AXIOMATISABILITY PROBLEMS FOR S-POSETS

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Abstract. Let $\mathcal{C}$ be a class of ordered algebras of a given fixed type τ. Associated with the type is a first order language L_τ, which must also contain a binary predicate to be interpreted by the ordering in members of $\mathcal{C}$. One can then ask the question, when is the class $\mathcal{C}$ axiomatisable by sentences of L_τ? In this paper we will be considering axiomatisability problems for classes of left S-posets over a pomonoid S (that is, a monoid S equipped with a partial order compatible with the binary operation). We aim to determine the pomonoids S such that certain categorically defined classes are axiomatisable. The classes we consider are the free S-posets, the projective S-posets and classes arising from flatness properties. Some of these cases have been studied in a recent article by Pervukhin and Stepanova. We present some general strategies to determine axiomatisability, from which their results for the classes of weakly po-flat and po-flat S-posets will follow. We also consider a number of classes not previously examined.

1. Introduction

A *pomonoid* is a monoid S with a partial order $\leq$ which is compatible with the binary operation. Just as the representation of a monoid M by mappings of sets gives us the theory of M-acts, representations of a pomonoid S by order-preserving maps of partially ordered sets give us S-posets. Thus a *left S-poset* is a non-empty partially ordered set A on which S-acts on the left, that is, there is a map $S \times A \to A$, where $(s, a) \mapsto sa$ such that for all $s, t \in S$ and $a \in A$,

$$s(t(a)) = (st)a \text{ and } 1\,a = a$$

such that the map is monotone in both co-ordinates, that is, for all $s, t \in S$ with $s \leq t$ and $a, b \in A$ with $a \leq b$,

$$sa \leq ta \text{ and } sa \leq sb.$$

The class of all left S-posets is denoted by S-Pos. It is worth pointing out in this Introduction that S-posets (indeed, pomonoids) are not merely algebras, they are relational structures. As such, care is needed to take account of the partial order relation, particularly when considering congruences.

A morphism $\phi : A \to B$ from a left S-poset A to a left S-poset B is called an *S-poset morphism* or more briefly, *S-pomorphism*, if it preserves the action of S (that is, it is an S-act morphism) and the ordering on A. In other words, for all $a, b \in A$ with $a \leq b$ and $s \in S$ we have

$$(as)\phi = (a\phi)s \text{ and } a\phi \leq b\phi.$$

It is an *S-po-isomorphism* if, in addition, it is a bijection such that the inverse is also an S-pomorphism, so that for all $a, b \in A$ we have $a \leq b$ if and only if $a\phi \leq b\phi$. We then say that A and B are *isomorphic* and write $A \cong B$. Note that a bijective S-pomorphism need not to be S-po-isomorphism. An S-pomorphism $f : A \to B$ between two left S-posets A and B is called an *S-embedding* if $f : A \to Af$ is an S-po-isomorphim.

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We denote the category of left S-posets and S-pomorphisms by **S-Pos**. Dual definitions give us the class Pos-S of right S-posets which together with the corresponding notion of S-pomorphisms yields the category **Pos-S** of right S-posets and S-pomorphisms.

The study of M-acts over a monoid M has been well established since the 1960s, and received a boost following the publication of the monograph [15] in 2000. On the other hand, the investigation of S-posets, initiated by Fakhruddin in the 1980s [7], [8], was not taken up again until this millenium, which has seen a burst of activity on this topic, mostly (but not exclusively) concentrating on projectivity and various notions of flatness for S-posets, as we do here. Definitions and concepts relating to flatness are given in Section 2: an excellent survey is given in [3].

Associated with the class S-Pos for a pomonoid S we have a first order language $L_S^{\leq}$, which has no constant symbols, a unary function symbol λ_s for each $s \in S$, and (other than $=$), a single relational symbol $\leq$ with $\leq$ being binary. An S-poset provides an interpretation of $L_S^{\leq}$ in the obvious way, indeed in $L_S^{\leq}$ we write sx for $\lambda_s(x)$. A class of $\mathcal{C}$ of left S-posets is *axiomatisable* (or *elementary*) if there is a set of sentences Π of $L_S^{\leq}$ such that for any interpretation A of $L_S^{\leq}$, A lies in $\mathcal{C}$ if and only if all sentences of Π are true in A, that is, A is a *model* of Π. We say in this case that Π *axiomatises* $\mathcal{C}$. We note that S-Pos itself is axiomatisable amongst all interpretations of $L_S^{\leq}$. For any $s, t \in S$ and $u, v \in S$ with $u \leq v$ we define sentences

$$\varphi_{s,t} := (\forall x)\big(s(t(x)) = (st)x\big),\ \theta_s := (\forall x, y)\big(x \leq y \to sx \leq sy\big)$$

and

$$\psi_{u,v} := (\forall x)(ux \leq vx).$$

Then Π_S axiomatises S-Pos where

$$\Pi_S = \{(\forall x)(1\,x = x)\} \cup \big\{\varphi_{s,t} : s, t \in S\big\} \cup \big\{\theta_s : s \in S\big\} \cup \big\{\psi_{u,v} : u, v \in S, u \leq v\big\}.$$

Some classes of left S-posets are axiomatisable for *any* monoid S. For example, the class $\mathcal{T}$ of left S-posets with the trivial partial order is axiomatised by

$$\Pi_S \cup \{(\forall x, y)\big(x \leq y \to x = y\big)\}.$$

To save repetition, we will assume from now on that when axiomatising a class of left S-posets, Π_S is understood, so that we would say $\{(\forall x, y)\big(x \leq y \to x = y\big)\}$ axiomatises $\mathcal{T}$. Other natural classes of left S-posets are axiomatisable for some pomonoids and not for others and it is our aim here to investigate the pomonoids that arise.

