathsf{y}, \mathsf{b}, \mathsf{r}_{bb'} \quad : \; a \in A, \; S \subseteq A \; |S| \leq 2, \; b, b' \in B$$

All self-converse except $\breve{\mathsf{r}}_{bb'} = \mathsf{r}_{b'b}$.

Forbidden triples

PTs of

$(1', x, y)$	$x \neq y$
$(\mathsf{g}_a, \mathsf{g}_{a'}, \gamma)$	$a, a' \in A, \; \gamma$ is white or green
$(\mathsf{y}, \mathsf{y}, \mathsf{y}), (\mathsf{y}, \mathsf{y}, \mathsf{b})$	
$(\mathsf{r}_{b_0 b_1}, \mathsf{r}_{b'_1 b'_2}, \mathsf{r}_{b''_0 b''_2})$	unless $b_0 = b''_0, \; b_1 = b'_1, \; b'_2 = b''_2$
$(\mathsf{g}_a, \mathsf{g}_{a'}, \mathsf{r}_{bb'})$	if $(a, a') \in r^A$ but $(b, b') \notin r^B$
$(\mathsf{w}_S, \mathsf{g}_a, \mathsf{y})$	unless $a \in S$

We now have

$$\exists \text{ has winning strategy in } \mathrm{EF}_r^p(A, B)$$
$$\Updownarrow$$
$$\exists \text{ has winning strategy in } G_{1+r}^{2+p}(\mathcal{A}_{A,B})$$

RRA is not finitely axiomatisable

- Let $\mathcal{A}_n = \mathcal{A}_{\mathsf{K}_{n+1}, \mathsf{K}_n}$.

- $\forall$ has winning strategy in $\mathsf{EF}_{n+1}(\mathsf{K}_{n+1}, \mathsf{K}_n)$ so $\forall$ has winning strategy in $G_{n+2}(\mathcal{A}_n)$ and $\mathcal{A}_n \notin$ **RRA**.

- But $\exists$ has winning strategy in $\mathsf{EF}_n(\mathsf{K}_{n+1}, \mathsf{K}_n)$ so $\exists$ has winning strategy in $G_{n+1}(\mathcal{A}_n)$. So $\mathcal{A}_n \models \sigma_{n+1}$.

- Let $\mathcal{A} = \Pi_U \mathcal{A}_n$ be a non-principal ultraproduct. Then $\mathcal{A} \models \sigma_n$, all n. Hence $\mathcal{A} \in$ **RRA**.

- No finite axiomatisation of **RRA** exists.

CRA is not elementary

Let $A = \mathsf{K}_\omega,\ B = \dot{\bigcup}_{n<\omega} \mathsf{K}_n$.

$$\begin{aligned}
&\forall \text{ has winning strategy in } \mathsf{EF}_\omega(A, B)\\
&\Rightarrow \quad \forall \text{ has winning strategy in } G_\omega(\mathcal{A}_{A,B})\\
&\Rightarrow \quad \mathcal{A}_{A,B} \notin \mathbf{CRA}
\end{aligned}$$

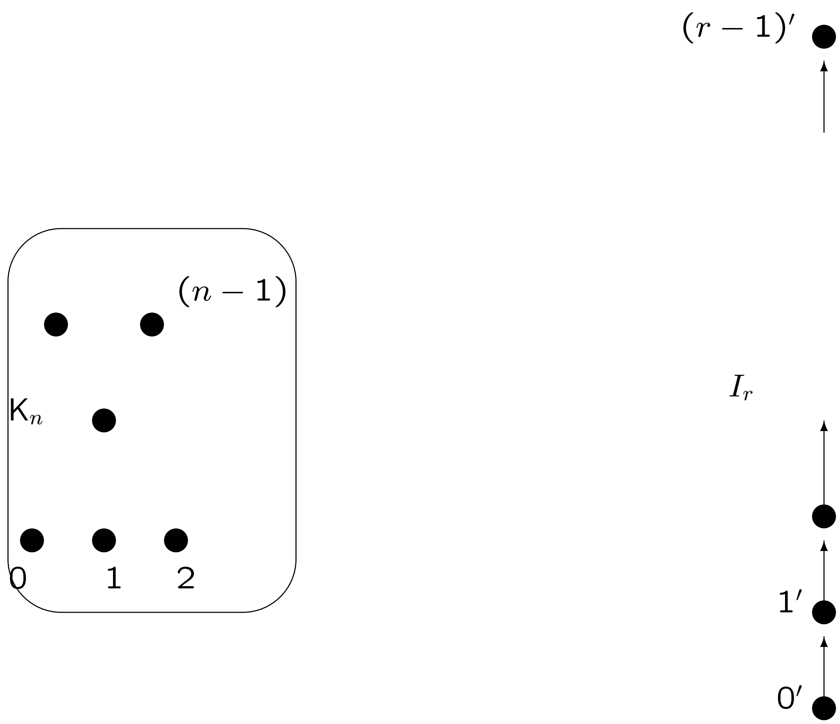
But

$$\begin{aligned}
&\exists \text{ has winning strategy in } \mathsf{EF}_n(A, B)\\
&\Rightarrow \quad \exists \text{ has winning strategy in } G_n(\mathcal{A}_{A,B})\\
&\Rightarrow \quad \mathcal{A}_{A,B} \models \sigma_n
\end{aligned}$$

Hence

$$\mathbf{CRA} \not\ni \mathcal{A}_{A,B} \equiv \Pi_U \mathcal{A}_{A,B} \succeq \mathcal{B} \in \mathbf{CRA}$$

RA_{n+1} not finitely axiomatisable over RA_n



K_n is complete irreflexive graph over $\{0, 1, \ldots, n-1\}$.
I_r is successor relation over $\{0', 1', \ldots, (r-1)'\}$.

A^n_r has nodes $n \cup r'$ and has edges

$$\{(i,j) : i \neq j < n\} \;\cup\; \{(i', (i+1)') : i < r\}$$
$$\cup\; \{(i,j'), (j',i) : i < n,\; j < r\}$$

Some corollaries

Rainbow construction produces relation algebras that we can use to prove:-

- Non-finite axiomatisability of **RRA** [Monk, 1964]

- Non-finite axiomatisability of the representation class of any sub-signature of **RA** including compostion, converse and intersection [Hodkinson Mikulas, 2000]

- No set of equations using a finite number of variables can define **RRA** [Jónsson, 1991]

- Class of completely representable relation algebras not closed under elementary equivalence.

- Can be extended to cover similar results for cylindric algebras.

Open Problems

- Is this decidable: does a given finite relation algebra have a representation on a finite base??

-

No k-variable first order axiomatisation of RRA?

Find two finite graphs A, B with $A \not\cong B$ but can't distinguish A, B using a k *colour* game.

Say A cannot embed in B. Then $\mathcal{A}_{A,B} \not\in$ **RRA** but $\mathcal{A}_{B,B} \in$ **RRA** and no k-variable formula distinguishes $\mathcal{A}_{A,B}$ from $\mathcal{A}_{B,B}$.

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