ON SOME FINITARY CONDITIONS ARISING FROM THE AXIOMATISABILITY OF CERTAIN CLASSES OF MONOID ACTS

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Abstract. This article considers those monoids S satisfying one or both of the finitary properties ($\mathbf{R}$) and ($\mathbf{r}$), focussing for the most part on inverse monoids. These properties arise from questions of axiomatisability of classes of S-acts, and appear to be of interest in their own right. If S *weakly right noetherian (WRN)*, that is, S has the ascending chain condition on right ideals, then certainly ($\mathbf{r}$) holds. Other than this, we show that ($\mathbf{R}$), ($\mathbf{r}$) and (WRN) are independent. Our most detailed results are for Clifford monoids, in which case we completely characterise those S with trivial structure homomorphisms satisfying ($\mathbf{R}$) or ($\mathbf{r}$).

1. Introduction

This article investigates finitary conditions for a monoid S arising from questions of axiomatisability of classes of right S-acts. The classes we consider are defined in terms of flatness properties. To explain the finitary conditions in question, we begin with a definition.

Definition 1.1. *If S is a monoid, $s, t \in S$, we define*

$$\mathbf{r}^S(s,t) = \{u : su = tu\} \text{ and } \mathbf{R}^S(s,t) = \{(u,v) : su = tv\}.$$

We usually suppress the superscript S unless there is danger of ambiguity. For convenience, we allow $\emptyset$ to be a right ideal of S and a subact of any right S-act. Then clearly, $\mathbf{r}(s,t)$ is a right ideal of S, and $\mathbf{R}(s,t)$ is an S-subact of the right S-act $S \times S$. We say that S satisfies ($\mathbf{r}$) (resp. ($\mathbf{R}$)) if each non-empty $\mathbf{r}(s,t)$ is finitely generated (resp. each non-empty $\mathbf{R}(s,t)$ is finitely generated).

The motivation for studying monoids satisfying ($\mathbf{r}$) and ($\mathbf{R}$) is given in the result below: here $\mathcal{SF}$ denotes the class of *strongly flat* right S-acts. It is known that a right S-act is strongly flat if and only if it satisfies interpolation conditions known as (E) and (P). We denote the class of right S-acts satisfying (E) (resp. (P)) by $\mathcal{E}$ (resp. $\mathcal{P}$). Further details

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of these concepts can be found in [2]. The following result is proven in [1] but stated in the form we require in [2].

Theorem 1.2. [2] *Let S be a monoid. Then*

(1) $\mathcal{E}$ is axiomatisable if and only if S satisfies $(\mathbf{r})$*;*
(2) $\mathcal{P}$ is axiomatisable if and only if S satisfies $(\mathbf{R})$*;*
(3) $\mathcal{SF}$ is axiomatisable if and only if S satisfies $(\mathbf{r})$ *and* $(\mathbf{R})$*.*

Certainly projective acts are strongly flat: discussion of the relationship between the S-act properties of Condition (P), strongly flat and projective can be found in, for example, [9] and [8].

A monoid S is *weakly right noetherian* (WRN) if every right ideal is finitely generated. If S is (WRN), then it certainly satisfies $(\mathbf{r})$, but is the converse true? Moreover, how does satisfaction of $(\mathbf{r})$ and $(\mathbf{R})$ affect the structure of the monoid? Such questions being too broad as posed, we focus here largely on the inverse case. Investigations into $(\mathbf{r})$ and $(\mathbf{R})$ were begun in [3] (where they were referred to as (FGr) and (FGR)). The Bicyclic semigroup must satisfy $(\mathbf{r})$ since it has (WRN). It is shown in [3] that the Bicyclic semigroup also satisfies $(\mathbf{R})$. On the other hand, if D is the extended Bicyclic semigroup, that is, $D = \mathbb{Z} \times \mathbb{Z}$ under the multiplication

$$(a, b)(c, d) = (a - b + \max\{a, b\}, d - c + \max\{a, b\}),$$

then D^1 satisfies neither (WRN) nor $(\mathbf{R})$, but does satisfy $(\mathbf{r})$. Preliminary investigations in [3] were also made into Clifford monoids. The complicated behaviour of $(\mathbf{r})$ and $(\mathbf{R})$ revealed in [3] prompted the investigations in this current article.

In Section 2 we consider closure properties of the classes of monoids satisfying $(\mathbf{r})$ and $(\mathbf{R})$. The classes are closed under finite direct product and retract, but not under taking substructures and homomorphic images.

Since some of the conditions on a semilattice Y of groups in Sections 4 and 5 are in terms of the corresponding conditions for Y, in Section 3 we consider semilattices satisfying $(\mathbf{r})$ and $(\mathbf{R})$. Whilst not true for an arbitrary monoid, any semilattice satisfying $(\mathbf{R})$ satisfies $(\mathbf{r})$. If a semilattice satisfies $(\mathbf{R})$ then it is finite above, hence a lattice. Conversely, any distributive lattice that is finite above satisfies $(\mathbf{R})$. We remark that as this is a paper about monoids, all our semilattices are *semilattice monoids*, that is, they have a greatest element.

We then turn our attention to Clifford monoids in Sections 4 and 5. Recall that an inverse semigroup is *Clifford* (that is, the idempotents of S are central) if and only if S is a (strong) semilattice Y of groups $G_\alpha, \alpha \in Y$. We will say that such an S has *trivial structure homomorphisms* if the structure homomorphisms $\varphi_{\alpha,\beta}$ where $\alpha > \beta$ are all trivial. Our most complete results are in the case where Y has a least element 0, or where the structure homomorphisms are trivial. If both of these conditions hold, then S satisfies $(\mathbf{R})$ if and only if $S \setminus G_0$ is finite.

For background details of the theory of acts over monoids, we refer the reader to [5].

2. (**R**), (**r**) AND CONSTRUCTIONS

In this section, we are going to investigate how the properties (**R**) and (**r**) behave with respect to certain universal algebraic operators. The following example shows that (**R**) and (**r**) are not preserved under taking submonoids and homomorphic images.

Example 2.1. *Let Y be the semilattice of finite subsets of $\mathbb{N}$ under union (that is, the free semilattice on $\mathbb{N}$), and let $T = Y \setminus \{\{1\}\}$. (Notice that T is both a monoid subsemilattice and homomorphic image of Y.) Then Y satisfies both* (**R**) *and* (**r**)*, however, T satisfies neither.*

Proof. That Y satisfies (**R**) and (**r**) follows from Lemmas 3.2 and 3.5. On the other hand,

$$\mathbf{r}^T(\{2\},\{1,2\}) = \{A \in T : \{1\} \subseteq A\},$$

and this ideal has infinitely many maximal elements ($\{1,n\}$ is maximal for all $n \neq 1$), so it cannot be finitely generated. Since T does not satisfy (**r**), by Lemma 3.2 neither can it satisfy (**R**). □

Definition 2.2. *Let S be a semigroup, and let $T \leq S$. We say that T is a* retract *of S if there exists a surjective homomorphism $\varphi\colon S \to T$ such that $\varphi^2 = \varphi$.*

Theorem 2.3. *If S is a monoid satisfying* (**R**) *(respectively,* (**r**)*), and T is a retract of S, then T also satisfies* (**R**) *(respectively,* (**r**)*).*

Proof. Since T is a retract of S, there exists a surjective homomorphism $\varphi\colon S \to T$ such that $\varphi^2 = \varphi$.

Let now $t, t' \in T$. Then $\mathbf{R}^S(t,t') = X \cdot S$ for some finite $X \subseteq S \times S$. We claim that $\mathbf{R}^T(t,t') = (X\varphi) \cdot T$, where

$$X\varphi = \{(u\varphi, v\varphi) : (u,v) \in X\}.$$

First note that if $(u,v) \in X$, then $tu = t'v$, thus

$$t(u\varphi) = t\varphi u\varphi = (tu)\varphi = (t'v)\varphi = t'(v\varphi),$$

so $(u\varphi, v\varphi) \in \mathbf{R}^T(t,t')$, that is, $X\varphi \subseteq \mathbf{R}^T(t,t')$, which implies that $(X\varphi) \cdot T \subseteq \mathbf{R}^T(t,t')$.

For the converse inclusion, let $(x,y) \in \mathbf{R}^T(t,t')$, that is, $tx = t'y$. This means that $(x,y) \in \mathbf{R}^S(t,t')$ also, so there exist $(u,v) \in X$ and $z \in S$ such that $(x,y) = (u,v)z$. However, in this case $(x,y) = (x\varphi, y\varphi) = (u\varphi, v\varphi)z\varphi$, that is, $(x,y) \in (X\varphi) \cdot T$.