Corresponding questions for classes of M-acts over a monoid M have been answered in [11, 21, 1] and [12], see also the survey article [13]. The classes of projective (strongly flat, po-flat, weakly po-flat) left S-posets $\mathcal{P}r\,(\mathcal{SF}, \mathcal{PF}, \mathcal{WPF})$ have recently been considered in [16] (which uses slightly different terminology; the results also appearing in [18]) as has the class $\mathcal{F}r$ of free left S-posets in the case where S has only finitely many right ideals. We note that many of the techniques of [16] follow those in the M-act case and, for this reason, we aim here to produce two general strategies that will deal with a number of axiomatisability questions for classes of S-posets (and, with minor adjustment, M-acts). In particular they may be applied to $\mathcal{PF}$ and $\mathcal{WPF}$. Just as many concepts of flatness that are equivalent for R-modules over a unital ring R are different for M-acts, so many concepts that coincide for M-acts split for S-posets. Thus [16] left a number of classes open; we address many of them here, with both our general techniques and ad hoc methods.

The structure of the paper is as follows. After Section 2 which gives brief details of the concepts required to follow this article, we consider in Section 3 classes defined by flatness properties that translate into so called 'interpolation conditions'. In these cases we can give arguments that are rather more direct than those of subsequent sections. In Sections 4 and 5 we present our *general* axiomatisability results, which apply to various classes defined by flatness properties. There are two kinds of results, both phrased in terms of 'replacement tossings'; we show how they may be applied to reproduce the results of [16] determining for which pomonoids $\mathcal{PF}$ or $\mathcal{PWF}$ are axiomatisable, together with a number of other applications. Section 6 briefly visits the question of axiomatisability of $\mathcal{F}r$ and $\mathcal{P}r$; the results here are easily deducible from the corresponding ones for M-acts. In Section 7 we illustrate our ideas with some straightforward pomonoids, which nevertheless indicate the subtlety of the ideas involved. Finally in Section 8 we present some open problems.

2. Preliminaries: flatness properties for S-posets

Free and projective S-posets have the standard categorical definitions. We remark that [16] distinguishes between S-posets over a pomonoid S that are free over posets and those free over sets: the free S-posets we consider here are what [16] would refer to as *free over sets*. The classes of free (projective) left S-posets are denoted by $\mathcal{F}r$ ($\mathcal{P}r$), respectively. The structure of S-posets in $\mathcal{F}r$ and $\mathcal{P}r$ is transparent.

First note that for a symbol x we let $Sx = \{sx : s \in S\}$ be a set in one-one correspondence with S such that Sx becomes a left S-poset (isomorphic to S) where we define $s(tx) = (st)x$ for all $s, t \in S$ and $sx \leq tx$ if and only if $s \leq t$ in S.

Theorem 2.1. [20] (i) *An S-poset A is free on a set X if and only if $A \cong \bigcup_{x \in X} Sx$ where for all $x, y \in X$ and $s, t \in S$,*

$$sx \leq ty \text{ if and only if } x = y \text{ and } s \leq t.$$

(ii) *An S-poset B is projective if and only if $B \cong \bigcup_{x \in X} Se_x x$ for some set X and idempotents e_x of S.*

As in the unordered case, it is clear that every free S-poset is projective and (provided S has idempotents other than 1), the converse is not true.

To define notions of flatness, we need to consider the tensor product of S-posets. Let A be a right S-poset and B a left S-poset. The tensor product, which is denoted by $A \otimes B$, is the quotient of $A \times B$, considered as an S-poset under trivial S-action, by the order congruence relation θ on $A \times B$ generated by

$$\{(as, b), (a, sb) : s \in S, a \in A, b \in B\}.$$

We will denote the equivalence class of $(a, b) \in A \times B$ with respect to congruence θ by $a \otimes b$. We say a little more about order congruences in Section 4. The following lemma explains the ordering in $A \otimes B$.

Lemma 2.2. [20] *Let S be a pomonoid, let A be a right S-poset, B a left S-poset, $a, a' \in A$, and $b, b' \in B$. Then $a \otimes b \leq a' \otimes b'$ in $A \otimes B$ if and only if there exists $a_2, a_3, \cdots, a_m \in A$, $b_1, b_2, \cdots, b_m \in B$ and $s_1, t_1, \cdots, s_m, t_m \in S$ such that $(*)$ holds:*

$$\begin{array}{rclrcl} & & & b & \leq & s_1 b_1 \\ a s_1 & \leq & a_2 t_1 & t_1 b_1 & \leq & s_2 b_2 \\ a_2 s_2 & \leq & a_3 t_2 & t_2 b_2 & \leq & s_3 b_3 \\ & \vdots & & & \vdots & \\ a_m s_m & \leq & a' t_m & t_m b_m & \leq & b' \end{array} \qquad (*)$$

It follows that $a' \otimes b' \leq a \otimes b$ if and only if there exists $c_2, \cdots c_n \in A$ and $d_1, \cdots, d_n \in B$ and $u_1, v_1, \cdots, u_n, v_n \in S$ such that $(**)$ holds:

$$\begin{array}{rclrcl} & & & b' & \leq & u_1d_1 \\ a'u_1 & \leq & c_2v_1 & v_1d_1 & \leq & u_2d_2 \\ c_2u_2 & \leq & c_3v_2 & v_2d_2 & \leq & u_3d_3 \\ & \vdots & & & \vdots & \\ c_nu_n & \leq & av_n & v_nd_n & \leq & b \end{array} \qquad (**)$$

Thus $a \otimes b = a' \otimes b'$ in $A \otimes B$ if and only if $(*)$ and $(**)$ hold.

Definition 2.3. The sequence $(*)$ is called an *ordered tossing* $\mathcal{T}$ of length m from (a,b) to (a',b'). The *ordered skeleton* $\mathcal{S}(\mathcal{T})$ of $\mathcal{T}$ is the sequence $\mathcal{S}(\mathcal{T}) = (s_1, t_1, \cdots, s_m, t_m)$. The two sequences $(*)$ and $(**)$ constitute a *double ordered tossing* $\mathcal{DT}$ of length $m+n$, from (a,b) to (a',b') with *double ordered skeleton*

$$\mathcal{S}(\mathcal{DT}) = (s_1, t_1, \cdots, s_m, t_m, u_1, v_1, \cdots, u_n, v_n).$$

We may also write $\mathcal{S}(\mathcal{DT}) = (\mathcal{S}_1, \mathcal{S}_2)$ where

$$\mathcal{S}_1 = (s_1, t_1, \cdots, s_m, t_m) \text{ and } \mathcal{S}_2 = (u_1, v_1, \ldots, u_n, v_n).$$

As in the case of M-acts different notions of flatness are drawn from the tensor functor

$$- \otimes B : \mathbf{Pos\text{-}S} \to \mathbf{Pos}$$

where if A, A' are right S-posets and $f : A \to A'$ is a pomorphism,

$$A \mapsto A \otimes B \text{ and } f \mapsto f \otimes I_B$$

and

$$(a \otimes b)(f \otimes I_B) = af \otimes b.$$

Elementary considerations of partially ordered sets (regarded as S-acts over a trivial pomonoid) tell us that S-monomorphisms and S-embeddings in **S-Pos**, and indeed bijective S-pomorphisms and S-po-isomorphisms, are not the same. This leads us to two variations on notions of flatness.

An S-poset A is called *flat* if the functor $- \otimes B$ takes S-embeddings in the category **Pos-S** to monomorphisms in the category **Pos** of posets. It is called *(principally) weakly flat* if the functor $- \otimes B$ takes S-embeddings of (principal) right ideals of S into S to monomorphisms. In [19] Shi defined notions of po-flat, weakly po-flat, principally weakly po-flat S-posets, as follows. An S-poset B is called *po-flat* if the functor $- \otimes B$ takes S-embeddings in the category of **Pos-S** to embeddings in **Pos**. It is *(principally) weakly po-flat* if the functor $- \otimes B$ preserves the S-embeddings of (principal) right ideals of S into S.

A left S-poset B is called *strongly flat* if the functor $- \otimes B$ preserves subpullbacks and subequalizers, or equivalently [2] if B satisfies Condition (P) and Condition (E) which are defined below. For details relating to subpullback diagrams in the category **S-Pos** we refer the reader to [10].

Condition (P): for all $b, b' \in B$ and $s, s' \in S$, if $s\,b \leq s'\,b'$ then there exists $b'' \in B$ and $u, u' \in S$ such that $b = u\,b''$, $b' = u'\,b''$ and $s\,u \leq s'\,u'$;

Condition (E): for all $b \in B$ and $s, s' \in S$, if $s\,b \leq s'\,b$ then there exists $b'' \in B$ and $u \in S$ such that $b = u\,b''$ and $s\,u \leq s'\,u$.

Such flatness conditions, i.e. using elements of S and S-posets rather than tossings explictly, we call *interpolation conditions*. Weaker than either (P) or (E) we have

Condition (EP): for all $b \in B$ and $s, s' \in S$, if $sb \leq s'b$ then there exists $b'' \in B$ and $u, u' \in S$ such that $b = ub'' = u'b''$ and $su \leq s'u'$. The unordered version of this condition was introduced for M-acts in [9].

In the theory of M-acts over a monoid M, it is true that all M-acts satisfy the unordered version of Condition (P) if and only if M is a group. We can, however, find an S-poset over an ordered group S which does not satisfy Condition (P). With this in mind, Shi [19] defined

another notion for a left S-poset B similar to Condition (P), called Condition (P_w):
Condition (P_w): for all $b, b' \in B$ and $s, s' \in S$, if $s\,b \leq s'\,b'$ then there exists $b'' \in B$, $u, u' \in S$ such that $s\,u \leq s'\,u'$, $b \leq u\,b''$ and $u'\,b'' \leq b'$.

Let G be an ordered group, then all G-posets satisfy Condition (P_w) [19]. Clearly (P) implies (P_w) and from [19], (P_w) implies po-flat.