The case of (**r**) is similar; if $\mathbf{r}^S(t,t') = X \cdot S$ for some $X \subseteq S$, then $\mathbf{r}^T(t,t') = (X\varphi) \cdot T$. □

Theorem 2.4. *If S and T are monoids satisfying* (**R**) *(respectively,* (**r**)*), then $S \times T$ also satisfies* (**R**) *(respectively,* (**r**)*).*

Proof. It is entirely routine to check that if $\mathbf{R}(s,s') = X \cdot S$ and $\mathbf{R}(t,t') = Y \cdot T$ for some finite sets $X \subseteq S \times S$ and $Y \subseteq T \times T$, then

$$\mathbf{R}((s,t),(s',t')) = (X \otimes Y) \cdot (S \times T),$$

where

$$X \otimes Y = \{((p,h),(q,k)) : (p,q) \in X, (h,k) \in Y\}.$$

Similarly, if $\mathbf{r}(s,s') = X \cdot S$ and $\mathbf{r}(t,t') = Y \cdot T$ for some finite sets $X \subseteq S$ and $Y \subseteq T$, then

$$\mathbf{r}((s,t),(s',t')) = (X \times Y) \cdot (S \times T).$$

□

It is more surprising that the conditions ($\mathbf{R}$) and ($\mathbf{r}$) are preserved by taking semidirect products of groups by monoids.

Theorem 2.5. *Let Y be any monoid satisfying ($\mathbf{R}$) (respectively, ($\mathbf{r}$)) and let G be a group acting on Y. Then $Y \rtimes G$ satisfies ($\mathbf{R}$) (respectively, ($\mathbf{r}$)).*

Proof. Suppose that Y satisfies ($\mathbf{R}$), and let $(\alpha,g),(\beta,h) \in Y \rtimes G$. Then $\mathbf{R}^Y(\alpha,\beta) = X \cdot Y$ for some finite $X \subseteq Y \times Y$. We claim that $\mathbf{R}((\alpha,g),(\beta,h))$ is generated by the finite set

$$X' = \{(({}^{g^{-1}}\mu, g^{-1}), ({}^{h^{-1}}\nu, h^{-1})) : (\mu,\nu) \in X\}.$$

Clearly, $X' \subseteq \mathbf{R}((\alpha,g),(\beta,h))$. Now let $((\gamma,i),(\delta,j)) \in \mathbf{R}((\alpha,g),(\beta,h))$, that is, $\alpha \cdot {}^{g}\gamma = \beta \cdot {}^{h}\delta$ and $gi = hj$. Since X generates $\mathbf{R}^Y(\alpha,\beta)$, the first equality implies that there exist $(\mu,\nu) \in X$ and $\epsilon \in Y$ such that $({}^{g}\gamma, {}^{h}\delta) = (\mu,\nu)\epsilon$. It is routine to check now that

$$(\gamma,i) = ({}^{g^{-1}}\mu, g^{-1}) \cdot (\epsilon, gi),\ (\delta,j) = ({}^{h^{-1}}\nu, h^{-1}) \cdot (\epsilon, hj),$$

and since $gi = hj$, this shows that $((\gamma,i),(\delta,j)) \in X' \cdot (Y \rtimes G)$.

For the other part, suppose that Y satisfies ($\mathbf{r}$), and let $(\alpha,g),(\beta,h) \in Y \rtimes G$. Note that if $g \neq h$, then $\mathbf{r}((\alpha,g),(\beta,h)) = \emptyset$, so we suppose that $g = h$. We have that $\mathbf{r}^Y(\alpha,\beta) = X \cdot Y$ for some finite $X \subseteq Y$. It is now routine to check that the finite set

$$X' = \{({}^{g^{-1}}\mu, 1) : \mu \in X\}$$

generates $\mathbf{r}((\alpha,g),(\beta,h))$. □

Note that the submonoid $\{(\alpha,1) : \alpha \in Y\}$ of $Y \rtimes G$ is not necessarily a retract, that is, we cannot conclude that if $Y \rtimes G$ satisfies ($\mathbf{R}$) or ($\mathbf{r}$), then so does Y. However, if Y is a semilattice, we can prove an equivalence.

Theorem 2.6. *Let Y be a semilattice, and let G be a group acting on Y. Then the semidirect product $Y \rtimes G$ satisfies ($\mathbf{R}$) (respectively, ($\mathbf{r}$)) if and only if Y satisfies ($\mathbf{R}$) (respectively, ($\mathbf{r}$)).*

Proof. Let $S = Y \rtimes G$. From Theorem 2.5, if Y satisfies ($\mathbf{R}$) (respectively ($\mathbf{r}$)), then so does S.

Conversely, suppose that S satisfies ($\mathbf{R}$), and let $\alpha,\beta \in Y$. Then

$$\mathbf{R}^S((\alpha,e),(\beta,e)) = X \cdot S$$

for some finite set $X \subseteq S \times S$, where e denotes the identity of G.

Suppose now that $((\gamma,g),(\delta,h)) \in X$. Then

$$(\alpha\gamma, g) = (\alpha,e)(\gamma,g) = (\beta,e)(\delta,h) = (\beta\delta,h),$$

that is, $(\gamma,\delta) \in \mathbf{R}^Y(\alpha,\beta)$ and $g = h$.

We claim that $\mathbf{R}^Y(\alpha,\beta)$ is generated by the finite set

$$X' = \{(\gamma,\delta) : ((\gamma,g),(\delta,g)) \in X\}.$$

We have just verified that $X' \subseteq \mathbf{R}^Y(\alpha,\beta)$, so $X' \cdot Y \subseteq \mathbf{R}^Y(\alpha,\beta)$.

For the converse inclusion, suppose that $\alpha\mu = \beta\nu$. Then $(\alpha,e)(\mu,e) = (\beta,e)(\nu,e)$, so

$$((\mu,e),(\nu,e)) = ((\gamma,g),(\delta,g)) \cdot (\tau,h) = ((\gamma\, {}^g\tau, gh),(\delta\, {}^g\tau, gh))$$

for some $((\gamma,g),(\delta,g)) \in X$. That is, we have $(\mu,\nu) = (\gamma,\delta) \cdot {}^g\tau$, which shows that $(\mu,\nu) \in X' \cdot Y$, so the direct part for $(\mathbf{R})$ is proved.

The case of $(\mathbf{r})$ is very similar, it is easy to check that if $\mathbf{r}^S((\alpha,e),(\beta,e))$ is generated by some finite set $X \subseteq S$, then the finite set

$$X' = \{\gamma \in Y : (\gamma,g) \in X\}$$

generates $\mathbf{r}^Y(\alpha,\beta)$. □

3. Semilattices

We now present some results regarding semilattices. We assume the reader is familiar with the basic notions of lattice theory, such as *distributive* and *complete* lattices (see for example [4] or [7]). Note that a semilattice satisfies the condition (WRN) if and only if its underlying set is well partially ordered by the dual of the partial order induced by the semilattice operation.

For an element a of a semilattice Y we will use the notation $a\!\uparrow$ and $a\!\downarrow$ to denote the principal filter and the principal left ideal generated by a, that is,

$$a\!\uparrow = \{b \in Y : b \geq a\} \text{ and } a\!\downarrow = \{b \in Y : b \leq a\}.$$

First we characterise the condition (WRN) for semilattices. The following proposition is almost folklore (see for example results in [6]). For completeness, we incorporate its proof.

Proposition 3.1. *A semilattice satisfies* (WRN) *if and only if it has neither infinite antichains nor infinite ascending chains.*

Proof. For the direct part, let Y be a semilattice satisfying (WRN). For every $\beta \in Y$, denote by $\beta\!\downarrow$ the ideal generated by β. Note that in a semilattice, the union of ideals is again an ideal. Now suppose that $\alpha_1, \alpha_2, \ldots$ is an infinite antichain. In this case

$$\alpha_1\!\downarrow \subset \alpha_1\!\downarrow \cup\, \alpha_2\!\downarrow \subset \ldots \subset \alpha_1\!\downarrow \cup\, \alpha_2\!\downarrow \cup \ldots \cup\, \alpha_i\!\downarrow \subset \ldots$$

is an infinite ascending chain of ideals of Y, contradicting (WRN). Similarly, if $\alpha_1 < \alpha_2 < \ldots$ is an infinite ascending chain, then the sequence $\alpha_1\!\downarrow \subset \alpha_2\!\downarrow \subset \ldots$ is again an infinite ascending chain of ideals. Thus, if Y satisfies (WRN), then it cannot have any infinite antichains or infinite ascending chains.

For the converse part, suppose that Y has no infinite ascending chains nor infinite antichains, and let $I_0 \subset I_1 \subset \ldots$ be an infinite ascending chain of ideals. For every $i \geq 1$, let $\alpha_i \in I_i \setminus I_{i-1}$. Since the I_i's are ideals, we have that $\alpha_i \not\geq \alpha_j$ for every $j > i$.

We show first that for every $i \geq 1$, there exists $j > i$ such that $\alpha_j \perp \alpha_k$ for every $k > j$. Suppose on the contrary, that such a j does not exist for some i. That is, there exists an i such that for every $j > i$ there exists $k_j > j$ such that $\alpha_j < \alpha_{k_j}$. Let $j_0 = i + 1$. In this case the sequence $\alpha_{j_0} < \alpha_{k_{j_0}} < \alpha_{k_{k_{j_0}}} < \ldots$ is an ifinite ascending chain, contradicting our assumptions.

So we have shown that for every i, there exists $j_i > i$ such that $\alpha_{j_i} \perp \alpha_k$ for every $k > j_i$. However, this property implies the existence of an infinite antichain, namely $\alpha_{j_1} \perp \alpha_{j_{j_1}} \perp \alpha_{j_{j_{j_1}}} \perp \ldots$, which is a contradiction. So Y cannot have an infinite ascending chain of ideals, thus it satisfies (WRN). □

Lemma 3.2. *If a semilattice Y satisfies* $(\mathbf{R})$*, then it satisfies* $(\mathbf{r})$*.*

Proof. Let $\alpha, \beta \in Y$, and let $\mathbf{R}(\alpha, \beta) = X \cdot Y$ for some finite $X \subseteq Y \times Y$. Then define the set

$$X' = \{\mu_i \nu_i : (\mu_i, \nu_i) \in X\}.$$

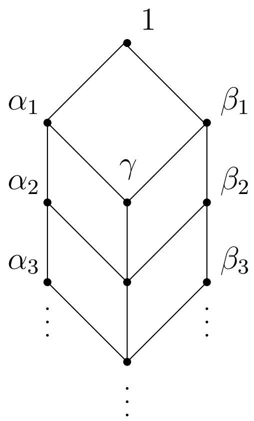
We claim that the right ideal $\mathbf{r}(\alpha, \beta)$ is generated by X'. Clearly $X' \subseteq \mathbf{r}(\alpha, \beta)$. Let $\gamma \in \mathbf{r}(\alpha, \beta)$, that is, $\alpha\gamma = \beta\gamma$. This equality implies that $(\gamma, \gamma) \in \mathbf{R}(\alpha, \beta)$, so there exist $(\mu_i, \nu_i) \in X$ and $\epsilon \in Y$ such that $\gamma = \mu_i \epsilon = \nu_i \epsilon$. In this case $\gamma \leq \mu_i, \nu_i$, so $\gamma = \mu_i \nu_i \cdot \epsilon \in X' \cdot Y$. □

Definition 3.3. *We say that a semilattice Y is* finite above *if its principal filters are finite.*

Lemma 3.4. *If a semilattice Y satisfies* $(\mathbf{R})$*, then it is finite above.*

Proof. Let $\alpha \in Y$. In this case $(1, \beta) \in \mathbf{R}(\alpha, \alpha)$ for every $\beta \geq \alpha$. However, if $(1, \beta) = (\mu, \nu)\epsilon$, then necessarily $\mu = \epsilon = 1$ and $\nu = \beta$, that is, the pairs of the form $(1, \beta)$ cannot be consequences of any other pairs. These two facts imply that if $\mathbf{R}(\alpha, \alpha)$ is finitely generated, then the principal filter generated by α must be finite. □

Denote by $\mathbb{N}^\infty$ the set of natural numbers with the infinity adjoined. As remarked in Section 1, every chain having a greatest element satisfies $(\mathbf{r})$, so in particular, $(\mathbb{N}^\infty, \min)$ satisfies $(\mathbf{r})$. However, it is not finite above, so cannot satisfy $(\mathbf{R})$. For a more sophisticated counterexample, note that the following semilattice Y satisfies $(\mathbf{r})$ since from Proposition 3.1 it satisfies (WRN), and it is finite above also. However, $(\alpha_i, \beta_i) \in \mathbf{R}(\gamma, \gamma)$ for each i, but if $(\alpha_i, \beta_i) = (\mu, \nu)\epsilon$, then $\epsilon = 1$ and $\mu = \alpha_i$, $\nu = \beta_i$. Thus it is impossible to find a finite set of generators for $\mathbf{R}(\gamma, \gamma)$, so that $(\mathbf{R})$ does not hold. This example shows that condition (WRN) does not imply $(\mathbf{R})$.



Note that if a semilattice is finite above, then it is necessarily a lattice, where the join operation is defined by

$$\alpha \vee \beta = \prod \{\gamma \in Y : \gamma \geq \alpha, \beta\}.$$

In the sequel we are going to investigate lattices, where the meet operation will be multiplication. That is, the operation of the semilattice Y is multiplication, but we have an additional operation $\vee$ such that $(Y, \vee, \cdot)$ becomes a lattice.

Lemma 3.5. *If Y is a distributive lattice that is finite above, then Y satisfies* $(\mathbf{R})$.

Proof. Let $\alpha, \beta \in Y$, and let

$$X = \{(\mu, \nu) : \mu, \nu \geq \alpha\beta \text{ and } \alpha\mu = \beta\nu\}.$$

We claim that $\mathbf{R}(\alpha, \beta)$ is generated by the finite set X.

Clearly $X \cdot Y \subseteq \mathbf{R}(\alpha, \beta)$. To show the converse inclusion, let $(\gamma, \delta) \in \mathbf{R}(\alpha, \beta)$, that is, $\alpha\gamma = \beta\delta$. Note that $\gamma \geq \alpha\gamma = \beta\delta$, so

$$\gamma = (\beta\delta) \vee \gamma = (\beta \vee \gamma)(\gamma \vee \delta),$$

and similarly $\delta = (\alpha \vee \delta)(\gamma \vee \delta)$. Furthermore,

$$\alpha(\beta \vee \gamma) = (\alpha\beta) \vee (\alpha\gamma) = (\alpha\beta) \vee (\beta\delta) = \beta(\alpha \vee \delta),$$

and since $\beta \vee \gamma, \alpha \vee \delta \geq \alpha\beta$, we have that $(\beta \vee \gamma, \alpha \vee \delta) \in X$, and so $(\gamma, \delta) \in X \cdot Y$. □

Definition 3.6. *We say that a distributive lattice Y is a* Boolean lattice *if it has a smallest element, denoted by* 0*, and if there exists a unary operation $'$ on B such that $\alpha\alpha' = 0$ for every $\alpha \in Y$.*

Notice that if Y is a Boolean lattice, then for any $\alpha \in Y$, we have $\alpha \vee \alpha' = 1$. This is used in the proof of the next result.

Theorem 3.7. *If Y is a Boolean lattice, or a completely distributive lattice, then Y satisfies* $(\mathbf{r})$.

Proof. Note that if Y is a (completely) distributive lattice, then $\mathbf{r}(\alpha, \beta)$ is closed under (infinite) finite joins. This implies that if Y is completely distributive, then $\mathbf{r}(\alpha, \beta)$ is generated by $\bigvee \mathbf{r}(\alpha, \beta)$.

On the other hand, if Y is a Boolean lattice, then it is easy to check that $\mathbf{r}(\alpha,\beta)$ is generated by $(\alpha \bigtriangleup \beta)'$, where

$$\gamma \bigtriangleup \delta = (\gamma\delta') \vee (\gamma'\delta).$$

□

As the following example shows, distributive lattices need not satisfy $(\mathbf{r})$ in general.

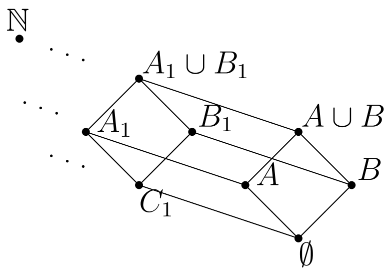
Example 3.8. *Let Y be the sublattice (not complete sublattice) of $\mathcal{P}(\mathbb{N})$ generated by the following sets, together with $\mathbb{N}$:*

$$A = \{n : n \equiv 1 \ (mod\ 3)\}$$
$$B = \{n : n \equiv 2 \ (mod\ 3)\}$$
$$C_i = \{n : 3 \mid n, n \leq 3i\} \text{ for all } i \geq 1$$

Then the lattice Y is a distributive lattice such that $(Y,\cap)$ does not satisfy $(\mathbf{r})$.

Proof. Since $A\cap B = A\cap C_i = B\cap C_i = \emptyset$ for all i, it is easy to check that Y is isomorphic to the direct product of the chain $\emptyset \subset C_1 \subset C_2 \subset \dots$ by the diamond $\{\emptyset, A, B, A\cup B\}$, with a greatest element $(\mathbb{N})$ adjoined.

The Hasse diagram of Y is the following (where $A_i = C_i \cup A$ and $B_i = C_i \cup B$):



Being a sublattice of $\mathcal{P}(\mathbb{N})$, Y is clearly a distributive lattice, and it is easy to see that Y is a complete lattice also. The join operation in Y is the finite union inherited from $\mathcal{P}(\mathbb{N})$, however the infinite join in Y is different from the one in $\mathcal{P}(\mathbb{N})$, for example in Y, $\bigvee_{i=1}^{\infty} C_i = \mathbb{N}$. Note that Y is not a completely distributive lattice, for $\bigvee_{i=1}^{\infty}(A\cap C_i) = \emptyset \neq A = A \cap \bigvee_{i=1}^{\infty} C_i$.