We will denote the classes strongly flat, flat, weakly flat, principally weakly flat, po-flat, weakly po-flat, principally weakly po-flat left S-posets by

$$\mathcal{SF},\ \mathcal{F},\ \mathcal{WF},\ \mathcal{PWF},\ \mathcal{PF},\ \mathcal{WPF},\ \mathcal{PWPF}$$

respectively. We will denote the classes of left S-posets satisfying Conditions (P), (E), (EP) and (P_w) by

$$\mathcal{P}, \mathcal{E}, \mathcal{EP} \text{ and } \mathcal{P}_w.$$

Finally in our list of flatness properties we turn our attention to those introduced in [10] by Golchin and Rezaei. They define Conditions (WP),(WP_w),(PWP) and (PWP_w) for S-posets, which are derived from the concepts of subpullback diagrams in **S-Pos**. For our purposes here it is enough to define (PWP) and (PWP_w) for a left S-poset B:
Condition (PWP): for all $b, b' \in B$ and $s \in S$, if $sb \leq sb'$ then there exist $u, u' \in S$ and $b'' \in B$ such that $b = ub''$, $b' = u'b''$ and $su \leq su'$;
Condition (PWP_w): for all $b, b' \in B$ and $s \in S$, if $sb \leq sb'$ then there exist $u, u' \in S$ and $b'' \in B$ such that $b \leq ub''$, $u'b'' \leq b'$ and $su \leq su'$.

We denote by

$$\mathcal{WP}, \mathcal{WP}_w, \mathcal{PWP} \text{ and } \mathcal{PWP}_w$$

the classes of left S-posets satisfying Conditions (WP),(WP_w),(PWP) and (PWP_w), respectively.

Remark 2.4. [10][20] In **S-POS** we have the following implications, all of which are known to be strict except for Condition (P_w) implies po-flat:

$$\begin{array}{ccccccccccc}
\mathcal{F}r & \Rightarrow & \mathcal{P}r & \Rightarrow & \mathcal{SF} & \Rightarrow & \mathcal{P} & \Rightarrow & \mathcal{WP} & \Rightarrow & \mathcal{PWP} \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & & & & & \mathcal{P}_w & \Rightarrow & \mathcal{WP}_w & \Rightarrow & \mathcal{PWP}_w \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & & & & & \mathcal{PF} & \Rightarrow & \mathcal{WPF} & \Rightarrow & \mathcal{PWPF} \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & & & & & \mathcal{F} & \Rightarrow & \mathcal{WF} & \Rightarrow & \mathcal{PWF}
\end{array}$$

We are interested in determining for which pomonoids these classes are axiomatisable. Our major tool is that of an *ultraproduct*; further details may be found in [5]. The next result is crucial.

Theorem 2.5. (Łos's Theorem)[5] *Let L be a first order language, and let $\mathcal{C}$ be a class of L-structures. If $\mathcal{C}$ is axiomatisable, then $\mathcal{C}$ is closed under ultraproducts.*

We now introduce a new notion of flatness, that can be adapted to many of the classes above. Let $\mathcal{C}$ be a class of S-embeddings of right S-posets. For example, $\mathcal{C}$ could be all S-embeddings, or all S-embeddings of right ideals into S via inclusion maps. We say that a left S-poset B is $\mathcal{C}$*-flat* if the functor $-\otimes B$ maps every S-embedding $\mu : A \to A' \in \mathcal{C}$ to a monomorphism $\mu \otimes I_B : A \otimes B \to A' \otimes B$. The class of $\mathcal{C}$-flat left S-posets is denoted by $\mathcal{CF}$. Similarly, if $-\otimes B$ maps every S-embedding $\mu : A \to A' \in \mathcal{C}$ to an embedding $\mu \otimes I_B : A \otimes B \to A' \otimes B$, then we say that B is $\mathcal{C}$*-po-flat* and we denote the class of $\mathcal{C}$-po-flat left S-posets by $\mathcal{CPF}$. Thus, if $\mathcal{C}$ is the class of all S-embeddings of right S-posets, then $\mathcal{CF} = \mathcal{F}$ and $\mathcal{CPF} = \mathcal{PF}$.

3. Axiomatisability of some specific classes of S-posets

We now concentrate on axiomatisability problems for certain classes of left S-posets, in the cases that we can avoid the 'replacement tossings' arguments of Sections 4 and 5. We consider the classes of left S-posets satisfying Condition (P) and (E) (which together give us the class of strongly flat left S-posets), and the classes of left S-posets satisfying Condition (EP), (P_w), (PWP) and (PWP_w).

Let S be a pomonoid and let $(s,t) \in S \times S$. We define

$$\mathbf{R}^{\leq}(s,t) = \{(u,v) \in S \times S : su \leq tv\} \text{ and } \mathbf{r}^{\leq}(s,t) = \{u \in S : su \leq tu\}$$

so that $\mathbf{R}^{\leq}(s,t)$ is either empty or is an S-subposet of the right S-poset $S \times S$, and $\mathbf{r}^{\leq}(s,t)$ is either empty or is a right ideal of S. Note that in [16], $\mathbf{R}^{\leq}(s,t)$ and $\mathbf{r}^{\leq}(s,t)$ are written as $R^{<}(s,t)$ and $r^{<}(s,t)$.

3.1. Conditions (P) and (E) and the class $\mathcal{SF}$.

For completeness we give the following results from [16]; they may also be found in the thesis of the second author [18]. The proofs follow closely those of the unordered case in [11],[12] and [13].

Theorem 3.1. [16] *Let S be a pomonoid.*

(1) The class of left S-posets satisfying Condition (P) is axiomatisable if and only if for every $s,t \in S$, $\mathbf{R}^{\leq}(s,t)$ is empty or is finitely generated.

(2) The class of left S-posets satisfying Condition (E) is axiomatisable if and only if for every $s,t \in S$, $\mathbf{r}^{\leq}(s,t)$ is empty or is finitely generated.

(3) The class of $\mathcal{SF}$ of strongly flat left S-posets is axiomatisable if and only if for every $s,t \in S$, $\mathbf{R}^{\leq}(s,t)$ is empty or is finitely generated and $\mathbf{r}^{\leq}(s,t)$ is empty or is finitely generated.

Note that in each case in Theorem 3.1, the axiomatisability of the class is equivalent to it being closed under ultrapowers of copies of S.

3.2. Condition (EP).

We recall from Section 1 that, in the terminology introduced above, a left S-poset A satisfies Condition (EP) if, given $sa \leq ta$ for any $s,t \in S$ and $a \in A$, we have that $a = ua' = va'$ for some $(u,v) \in \mathbf{R}^{\leq}(s,t)$ and $a' \in A$.

Theorem 3.2. *The following conditions are equivalent for a pomonoid S:*

(1) the class $\mathcal{EP}$ is axiomatisable;

(2) the class $\mathcal{EP}$ is closed under ultraproducts;

(3) for any $s,t \in S$ either $sa \not\leq ta$ for all $a \in A \in \mathcal{EP}$ or there exists a finite subset f of $\mathbf{R}^{\leq}(s,t)$, such that for any $a \in A \in \mathcal{EP}$

$$sa \leq ta \Rightarrow (a,a) = (u,v)b \text{ for some } (u,v) \in f, b \in A.$$

Proof. (1) $\Rightarrow$ (2) This follows from Łos's Theorem.

(2) $\Rightarrow$ (3) Suppose $sa \leq ta$ for some $a \in A \in \mathcal{EP}$ and for each finite subset f of $\mathbf{R}^{\leq}(s,t)$, there exists $A_f \in \mathcal{EP}$, $a_f \in A_f$ with $sa_f \leq ta_f$ and $(a_f, a_f) \notin fA_f$.

Let J be the set of finite subsets of $\mathbf{R}^{\leq}(s,t)$. For each $(u,v) \in \mathbf{R}^{\leq}(s,t)$ we define

$$J_{(u,v)} = \{f \in J : (u,v) \in f\}$$

As each intersection of finitely many of the sets $J_{(u,v)}$ is non-empty, we are able to define an ultrafilter Φ on J, such that each $J_{(u,v)} \in \Phi$ for all $(u,v) \in \mathbf{R}^{\leq}(s,t)$.

Now $s(a_f) \leq t(a_f)$ in A where $A = \prod_{f \in J} A_f$, and it follows that the inequality $s(a_f)_\Phi \leq t(a_f)_\Phi$ holds in $\mathcal{U}$ where $\mathcal{U} = \prod_{f \in J} A_f/\Phi$. By assumption, $\mathcal{U}$ lies in $\mathcal{EP}$, so there exists $(u,v) \in \mathbf{R}^{\leq}(s,t)$, and $r_f \in A_f$ such that $(a_f)_\Phi = u(r_f)_\Phi = v(r_f)_\Phi$.

As Φ is closed under finite intersections, there must exist $T \in \Phi$ such that $a_f = ur_f = vr_f$ for all $f \in T$.

Now suppose that $f \in T \cap J_{(u,v)}$, then $(u,v) \in f$ and

$$(a_f, a_f) = (u,v)r_f \in fA_f$$

a contradiction to our assumption, hence (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1) Given that (3) holds, we give an explicit set of sentences that axiomatises $\mathcal{EP}$.

For any element $\rho = (s,t) \in S \times S$ with $sa \leq ta$, for some $a \in A$ where $A \in \mathcal{EP}$, we choose and fix a finite set of elements $\{(u_{\rho 1}, v_{\rho 1}), \cdots, (u_{\rho n(\rho)}, v_{\rho n(\rho)})\}$ of $\mathbf{R}^{\leq}(\rho)$ as guaranteed by (3). We define sentences ϕ_ρ of $L_S^{\leq}$ as follows:

If $sa \nleq ta$ for all $a \in A \in \mathcal{EP}$, let

$$\phi_\rho = (\forall x)(sx \nleq tx);$$

otherwise,

$$\phi_\rho = (\forall x)\big(sx \leq tx \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x = u_{\rho i}z = v_{\rho i}z))\big).$$

Let

$$\sum_{\mathcal{EP}} = \{\phi_\rho : \rho \in S \times S\}.$$

We claim that $\sum_{\mathcal{EP}}$ axiomatises the class $\mathcal{EP}$.

Suppose that $A \in \mathcal{EP}$ and $\rho = (s,t) \in S \times S$. If $sb \nleq tb$, for all $b \in B \in \mathcal{EP}$, then certainly this is true for A, so that $A \models \phi_\rho$.

Suppose on the other hand that $sb \leq tb$, for some $b \in B \in \mathcal{EP}$; then

$$\phi_\rho = (\forall x)\big(sx \leq tx \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x = u_{\rho i}z = v_{\rho i}z))\big).$$

Suppose $sa \leq ta$ where $a \in A$. As $A \in \mathcal{EP}$, (3) tells us that there is an element $b \in A$ and $(u_{\rho i}, v_{\rho i})$ for some $i \in \{1, \ldots, n(\rho)\}$ with $a = u_{\rho i}b = v_{\rho i}b$. Hence $A \models \phi_\rho$.