To see that Y does not satisfy $(\mathbf{r})$, note that

$$\mathbf{r}(A,B) = \{C_i : i = 1, 2, \dots\},$$

which is not a finitely generated right ideal of $(Y,\cap)$. □

4. Clifford monoids with finite semilattice of idempotents

If the semilattice of idempotents of a Clifford monoid is finite, we can give a local condition which is equivalent to $(\mathbf{R})$, while $(\mathbf{r})$ holds always in these cases. In the sequel, S denotes a Clifford monoid having a finite semilattice of idempotents E. The structure semilattice of S (which is isomorphic to E) is denoted by Y. If $\alpha \in Y$, we denote by G_α the $\mathcal{H} = \mathcal{J}$-class corresponding to α, and denote by e_α the identity of G_α. We identify Y with $S/\mathcal{J}$ so that for $a \in G_\alpha$ we have $a\mathcal{J} = \alpha$ where $a\mathcal{J}$ is the image of a under the natural morphism. Notice that if a semilattice has a greatest element, and it is finite above, then least upper bounds exist, thus it has an operation $\vee$ such that $(Y, \cdot, \vee)$ is a lattice.

At this point it is useful to introduce the following concept.

Definition 4.1. *Let $s, t \in S$ and $\alpha, \beta \in Y$. Then we say that the set*

$$R(s,t,\alpha,\beta) = R(s,t) \cap (G_\alpha \times G_\beta)$$

is locally finitely generated *if there exists a finite subset $X \subseteq R(s,t,\alpha,\beta)$ such that $R(s,t,\alpha,\beta) = X \cdot G_{\alpha\vee\beta}$.*

Lemma 4.2. *The set $\mathbf{R}(s,t)$ is finitely generated if and only if $\mathbf{R}(s,t,\alpha,\beta)$ is locally finitely generated for every $\alpha, \beta \in Y$.*

Proof. For the converse part, suppose that for every $\alpha, \beta \in Y$, we have that $\mathbf{R}(s,t,\alpha,\beta) = X_{\alpha,\beta} \cdot G_{\alpha\vee\beta}$ for some finite set $X_{\alpha,\beta}$. Let

$$X = \bigcup_{\alpha,\beta\in Y} X_{\alpha,\beta},$$

so that X is a finite subset of $\mathbf{R}(s,t)$. For every $\alpha, \beta \in Y$, we have that

$$\mathbf{R}(s,t) \cap (G_\alpha \times G_\beta) = \mathbf{R}(s,t,\alpha,\beta) = X_{\alpha,\beta} \cdot G_{\alpha\vee\beta} \subseteq X \cdot S,$$

thus $\mathbf{R}(s,t) = \bigcup_{\alpha,\beta\in Y} \mathbf{R}(s,t,\alpha,\beta) \subseteq X \cdot S$ and $\mathbf{R}(s,t)$ is generated by X.

Note that in the following argument for the direct part, to show that $\mathbf{R}(s,t,\alpha,\beta)$ is finitely generated, we need a weaker condition than Y being finite, namely that $\alpha \vee \beta$ exists. Suppose that $\mathbf{R}(s,t) = X \cdot S$ for some finite set $X \subseteq S \times S$. Let $\alpha, \beta \in Y$ be fixed and let

$$X_{\alpha,\beta} = \{(ue_\alpha, ve_\beta) : (u,v) \in X, ue_\gamma \in G_\alpha, ve_\gamma \in G_\beta \text{ for some } \gamma \in Y\}.$$

Notice that $X_{\alpha,\beta}$ is contained in $\mathbf{R}(s,t)$ and is finite, since X is finite. Of course, $X_{\alpha,\beta}$ can be empty if there do not exist suitable γ's.

We claim that

$$\mathbf{R}(s,t,\alpha,\beta) = X_{\alpha,\beta} \cdot G_{\alpha\vee\beta}.$$

To show this, first let $(a,b) \in \mathbf{R}(s,t,\alpha,\beta)$. Then there exist $(u,v) \in X$ and $z \in S$ such that $(a,b) = (u,v)z$, that is, $uz \in G_\alpha$ and $vz \in G_\beta$, which in turn implies that $(ue_\alpha, ve_\beta) \in X_{\alpha,\beta}$. Also, $a = uz$ implies that $z\mathcal{J} \geq \alpha$, and similarly, $z\mathcal{J} \geq \beta$. Using these facts we deduce that $a = uz = ue_\alpha ze_{\alpha\vee\beta}$ and similarly $b = ve_\beta ze_{\alpha\vee\beta}$. That is,

$$(a,b) = (u,v)z = (ue_\alpha, ve_\beta)ze_{\alpha\vee\beta} \in X_{\alpha,\beta} \cdot G_{\alpha\vee\beta}.$$

Conversely, if $(a,b) \in X_{\alpha,\beta} \cdot G_{\alpha\vee\beta}$, then clearly $a \in G_\alpha$ and $b \in G_\beta$, and since $X_{\alpha,\beta} \subseteq \mathbf{R}(s,t)$, we have that $(a,b) \in \mathbf{R}(s,t) \cap G_\alpha \times G_\beta$, completing the proof that $\mathbf{R}(s,t,\alpha,\beta)$ is locally finitely generated. ∎

Though the following theorem concerns all Clifford monoids (not just those which have a finite structure semilattice), we include it here, for it follows from the proof of Theorem 4.2.

Corollary 4.3. *If S is any Clifford monoid satisfying* $(\mathbf{R})$*, then the kernels of the structure homomorphisms must be finite.*

Proof. Let $\beta \geq \alpha$, and let φ be the structure homomorphism $G_\beta \to G_\alpha$. It is easy to check that

$$R = \mathbf{R}(e_\alpha, e_\alpha, \beta, \beta) = \{(a, b) \in G_\beta \times G_\beta : a\varphi = b\varphi\}.$$

From the direct part of Theorem 4.2, R is locally finitely generated, for $\beta \vee \beta$ always exists in any semilattice, and it equals β. Let $X \subseteq G_\beta \times G_\beta$ be such that $R = X \cdot G_\beta$ with X finite. For any $g \in \mathrm{Ker}\ \varphi$, $(g, e_\beta) \in R$, so $(g, e_\beta) = (u, v)z$ for some $(u, v) \in X$, giving $g = ge_\beta^{-1} = (uz)(vz)^{-1} = uv^{-1}$. Consequently, Ker φ is finite.

Note that if Ker $\varphi = \{g_1, \ldots, g_n\}$ is finite, then $\{(e_\beta, g_i) : 1 \leq i \leq n\}$ locally finitely generates R. ☐

By making use of the previous theorems, we can concentrate on the question of when $\mathbf{R}(s, t, \alpha, \beta)$ is locally finitely generated. For this, the following lemma will be useful.

Lemma 4.4. *If $s, t \in S$ and $\alpha, \beta \in Y$ such that $\mathbf{R}(s, t, \alpha, \beta) \neq \emptyset$, then*

$$\mathbf{R}(s, t, \alpha, \beta) = \mathbf{R}(se_\gamma, te_\gamma, \alpha, \beta)$$

for $\gamma = (s\mathcal{J})\alpha = (t\mathcal{J})\beta$.

Proof. Let $(a, b) \in \mathbf{R}(s, t, \alpha, \beta)$. Then $sa = tb$, and let $\gamma = (sa)\mathcal{J} = s\mathcal{J} \cdot \alpha = t\mathcal{J} \cdot \beta$. Notice that for every $(a', b') \in G_\alpha \times G_\beta$, we have that $sa', tb' \in G_\gamma$.

Now if $(a', b') \in \mathbf{R}(s, t, \alpha, \beta)$, then $sa' = sa'e_\gamma = se_\gamma a'$, and $tb' = tb'e_\gamma = te_\gamma b'$, so $(a', b') \in \mathbf{R}(se_\gamma, te_\gamma, \alpha, \beta)$. Conversely, if $(a', b') \in \mathbf{R}(se_\gamma, te_\gamma, \alpha, \beta)$, then $se_\gamma a' = te_\gamma b'$, but $(sa')\mathcal{J} = \gamma = (tb')\mathcal{J}$, so $sa' = se_\gamma a' = te\gamma b' = tb'$, giving $(a', b') \in \mathbf{R}(s, t, \alpha, \beta)$. ☐

As a consequence of Theorem 4.2 and Lemma 4.4, we have the following theorem.

Theorem 4.5. *Let S be a Clifford monoid having a finite structure semilattice Y. Then $\mathbf{R}(s, t)$ is finitely generated for all $s, t \in S$ if and only if $\mathbf{R}(s', t', \alpha, \beta)$ is locally finitely generated for all $s', t' \in S$ and $\alpha, \beta \in Y$ satisfying $s'\mathcal{J} = t'\mathcal{J} \leq \alpha\beta$.*

Corollary 4.6. *If S is a Clifford monoid having a finite structure semilattice Y with trivial structure homomorphisms, then S satisfies* $(\mathbf{R})$ *if and only if $S \setminus G_0$ is finite where 0 is the least element of Y.*

Proof. If S satisfies $(\mathbf{R})$, then by Corollary 4.3, the kernels of the structure homomorphisms must be finite, and so as the (non-identity) ones are trivial, G_α is finite for all $\alpha \neq 0$.