Conversely suppose that A is a model of $\sum_{\mathcal{EP}}$ and $sa \leq ta$ where $s,t \in S$ and $a \in A$. We cannot have that ϕ_ρ is $(\forall x)(sx \nleq tx)$. It follows that for some $b \in B \in \mathcal{EP}$ we have $sb \leq tb$, $f = \{(u_{\rho\, 1}, v_{\rho\, 1}), \cdots, (u_{\rho\, n(\rho)}, v_{\rho\, n(\rho)})\}$ exists as in (3) and ϕ_ρ is

$$(\forall x)\big(sx \leq tx \to (\exists\, z)(\bigvee_{i=1}^{n(\rho)} (\, x = u_{\rho i}z = v_{\rho i}\, z))\big).$$

Hence there exists an element $c \in A$ with $a = u_{\rho i}c = v_{\rho i}c$ for some $i \in \{1, 2, \ldots, n(\rho)\}$. By definition of $u_{\rho i}, v_{\rho i}$ we have $su_{\rho i} \leq tv_{\rho i}$. Thus A satisfies Condition (EP) and so $\sum_{\mathcal{EP}}$ axiomatises $\mathcal{EP}$. $\square$

Notice that for any $a \in A \in \mathcal{EP}$, if $sa \leq sa$ then certainly $(a,a) = (1,1)a$ and $(1,1) \in \mathbf{R}^{\leq}(s,s)$. Thus to check that the condition in (3) of Theorem 3.2 holds, it is enough to consider the cases where $s \neq t$.

Lemma 3.3. *If S is a pomonoid such that $\mathbf{R}^{\leq}(s,t)$ is finitely generated for all $s,t \in S$ with $s \neq t$, then $\mathcal{EP}$ is axiomatisable. Hence if $\mathcal{P}$ is axiomatisable, then so is $\mathcal{EP}$.*

3.3. **Axiomatisability of Condition (PWP).** We solve the axiomatisability problem for $\mathcal{PWP}$ by following similar lines to those for $\mathcal{EP}$. For any $s \in S$ we have that $\mathbf{R}^{\leq}(s,s) \neq \emptyset$ and this enables us to concentrate on ultrapowers of S, resulting in a slight simplification as compared to the final condition in Theorem 3.2.

Theorem 3.4. *The following conditions are equivalent for a pomonoid S:*

(1) the class $\mathcal{PWP}$ is axiomatisable;
(2) the class $\mathcal{PWP}$ is closed under ultraproducts;
(3) every ultrapower of S lies in $\mathcal{PWP}$;
(4) $\mathbf{R}^{\leq}(s,s)$ is finitely generated for any $s \in S$.

3.4. **Axiomatisability of Condition (P_w).** We recall that a left S-poset A satisfies Condition (P_w) if for any $a, a' \in A$ and $s, s' \in S$, if $sa \leq s'a'$, then there exists $a'' \in A$ and $u, u' \in S$ with $(u,u') \in \mathbf{R}^{\leq}(s,s')$, $a \leq ua''$ and $u'a'' \leq a'$.

Theorem 3.5. *The following conditions are equivalent for a pomonoid S:*

(1) the class $\mathcal{P}_w$ is axiomatisable;
(2) the class $\mathcal{P}_w$ is closed under ultraproducts;
(3) every ultrapower of S satisfies Condition (P_w);
(4) for any $\rho = (s,t) \in S \times S$, either $\mathbf{R}^{\leq}(s,t) = \emptyset$ or there exists finitely many

$$(u_{\rho 1}, v_{\rho 1}), \dots, (u_{\rho n(\rho)}, v_{\rho n(\rho)}) \in \mathbf{R}^{\leq}(s,t)$$

such that for any $(x,y) \in \mathbf{R}^{\leq}(s,t)$,

$$x \leq u_{\rho i} h \text{ and } v_{\rho i} h \leq y$$

for some $i \in \{1, \dots, n(\rho)\}$ and $h \in S$.

Proof. (3) $\Rightarrow$ (4) Suppose that every ultrapower of S has (P_w) but that (4) does not hold. Then there exists $\rho = (s,t) \in S \times S$ with $\mathbf{R}^{\leq}(s,t) \neq \emptyset$ but such that no finite subset of $\mathbf{R}^{\leq}(s,t)$ exists as in (4).

Let $\{(u_\beta, v_\beta) : \beta < \gamma\}$ be a set of minimal (infinite) cardinality γ contained in $\mathbf{R}^{\leq}(s,t)$ such that if $(x,y) \in \mathbf{R}^{\leq}(s,t)$, then

$$x \leq u_\beta h \text{ and } v_\beta h \leq y$$

for some $\beta < \gamma$ and $h \in S$. From the minimality of γ we may assume that for any $\alpha < \beta < \gamma$, it is not true that both

$$u_\beta \leq u_\alpha h \text{ and } v_\alpha h \leq v_\beta$$

for any $h \in S$.

Let Φ be a uniform ultrafilter on γ, that is, Φ is an ultrafilter on γ such that all sets in Φ have cardinality γ. Let $\mathcal{U} = S^\gamma/\Phi$, by assumption $\mathcal{U}$ satisfies Condition (P_w).

Since $su_\beta \leq tv_\beta$ for all $\beta < \gamma$, $s(u_\beta)_\Phi \leq t(v_\beta)_\Phi$. As $\mathcal{U}$ satisfies condition (P_w), there exists $(u,v) \in \mathbf{R}^{\leq}(s,t)$ and $(w_\beta)_\Phi \in \mathcal{U}$ such that

$$(u_\beta)_\Phi \leq u(w_\beta)_\Phi \text{ and } v(w_\beta)_\Phi \leq (v_\beta)_\Phi.$$

Let $D \in \Phi$ be such that

$$u_\beta \leq uw_\beta \text{ and } vw_\beta \leq v_\beta$$

for all $\beta \in D$. Now $(u,v) \in \mathbf{R}^{\leq}(s,t)$ so that

$$u \leq u_\sigma h \text{ and } v_\sigma h \leq v$$

for some $\sigma < \gamma$. Choose $\beta \in D$ with $\beta > \sigma$. Then

$$u_\beta \leq uw_\beta \leq u_\sigma hw_\beta \text{ and } v_\sigma hw_\beta \leq vw_\beta \leq v_\beta,$$

a contradiction. Thus (4) holds.

(4) $\Rightarrow$ (1) Suppose that (4) holds.

Let $\rho = (s,t) \in S \times S$. If $\mathbf{R}^{\leq}(s,t) = \emptyset$ we put

$$\Omega_\rho := (\forall x)(\forall y)(sx \not\leq ty).$$

If $\mathbf{R}^{\leq}(s,t) \neq \emptyset$, let

$$(u_{\rho 1}, v_{\rho 1}), \dots, (u_{\rho n(\rho)}, v_{\rho n(\rho)}) \in \mathbf{R}^{\leq}(s,t)$$

be the finite set given by our hypothesis, and put

$$\Omega_\rho := (\forall x)(\forall y)\big(sx \leq ty \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x \leq u_{\rho i}\, z \wedge v_{\rho i}\, z \leq y))$$

Let

$$\sum_{\mathcal{P}_w} = \{\Omega_\rho : \rho \in S \times S\}.$$

We claim that $\sum_{\mathcal{P}_w}$ axiomatises $\mathcal{P}_w$.

Let $A \in \mathcal{P}_w$ and let $\rho = (s,t) \in S \times S$. Suppose first that $\mathbf{R}^{\leq}(s,t) = \emptyset$. If $sa \leq tb$ for some $a, b \in S$, then as A satisfies (P_w) we have, in particular, that $\mathbf{R}^{\leq}(s,t) \neq \emptyset$, a contradiction. Hence $A \models \Omega_\rho$.

On the other hand, if $\mathbf{R}^{\leq}(s,t) \neq \emptyset$, then

$$\Omega_\rho = (\forall x)(\forall y)\big(sx \leq ty \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x \leq u_{\rho i}\, z \wedge v_{\rho i}\, z \leq y))$$

If $sa \leq tb$ where $a, b \in A$, then there exists $(u,v) \in \mathbf{R}^{\leq}(s,t)$ and $c \in A$ with

$$a \leq uc \text{ and } vc \leq b.$$

By hypothesis we have that

$$u \leq u_{\rho i}h \text{ and } v_{\rho i}h \leq v$$

for some $h \in S$ and $i \in \{1, \ldots, n(\rho)\}$. Now

$$a \leq uc \leq u_{\rho i}hc \text{ and } v_{\rho i}hc \leq vc \leq b$$

so that (with $z = hc$), $A \models \Omega_\rho$. Hence $A \models \sum_{\mathcal{P}_w}$.

Conversely, suppose that $A \models \sum_{\mathcal{P}_w}$ and $sa \leq tb$ for some $\rho = (s,t) \in S \times S$ and $a, b \in S$. We must therefore have that $\mathbf{R}^{\leq}(s,t) \neq \emptyset$ and consequently, Ω_ρ is

$$(\forall x)(\forall y)\big(sx \leq ty \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x \leq u_{\rho i}\, z \wedge v_{\rho i}\, z \leq y))$$

Hence $a \leq u_{\rho i}c$ and $v_{\rho i}c \leq b$ for some $c \in A$. By definition, $(u_{\rho i}, v_{\rho i}) \in \mathbf{R}^{\leq}(s,t)$, so that A lies in $\mathcal{P}_w$. □

Notice that if $\mathbf{R}^{\leq}(s,t)$ is either empty or finitely generated, then certainly (4) of Theorem 3.5 holds. If the ordering on S is equality, then $\mathcal{P}_w$ is axiomatisable if and only if $\mathbf{R}^{\leq}(s,t)$ is either empty or finitely generated, for all $s, t \in S$.

3.5. **Axiomatisability of Condition (PWP$_w$).** We solve the axiomatisability problem for Condition (PWP$_w$) by following similar lines to those for Condition (P$_w$). Of course in this case $\mathbf{R}^{\leq}(s,s) \neq \emptyset$ for any $s \in S$ and so our result is as follows.

Theorem 3.6. *The following conditions are equivalent for a pomonoid S:*
(1) the class $\mathcal{PWP}_w$ is axiomatisable;
(2) the class $\mathcal{PWP}_w$ is closed under ultraproducts;
(3) every ultrapower of S satisfies Condition (PWP$_w$);
(4) for any $s \in S$ there exists finitely many

$$(u_{\rho 1}, v_{\rho 1}), \ldots, (u_{\rho n(\rho)}, v_{\rho n(\rho)}) \in \mathbf{R}^{\leq}(s,s)$$

such that for any $(x,y) \in \mathbf{R}^{\leq}(s,s)$,

$$x \leq u_{\rho i}h \text{ and } v_{\rho i}h \leq y$$

for some $i \in \{1, \ldots, n(\rho)\}$ and $h \in S$.