For the converse, note that if G_α is finite for all $\alpha \neq 0$, then $\mathbf{R}(s, t, \alpha, \beta)$ is clearly finite for every $\alpha, \beta > 0$. If $\alpha > 0$ and $\beta = 0$, and $(u, v) \in \mathbf{R}(s, t, \alpha, \beta)$, then $su = tv$, and since $u \in G_\alpha$, but $su \in G_0$, we have that $se_0 = su = tv$, and since $v \in G_0$, this yields $v = t^{-1}se_0$, that is, $\mathbf{R}(s, t, \alpha, \beta) = G_\alpha \times \{t^{-1}se_0\}$, which is finite. Similarly, if $\alpha = 0 < \beta$, then $\mathbf{R}(s, t, \alpha, \beta)$ is finite. Furthermore, if $\alpha = \beta = 0$, then $\mathbf{R}(s, t, 0, 0)$ is generated by $(e_0 s^{-1} t, e_0)$. ☐

By making use of Theorem 4.5, one can give a necessary and sufficient condition on the structure homomorphisms.

Theorem 4.7. *Let S be a Clifford monoid with a finite structure semilattice Y, let $\alpha, \beta, \gamma \in Y$ be such that $\gamma \leq \alpha\beta$, and let $s, t \in G_\gamma$. Denote the structure homomorphisms $\varphi_{\alpha\vee\beta,\alpha}, \varphi_{\alpha\vee\beta,\beta}, \varphi_{\alpha,\gamma}$ and $\varphi_{\beta,\gamma}$ by $\varphi_\alpha, \varphi_\beta, \psi_\alpha$ and ψ_β, respectively. Define the subgroups*

$$H = \{(u, v) : u\psi_\alpha = v\psi_\beta\} \subseteq G_\alpha \times G_\beta$$
$$K = \{(g\varphi_\alpha, g\varphi_\beta) : g \in G_{\alpha\vee\beta}\} \subseteq G_\alpha \times G_\beta.$$

Then $K \leq H$, and for every $s, t \in G_\gamma$, we have that if $\mathbf{R}(s, t, \alpha, \beta)$ is non-empty, then it is locally finitely generated if and only if $[H : K]$ is finite.

Proof. Note that if $(u, v) \in G_\alpha \times G_\beta$, then

$$\{(u, v)g : g \in G_{\alpha\vee\beta}\} = \{(u(g\varphi_\alpha), v(g\varphi_\beta)) : g \in G_{\alpha\vee\beta} = (u, v)K.$$

This equality shows that any subset of $G_\alpha \times G_\beta$ that is locally finitely generated, is a union of left cosets of K, and one pair generates one left coset. That is, $\mathbf{R}(s, t, \alpha, \beta)$ is locally finitely generated if and only if it is a finite union of left cosets of K.

Suppose now that $(u, v) \in \mathbf{R}(s, t, \alpha, \beta)$, and let $(u', v') \in H$. Then

$$suu' = su \cdot u'\psi_\alpha = tv \cdot v'\psi_\beta = tvv',$$

that is, $(u, v)(u', v') \in \mathbf{R}(s, t, \alpha, \beta)$ as well, so $\mathbf{R}(s, t, \alpha, \beta)$ is a union of left cosets of H. However, if $(u, v), (u', v') \in \mathbf{R}(s, t, \alpha, \beta)$, then $su = tv$ and $su' = tv'$, so that

$$(u^{-1}u')\psi_\alpha = u^{-1}e_\gamma u' = (u^{-1}s^{-1}) \cdot su' = (v^{-1}t^{-1}) \cdot tv' = v^{-1}e_\gamma v' = (v^{-1}v')\psi_\beta,$$

that is, if $\mathbf{R}(s, t, \alpha, \beta)$ is non-empty, then it is a left coset of H. Summing up: if $\mathbf{R}(s, t, \alpha, \beta)$ is non-empty, then it is locally finitely generated if and only if $[H : K]$ is finite. ☐

Note that in case $\gamma < \alpha \leq \beta$, if $\text{Ker } \varphi_{\alpha,\gamma} = \{g_i : i \in I\}$, then $H = \dot{\bigcup}_{i\in I}(g_i, e_\beta)K$, so that $[H : K] = |\text{Ker } \varphi_{\alpha,\gamma}|$. Thus in this case, $\mathbf{R}(s, t, \alpha, \beta)$ is locally finitely generated if and only if the kernel of $\varphi_{\alpha,\gamma}$ is finite. If $\gamma = \alpha \leq \beta$, then actually $H = K$, so $\mathbf{R}(s, t, \alpha, \beta)$ is locally finitely generated. That is, the previous theorem gives a new condition only if $\alpha \perp \beta$. We summarize the results of this section in the following theorem.

Theorem 4.8. *If S is a Clifford monoid with finite structure semilattice Y, then*

- *S satisfies $(\mathbf{r})$,*
- *S satisfies $(\mathbf{R})$ if and only if for every $\alpha, \beta, \gamma \in Y$ such that $\gamma \leq \alpha\beta$, the index $[H_{\alpha,\beta,\gamma} : K_{\alpha,\beta,\gamma}]$ is finite where*

$$H_{\alpha,\beta,\gamma} = \{(u, v) : u\varphi_{\alpha,\gamma} = v\varphi_{\beta,\gamma}\} \leq G_\alpha \times G_\beta,$$
$$K_{\alpha,\beta,\gamma} = \{(g\varphi_{\alpha\vee\beta,\alpha}, g\varphi_{\alpha\vee\beta,\beta}) : g \in G_{\alpha\vee\beta}\} \leq G_\alpha \times G_\beta.$$

As a special case, if all homomorphisms are trivial, then S satisfies $(\mathbf{R})$ if and only if G_α is finite for all $0 \neq \alpha \in Y$, and if all homomorphisms are injective, then S satisfies $(\mathbf{R})$ if and only if $[G_\alpha\varphi_{\alpha,\gamma} \cap G_\beta\varphi_{\beta,\gamma} : G_\delta\varphi_{\delta,\gamma}]$ is finite for all $\alpha, \beta \geq \gamma \in Y$, where $\delta = \alpha \vee \beta$.

Proof. For the first part, notice that in S all right ideals are finitely generated, so S clearly satisfies $(\mathbf{r})$.

For the second part, first note that if $[H_{\alpha,\beta,\gamma} : K_{\alpha,\beta,\gamma}]$ is finite for all α, β, γ satisfying $\gamma \leq \alpha\beta$, then by Theorem 4.7, $\mathbf{R}(s, t, \alpha, \beta)$ is locally finitely generated for all $s, t \in S$ and $\alpha, \beta \in Y$, which by Lemma 4.2 implies that S satisfies ($\mathbf{R}$).

Conversely, if S satisfies ($\mathbf{R}$), then let $\alpha, \beta, \gamma \in Y$ such that $\gamma \leq \alpha\beta$. In this case $\mathbf{R}(e_\gamma, e_\gamma, \alpha, \beta)$ is not empty (because it contains (e_α, e_β), so $[H_{\alpha,\beta,\gamma} : K_{\alpha,\beta,\gamma}]$ must be finite by Theorem 4.7.

Suppose now that the structure homomorphisms are injective, let $\alpha, \beta, \gamma \in Y$ such that $\gamma \leq \alpha\beta$, and let $\delta = \alpha \vee \beta$. In this case it is routine to check that the map

$$\iota \colon H_{\alpha,\beta,\gamma} \to G_\gamma, (u, v) \mapsto u\varphi_{\alpha,\gamma} = v\varphi_{\beta,\gamma}$$

is an injective homomorphism with image $G_\alpha\varphi_{\alpha,\gamma} \cap G_\beta\varphi_{\beta,\gamma}$ which maps $K_{\alpha,\beta,\gamma}$ onto $G_\delta\varphi_{\delta,\gamma}$, thus $[H_{\alpha,\beta,\gamma} : K_{\alpha,\beta,\gamma}] = [G_\alpha\varphi_{\alpha,\gamma} \cap G_\beta\varphi_{\beta,\gamma} : G_\delta\varphi_{\delta,\gamma}]$.

The case when the structure homomorphisms are trivial was already settled in Corollary 4.6. ☐

5. Clifford monoids with trivial structure homomorphisms

In this section we suppose that the structure homomorphisms of the Clifford monoid S are all trivial, but its structure semilattice Y does not have to be finite. We introduce notation as follows:

$$\mathbf{R}(s, t)\mathcal{J} = \{(u\mathcal{J}, v\mathcal{J}) : su = tv\}$$

Note that $\mathbf{R}(s, t)\mathcal{J}$ is a subact of $Y \times Y$, and it is contained in $\mathbf{R}^Y(s\mathcal{J}, t\mathcal{J})$, but in general this containment is strict.

Theorem 5.1. *Let S be a Clifford monoid with trivial structure homomorphisms and structure semilattice Y. Then S satisfies ($\mathbf{R}$) if and only if the following are true:*

(1) Y is finite above;
(2) for every $0 \neq \alpha \in Y$, G_α is finite;
(3) for every $\alpha \in Y$, the set $\{\beta : \beta \perp \alpha, |G_\beta| > 1\}$ is finite;
(4) for every $s \in G_\alpha, t \in G_\beta$ there exists a finite set $X \subseteq Y \times Y$ such that $\mathbf{R}(s, t)\mathcal{J} = X \cdot Y$.

Proof. Suppose first that S satisfies ($\mathbf{R}$). Since E is a retract of S, by Theorem 2.3, it satisfies ($\mathbf{R}$), which by Lemma 3.4 implies that $E \cong Y$ is finite above. Corollary 4.3 shows that Condition (2) also holds.

For Condition (3) note that if $\beta \perp \alpha$ then $G_\beta \times G_\beta \subseteq \mathbf{R}(e_\alpha, e_\alpha)$. However, if $s \neq t \in G_\beta$, and $(s, t) = (u, v)z$ for some $u, v, z \in S$, then it is easy to check that either $u \in G_\beta$ or $v \in G_\beta$ holds (otherwise $uz = vz$, contradicting $s \neq t$). This fact shows that if $\mathbf{R}(e_\alpha, e_\alpha) = X \cdot S$ for some set X, then for every $\beta \perp \alpha$, $|G_\beta| > 1$, X must contain a pair having an element of G_β. This implies that if X is finite, then the set $\{\beta : \beta \perp \alpha, |G_\beta| > 1\}$ is finite also.

To show that Condition (4) holds, let $s \in G_\alpha, t \in G_\beta$. Then $\mathbf{R}(s, t) = X' \cdot S$ for some finite $X' \subseteq S \times S$. Let

$$X = \{(u\mathcal{J}, v\mathcal{J}) : (u, v) \in X'\}.$$

We claim that X satisfies Condition (4). By definition, $X \subseteq \mathbf{R}(s,t)\mathcal{J}$, and so $X \cdot Y \subseteq \mathbf{R}(s,t)\mathcal{J}$. On the other hand, if $(\gamma, \delta) \in \mathbf{R}(s,t)\mathcal{J}$, then there exists $(u, v) \in \mathbf{R}(s,t)$ such that $u\mathcal{J} = \gamma$ and $v\mathcal{J} = \delta$. Since $(u, v) \in \mathbf{R}(s,t)$, we have that there exists $(u', v') \in X'$ and $z \in S$ such that $(u, v) = (u', v')z$, which implies that $(\gamma, \delta) = (u'\mathcal{J}, v'\mathcal{J})z\mathcal{J}$, and since $(u'\mathcal{J}, v'\mathcal{J}) \in X$, this shows that $(\gamma, \delta) \in X \cdot Y$.

For the converse part,we suppose that the conditions are satisfied. Note that if Y has a 0, then by Condition (1) it is finite. In this case, by Corollary 4.6, Condition (2) implies that S satisfies ($\mathbf{R}$). Therefore, in the sequel, we assume that Y does not have a 0, and so G_α is finite for each $\alpha \in Y$.

Let $s \in G_\alpha, t \in G_\beta$ and let X be as in (4). Define the following sets:

$$X_1 = \{(u, v) : su = tv, (u\mathcal{J}, v\mathcal{J}) \in X\},$$
$$X_2 = \{(u, v) : su = tv, u\mathcal{J}, v\mathcal{J} \geq \alpha\beta\},$$
$$X_3 = \{(u, v) : su = tv, u \notin E, u\mathcal{J} \perp \alpha\}, \quad X_3^d = \{(u, v) : su = tv, v \notin E, v\mathcal{J} \perp \beta\}$$

Notice that X is finite, therefore X_1 is finite by (2). Since Y is finite above and every G_α is finite, we have that X_2 is finite also. Furthermore, X_3 is finite, because there are only finitely many non-idempotent u's such that $u\mathcal{J} \perp \alpha$, and for any such fixed u, there are only finitely many v's such that $su = tv$, because in this case $v\mathcal{J} \geq (su)\mathcal{J}$, and there are only finitely many v's satisfying this property. Dually, X_3^d is finite. We claim that the finite set

$$X' = X_1 \cup X_2 \cup X_3 \cup X_3^d$$

generates $\mathbf{R}(s,t)$.

By definition, $X' \subseteq \mathbf{R}(s,t)$. Suppose now that $(u, v) \in \mathbf{R}(s,t)$, that is, $su = tv$. By the definition of X, there exists a pair $(\gamma, \delta) \in X$ and $\epsilon \in Y$ such that $(u\mathcal{J}, v\mathcal{J}) = (\gamma, \delta)\epsilon$. Furthermore, there exists a pair $(u', v') \in \mathbf{R}(s,t)$ such that $u' \in G_\gamma$ and $v' \in G_\delta$. There are several different cases.

(1) Suppose that both u and v are idempotents. If $(e_\gamma, e_\delta) \in \mathbf{R}(s,t)$, then $(e_\gamma, e_\delta) \in X_1$, and since $(u, v) = (e_\gamma, e_\delta)e_\epsilon$, we have that $(u, v) \in X_1 \cdot S$.

If $(e_\gamma, e_\delta) \notin \mathbf{R}(s,t)$, then we have that $se_\gamma \neq te_\delta$, however $su = se_\gamma e_\epsilon = te_\delta e_\epsilon = tv$. Note that se_γ and te_δ are contained in the same $\mathcal{J}$-class, because $(\gamma, \delta) \in \mathbf{R}^Y(\alpha, \beta)$. Now if we multiply se_γ and te_δ by e_ϵ, they become equal, and this can only happen if multiplication by ϵ brings these elements into a strictly lower $\mathcal{J}$-class, that is, $(su)\mathcal{J} = (se_\gamma e_\epsilon)\mathcal{J} < (se_\gamma)\mathcal{J}$ and $(tv)\mathcal{J} = (te_\delta e_\epsilon)\mathcal{J} < (te_\delta)\mathcal{J}$. However, in this case we must have that $u\mathcal{J} = \gamma\epsilon < \gamma$ and $v\mathcal{J} = \delta\epsilon < \delta$. Since u and v are idempotents, these inequalities imply that $u = u'e_\epsilon$ and $v = v'e_\epsilon$, so $(u, v) = (u', v')e_\epsilon \in X_1 \cdot S$.

(2) If $u\mathcal{J} \geq \alpha$, then $\alpha = (su)\mathcal{J} = (tv)\mathcal{J} \leq v\mathcal{J}$, which shows that $(u, v) \in X_2$. Note that we have just shown that $u\mathcal{J} \geq \alpha$ implies that $v\mathcal{J} \geq \alpha$, and dually one can show that $v\mathcal{J} \geq \beta$ implies that $u\mathcal{J} \geq \beta$ as well.

(3) If $u\mathcal{J} \perp \alpha$, $u \notin E$, then $(u, v) \in X_3$.

(4) If $u\mathcal{J} \perp \alpha$, $u \in E$, then either $v \in E$, which case is already settled. Suppose now that $v \notin E$. If $v\mathcal{J} \geq \beta$, then also $u\mathcal{J} \geq \beta$, and in this case $(u, v) \in X_2$. If $v\mathcal{J} \perp \beta$,

then $(u,v) \in X_3^d$. So the only remaining case is when $v \notin E$, $v\mathcal{J} < \beta$. But in this case note that $v = tv = su \in E$, a contradiction.

(5) By duality the only remaining case is when $u\mathcal{J} < \alpha$ and $v\mathcal{J} < \beta$. However, in this case $u = su = tv = v \in G_\epsilon$ for some $\epsilon \in Y$. This implies that $se_\epsilon = te_\epsilon$ and $(u,v) = (e_\epsilon, e_\epsilon)u$. The pair (e_ϵ, e_ϵ) is contained in $X_1 \cdot S$ as we have already seen in Part 1, so $(u,v) \in X_1 \cdot S$ as well.

So we have shown that $(u,v) \in X' \cdot S$, which means that $\mathbf{R}(s,t) = X' \cdot S$, and the theorem is proved. □

The condition that $\mathbf{R}(s,t)\mathcal{J}$ has to be finitely generated in fact means that some pairs of $\mathbf{R}^Y(\alpha,\beta)$ for certain $\alpha, \beta \in Y$ are excluded from the generating set, because they cannot be realized in S. This view allows the following reformulation of Condition (4) of the previous theorem.

Theorem 5.2. *Let S be a Clifford monoid with trivial structure homomorphisms such that conditions (1)-(3) of Theorem 5.1 hold, and let $s \in G_\alpha, t \in G_\beta$. Then the following are true.*

(1) If $\alpha \leq \beta$ and $s \neq te_\alpha$, or dually, if $\alpha \geq \beta$ and $t \neq se_\beta$, then $\mathbf{R}(s,t)\mathcal{J}$ is finitely generated if and only if there exists a finite set $X \subseteq \mathbf{R}^Y(\alpha,\beta)$ satisfying the property that for every $(\gamma,\delta) \in \mathbf{R}^Y(\alpha,\beta)$, there exist $(\mu,\nu) \in X$ and $\epsilon \in Y$ with $(\gamma,\delta) = (\mu,\nu)\epsilon$ such that if $\gamma \perp \alpha$ and $\delta \perp \beta$, then also $\mu \perp \alpha$ and $\nu \perp \beta$.

(2) In every other case $\mathbf{R}(s,t)\mathcal{J} = \mathbf{R}^Y(\alpha,\beta)$.

Proof. (1) Suppose $\alpha \leq \beta$ and $s \neq te_\alpha$. First we prove that

(1)
$$\mathbf{R}(s,t)\mathcal{J} = \mathbf{R}^Y(\alpha,\beta) \setminus \{(\gamma,\delta) : \gamma,\delta > \alpha\}.$$

Note that $\mathbf{R}(s,t)\mathcal{J} \subseteq \mathbf{R}^Y(\alpha,\beta)$ in every case, and also note that if $\gamma,\delta > \alpha$ such that $\alpha\gamma = \beta\delta = \alpha$, then $su = s \neq te_\alpha = tv$ for every $u \in G_\gamma$ and $v \in G_\delta$, thus $(\gamma,\delta) \notin \mathbf{R}(s,t)\mathcal{J}$. This shows that $\mathbf{R}(s,t)\mathcal{J} \subseteq \mathbf{R}^Y(\alpha,\beta) \setminus \{(\gamma,\delta) : \gamma,\delta > \alpha\}$. For the converse part, let $(\gamma,\delta) \in \mathbf{R}^Y(\alpha,\beta) \setminus \{(\mu,\nu) : \mu,\nu > \alpha\}$, so that $\alpha\gamma = \beta\delta$. There are several cases now, and all have the same conclusion that $(\gamma,\delta) \in \mathbf{R}(s,t)\mathcal{J}$:

(a) If $\gamma > \alpha$, then $\delta \not> \alpha$, however, $\alpha = \beta\delta$, that is, $\delta \geq \alpha$, which implies that $\delta = \alpha$. In this case $s \cdot e_\gamma = t \cdot t^{-1}s$.

(b) If $\gamma = \alpha$, then again $\alpha = \beta\delta$, and we have $s \cdot s^{-1}t = t \cdot e_\delta$.

(c) If $\gamma < \alpha$ or $\gamma \perp \alpha$, then $s \cdot e_\gamma = e_{\alpha\gamma} = e_{\beta\delta} = t \cdot e_\delta$.

Suppose now that $\mathbf{R}(s,t)\mathcal{J}$ is generated by a finite set $X' \subseteq Y \times Y$. Define the finite set

$$X = X' \cup \{(\mu,\nu) : \mu,\nu > \alpha, \alpha\mu = \beta\nu\}.$$

By the previous observation it is straightforward that X generates $\mathbf{R}^Y(\alpha,\beta)$. However, we still need to check that X satisfies the required property. For this, let $(\gamma,\delta) \in \mathbf{R}^Y(\alpha,\beta)$ such that $\gamma \perp \alpha$ and $\delta \perp \beta$. In this case $s \cdot e_\gamma = e_{\alpha\gamma} = e_{\beta\delta} = t \cdot e_\delta$, that is, $(\gamma,\delta) \in \mathbf{R}(s,t)\mathcal{J}$. This implies that there exist $(\mu,\nu) \in X'$ and $\epsilon \in Y$ such that $(\gamma,\delta) = (\mu,\nu)\epsilon$. Note that if $\mu \leq \alpha$, then clearly $\gamma = \mu\epsilon \leq \alpha$, a contradiction. So $\mu \not\leq \alpha$, and similarly, $\nu \not\leq \beta$. If $\mu > \alpha$, then necessarily $\nu \not> \alpha$. However,

$\alpha = \alpha\mu = \beta\nu$ implies that $\nu \geq \alpha$, thus $\nu = \alpha$, contradicting the previous observation. That is, we have proven that $\mu \perp \alpha$, but in this case $\beta\nu = \alpha\mu < \alpha \leq \beta$, which implies that either $\nu < \beta$ or $\beta \perp \nu$. Since the first case leads to contradiction, we have that $\nu \perp \beta$, which shows that X has the required property.

For the converse part, suppose now that $\mathbf{R}^Y(\alpha, \beta)$ is generated by X satisfying the required property. Define the finite set

$$X' = (X \setminus \{(\mu, \nu) : \mu, \nu > \alpha\}) \cup \{(\alpha, \nu) : \alpha = \beta\nu\} \cup \{(\mu, \alpha) : \mu \geq \alpha\}.$$

We claim that X' generates $\mathbf{R}(s,t)\mathcal{J}$. First, note that by equation (1), $X' \subseteq \mathbf{R}(s,t)\mathcal{J}$. Now let $(\gamma, \delta) \in \mathbf{R}(s,t)\mathcal{J}$. Recall that in this case $\gamma > \alpha$ implies that $\delta \not> \alpha$. Furthermore, since $\mathbf{R}(s,t)\mathcal{J} \subseteq \mathbf{R}^Y(\alpha, \beta)$, there exist $(\mu, \nu) \in X$ and $\epsilon \in Y$ such that $(\gamma, \delta) = (\mu, \nu)\epsilon$. If $\mu \not> \alpha$ or $\nu \not> \alpha$, then $(\mu, \nu) \in X'$, which implies that $(\gamma, \delta) \in X' \cdot Y$, so we can suppose that $\mu > \alpha$ and $\nu > \alpha$. There are several different cases.

(a) If $\gamma > \alpha$, then $\delta \not> \alpha$. However, $\alpha = \alpha\gamma = \beta\delta$, so $\delta \geq \alpha$, which implies that $\delta = \alpha$, thus $(\gamma, \delta) = (\gamma, \alpha) \in X'$.

(b) If $\gamma = \alpha$, then similarly to the previous case we have that $\alpha = \beta\delta$, and so $(\gamma, \delta) = (\alpha, \delta) \in X'$ also.

(c) If $\gamma < \alpha$, then the facts that $\mu > \alpha, \gamma = \mu\epsilon$ imply that $\alpha\epsilon \leq \mu\epsilon = \gamma \leq \alpha\epsilon$, that is, $\gamma = \alpha\epsilon$. In this case $(\gamma, \delta) = (\alpha, \nu)\epsilon$. Furthermore, $\alpha = \alpha\mu = \beta\nu$ implies that $(\alpha, \nu) \in X'$, thus we have that $(\gamma, \delta) \in X' \cdot Y$.

(d) If $\delta < \alpha$, then similarly to the previous case we have that $\delta = \alpha\epsilon$, and so $(\gamma, \delta) = (\mu, \alpha)\epsilon \in X' \cdot Y$ also.

(e) If $\gamma \perp \alpha$, then $\alpha\delta \leq \beta\delta = \alpha\gamma < \alpha \leq \beta$, so either $\delta < \alpha$, which case was settled before, or $\delta \perp \beta$, which means that there exist $(\mu', \nu') \in X$ and $\epsilon' \in Y$ such that $(\gamma, \delta) = (\mu', \nu')\epsilon'$ and $\mu' \perp \alpha$, $\nu' \perp \beta$. This latter property implies that $(\mu', \nu') \in X'$, thus $(\gamma, \delta) \in X' \cdot Y$.

(2) There are several subcases here:

(a) $\alpha = \beta$ and $s = t$: in this case we can in fact suppose that $s = t = e_\alpha$,

(b) $\alpha < \beta$ and $s = e_\alpha$,

(c) $\alpha > \beta$ and $t = e_\beta$,

(d) $\alpha \perp \beta$.

It is easy to check that in all these cases if $\alpha\gamma = \beta\delta$ for some $\gamma, \delta \in Y$, then necessarily $se_\gamma = te_\delta$ holds, which shows that $\mathbf{R}^Y(\alpha, \beta) = \mathbf{R}(s,t)\mathcal{J}$.

□

In the remainder of this section we are going to investigate Condition ($\mathbf{r}$). Note that for any $s, t \in S$, $\mathbf{r}(s,t)$ is an ideal of S, so is a union of maximal subgroups, that is, a union of G_α's. First we handle some basic cases.

Lemma 5.3. *If $\alpha, \beta \in Y$ then $\mathbf{r}(e_\alpha, e_\beta) = \bigcup_{\gamma \in \mathbf{r}^Y(\alpha,\beta)} G_\gamma$.*

Proof. Let $s \in \mathbf{r}(e_\alpha, e_\beta)$, with $s \in G_\gamma$. Then $e_\alpha s = e_\beta s$ so that $\alpha\gamma = \beta\gamma$ and $\gamma \in \mathbf{r}^Y(\alpha, \beta)$. Thus $\mathbf{r}(e_\alpha, e_\beta) \subseteq \bigcup_{\gamma \in \mathbf{r}^Y(\alpha,\beta)} G_\gamma$.

On the other hand, if $t \in G_\gamma$ where $\gamma \in \mathbf{r}^Y(\alpha,\beta)$, then $e_\alpha e_\gamma = e_\beta e_\gamma$ gives us $e_\alpha t = e_\beta t$ and so $t \in \mathbf{r}(e_\alpha, e_\beta)$. Hence $\mathbf{r}^S(e_\alpha, e_\beta) \supseteq \bigcup_{\gamma \in \mathbf{r}^Y(\alpha,\beta)} G_\gamma$. □

Lemma 5.4. *If $\alpha > \beta$, $s \in G_\alpha$, and $t \in G_\beta$, then $\mathbf{r}(s,t) = \mathbf{r}(e_\alpha, t)$.*

Proof. Let $u \in S$ such that $su = tu$. In this case $(su)\mathcal{J} = (e_\alpha u)\mathcal{J} < \alpha$, which implies that $e_\alpha u = su = tu$. Vice versa, if $e_\alpha u = tu$, then necessarily $su = tu$ also. Thus $\mathbf{r}(s,t) = \mathbf{r}(e_\alpha, t)$. □

To characterize condition ($\mathbf{r}$) we first need some notation. For every $\alpha, \beta \in Y$ let

$$D_\alpha = \{\tau \in Y : \tau \not\geq \alpha\} \text{ and } U_{\alpha,\beta} = \{\gamma \in Y : \alpha\gamma < \beta\}$$

so that $D_\alpha, U_{\alpha,\beta}$ are ideals of Y, and let

$$I_\alpha = \bigcup_{\tau \in D_\alpha} G_\tau \text{ and } J_{\alpha,\beta} = \bigcup_{\tau \in U_{\alpha,\beta}} G_\tau,$$

so that I_α and $J_{\alpha,\beta}$ are ideals of S.

Lemma 5.5. *Let $s, t \in G_\alpha$ with $s \neq t$. Then $I_\alpha = \mathbf{r}(s,t)$.*

Proof. Let $e_\tau \in I_\alpha$ so that $\tau \not\geq \alpha$, which implies that $\alpha\tau < \alpha$. We have

$$se_\tau = se_\alpha e_\tau = se_{\alpha\tau} = e_{\alpha\tau} = \ldots = te_\tau.$$

Therefore $e_\tau \in \mathbf{r}(s,t)$, and so $G_\tau \subseteq \mathbf{r}(s,t)$. Hence $I_\alpha \subseteq \mathbf{r}(s,t)$.

Conversely, suppose that $G_\kappa \subseteq \mathbf{r}(s,t)$ so that in particular $se_\kappa = te_\kappa$. From $s \neq t$ we deduce $\kappa \not\geq \alpha$, and so $G_\kappa \subseteq I_\alpha$. We therefore have $\mathbf{r}(s,t) \subseteq I_\alpha$. □

Lemma 5.6. *If $\alpha \perp \beta$ then for any $s \in G_\alpha, t \in G_\beta$, $\mathbf{r}(s,t) = \cup_{\gamma \in \mathbf{r}^Y(\alpha,\beta)} G_\gamma$.*

Proof. If $\gamma \in \mathbf{r}^Y(\alpha,\beta)$ then as $\alpha \perp \beta$ we have $\alpha\gamma = \beta\gamma < \alpha, \beta$, so $se_\gamma = e_{\alpha\gamma} = e_{\beta\gamma} = te_\gamma$, implying that $G_\gamma \subseteq \mathbf{r}(s,t)$ and so $\cup_{\gamma \in \mathbf{r}^Y(\alpha,\beta)} G_\gamma \subseteq \mathbf{r}(s,t)$.

Conversely, if $g \in \mathbf{r}(s,t) \cap G_\gamma$ then $sg = tg$ so $\alpha\gamma = \beta\gamma$ and $g \in \cup_{\gamma \in \mathbf{r}^Y(\alpha,\beta)} G_\gamma$. Therefore the claim is proved. □

Lemma 5.7. *Suppose $\beta < \alpha$ and $t \in G_\beta \setminus \{e_\beta\}$. Then $\mathbf{r}(e_\alpha, t) = J_{\alpha,\beta}$.*

Proof. Let $u \in G_\gamma$ be such that $u \in \mathbf{r}(e_\alpha, t)$. Then $e_\alpha u = tu$ implies $e_\alpha e_\gamma = te_\gamma$ so that $\alpha\gamma = \beta\gamma \leq \beta$. If $\alpha\gamma = \beta$ then $\beta \leq \gamma$ so that $e_\beta = te_\gamma = t$, a contradiction. Therefore $\alpha\gamma < \beta$, so that $\gamma \in U_{\alpha,\beta}$ and $u \in \bigcup_{\tau \in U_{\alpha,\beta}} G_\tau$. Hence $\mathbf{r}(e_\alpha, t) \subseteq J_{\alpha,\beta}$.

Conversely, let $u \in J_{\alpha,\beta}$. Then $u \in G_\tau$ for some τ with $\alpha\tau < \beta$, so that

$$\alpha\tau = \alpha\tau\beta = \beta\tau < \beta$$

and

$$e_\alpha u = e_\alpha e_\tau u = te_\alpha e_\tau u = te_\beta e_\tau u = tu$$

so that $u \in \mathbf{r}(e_\alpha, t)$ and $J_{\alpha,\beta} \subseteq \mathbf{r}(e_\alpha, t)$. □

Theorem 5.8. *Let S be a monoid which is a semilattice Y of groups G_α such that the structure homomorphisms are trivial. Then S satisfies* $(\mathbf{r})$ *if and only if*

(i) D_α is finitely generated for any $\alpha \in Y$ with $|G_\alpha| > 1$,

(ii) $\mathbf{r}^Y(\alpha, \beta)$ is finitely generated for any $\alpha, \beta \in Y$,

(iii) $U_{\alpha,\beta}$ is finitely generated for any $\alpha, \beta \in Y$ with $G_\beta \neq \{e_\beta\}$ and $\alpha > \beta$.

Proof. Let $a \in G_\alpha$ and $b \in G_\beta$. If $\alpha \perp \beta$, then by Lemma 5.6, $\mathbf{r}(a, b) = \bigcup_{\gamma \in \mathbf{r}^Y(\alpha,\beta)} G_\gamma$. If $\alpha > \beta$, then $\mathbf{r}(a, b) = \mathbf{r}(e_\alpha, b)$ by Lemma 5.4. If $b = e_\beta$, then $\mathbf{r}(a, b) = \mathbf{r}(e_\alpha, e_\beta) = \bigcup_{\gamma \in \mathbf{r}^Y(\alpha,\beta)} G_\gamma$ by Lemma 5.3. If $t \neq e_\beta$, then $\mathbf{r}(a, b) = J_{\alpha,\beta}$ by Lemma 5.7. Finally, if $\alpha = \beta$, then either $a = b$ so that $\mathbf{r}(a, b) = S$, or if $s \neq t$, then $\mathbf{r}(s, t) = I_\alpha$ by Lemma 5.5.

The result now follows from the observation that every (right) ideal K of S is of the form $K = \bigcup_{\alpha \in I} G_\alpha$ for some ideal I of Y, and that K is finitely generated if and only if I is finitely generated. □

Note that if Y satisfies (WRN) then clearly all these properties hold.

Consider the case where Y is a chain. Then for $\alpha < \beta$ we have that $\mathbf{r}^Y(\alpha, \beta) = \alpha Y$ is finitely generated. However, $D_\alpha = \{\gamma : \gamma < \alpha\}$ is finitely generated if and only if α has a greatest predecessor. Furthermore, if $\alpha > \beta$, then $U_{\alpha,\beta} = D_\beta$. Thus S satisfies $(\mathbf{r})$ if and only if for every $\alpha \in Y$, if $|G_\alpha| > 1$ then α has a greatest predecessor.

As we have seen before, $(\mathbf{R})$ implies $(\mathbf{r})$ in case of semilattices. As the following example shows, this is not true for Clifford monoids in general.

Example 5.9. *Let Y be the semilattice of finite subsets of $\mathbb{N}$ under union, let $G_\alpha = \{e_\alpha\}$ for all $\alpha \neq \emptyset$ and let $G_\emptyset = \{e_\emptyset, a\}$ be the two-element group. Let S be the Clifford monoid having Y as its structure semilattice, and the groups G_α as its $\mathcal{H}$-classes (so every structure homomorphism has to be trivial). Then S satisfies* $(\mathbf{R})$*, but not* $(\mathbf{r})$*.*

Proof. It is routine to check that $\mathbf{R}(s, t)\mathcal{J} = \mathbf{R}^Y(s\mathcal{J}, t\mathcal{J})$ for every $s, t \in S$, thus it is finitely generated Lemma 3.5. In this case S satisfies $(\mathbf{R})$ by Theorem 5.1. However, $D_\emptyset = Y \setminus \{\emptyset\}$, which is not finitely generated (it has infinitely many maximal elements), so S does not satisfy $(\mathbf{r})$. □

The examples and results given so far enables us to investigate the connection between the conditions $(\mathbf{R})$, $(\mathbf{r})$ and (WRN).

Theorem 5.10. *The only valid implication between the conditions* $(\mathbf{R})$, $(\mathbf{r})$ *and* (WRN) *is that* (WRN) *implies* $(\mathbf{r})$.

In case of semilattices, $(\mathbf{R})$ *implies* $(\mathbf{r})$ *as well, but no other implications become valid.*

Proof. By definition, (WRN) implies $(\mathbf{r})$, and if S is a semilattice, then $(\mathbf{R})$ implies $(\mathbf{r})$ by Lemma 3.2. We have already seen examples contradicting all other implications. □

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