Similar remarks apply to those following Theorem 3.5.

4. Axiomatisability of $\mathcal{CF}$

We describe our two general results involving 'replacement tossings'. The first characterise those pomonoids S such that $\mathcal{CF}$ is axiomatisable, for a class $\mathcal{C}$ of right S-poset embeddings, where $\mathcal{C}$ satisfies Condition (Free). This will enable us to specialise to the case where $\mathcal{C}$ is the class of all right S-poset embeddings. For the second we consider an arbitary class $\mathcal{C}$; we then specialise to the cases where $\mathcal{C}$ consists of all inclusions of (principal) right ideals into S. We remark that similar methods can be applied to axiomatisability problems for S-acts over a monoid S, as shown in [18].

4.1. Axiomatisability of $\mathcal{CF}$ with Condition (Free).

It is convenient to introduce some notation. Let

$$\mathcal{S} = (s_1, t_1, \dots, s_m, t_m)$$

be an ordered skeleton of length m.

We define a formula $\epsilon_{\mathcal{S}}$ of $R_{\mathcal{S}}^{\leq}$, where $R_{\mathcal{S}}^{\leq}$ is the first order language associated with right S-posets, as follows:

$$\epsilon_{\mathcal{S}}(x, x_2, \cdots, x_m, x') := \big(xs_1 \leq x_2t_1 \wedge x_2s_2 \leq x_3t_2 \wedge \dots \wedge x_ms_m \leq x't_m\big)$$

and a formula $\theta_{\mathcal{S}}$ of $L_{S}^{\leq}$ by

$$\theta_{\mathcal{S}}(x, x_1, \cdots, x_m, x') := \big(x \leq s_1x_1 \wedge t_1x_1 \leq s_2x_2 \wedge \dots \wedge t_mx_m \leq x'\big).$$

Suppose now that

$$\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2) = (s_1, t_1, \dots, s_m, t_m, u_1, v_1, \dots, u_n, v_n)$$

is a double ordered skeleton of length $m+n$. We put

$$\delta_{\mathcal{S}}(x, x') := (\exists x_2 \dots \exists x_m \exists y_2 \dots \exists y_n)\epsilon_{\mathcal{S}_1}(x, x_2, \dots, x_m, x') \wedge \epsilon_{\mathcal{S}_2}(x', y_2, \dots, y_n, x).$$

On the other hand we define the formula

$$\gamma_{\mathcal{S}}(x, x') := (\exists x_1 \cdots \exists x_m \exists y_1 \cdots \exists y_n)\theta_{\mathcal{S}_1}(x, x_1, \dots, x_m, x') \wedge \theta_{\mathcal{S}_2}(x', y_1, \dots, y_n, x').$$

Remark 4.1. Let A, B be right and left S-posets, respectively, let $a, a' \in A$ and $b, b' \in B$.

(i) The pair (a, b) is connected to the pair (a', b') via a double ordered tossing with double ordered skeleton $\mathcal{S}$ if and only if $\delta_{\mathcal{S}}(a, a')$ is true in A and $\gamma_{\mathcal{S}}(b, b')$ is true in B.

(ii) If $\delta_{\mathcal{S}}(a, a')$ is true in A and $\psi : A \to A'$ is a (right) S-pomorphism, then $\delta_{\mathcal{S}}(a\psi, a'\psi)$ is true in A'.

(iii) If $\gamma_{\mathcal{S}}(b, b')$ is true in B and $\tau : B \to B'$ is an S-pomorphism, then $\gamma_{\mathcal{S}}(b\tau, b'\tau)$ is true in $B\tau$.

Definition 4.2. We say that $\mathcal{C}$ *satisfies Condition (Free)* if for each double ordered skeleton $\mathcal{S}$ there is an S-embedding $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$ in $\mathcal{C}$ and $u_{\mathcal{S}}, u'_{\mathcal{S}} \in W_{\mathcal{S}}$ such that $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$ and further for any S-embedding $\mu : A \to A' \in \mathcal{C}$ and any $a, a' \in A$ such that $\delta_{\mathcal{S}}(a\mu, a'\mu)$ is true in A' there is an S-pomorphism $\nu : W'_{\mathcal{S}} \to A'$ such that $u_{\mathcal{S}}\tau_{\mathcal{S}}\nu = a\mu$, $u'_{\mathcal{S}}\tau_{\mathcal{S}}\nu = a'\mu$ and $W_{\mathcal{S}}\tau_{\mathcal{S}}\nu \subseteq A\mu$.

Lemma 4.3. *Let $\mathcal{C}$ be a class of S-embeddings of right S-posets satisfying Condition (Free). Then the following are equivalent for a left S-poset B:*

(i) B is $\mathcal{C}$-flat;

(ii) $-\otimes B$ maps the S-embeddings $\nu_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$ in the category **Pos-S** *to monomorphisms in the category of* **Pos**, *for every double ordered skeleton $\mathcal{S}$;*

(iii) if $(\mu_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ and $(\mu'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ are connected by a double ordered tossing over $W'_{\mathcal{S}}$ and B with double ordered skeleton $\mathcal{S}$, then $(u_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}, b')$ are connected by a double ordered tossing over $W_{\mathcal{S}}$ and B.

Proof. Clearly we need only show that (iii) implies (i). Suppose that (iii) holds, let $\mu : A \to A'$ lie in $\mathcal{C}$ and suppose that

$$(a\mu, b), (a'\mu, b') \in A' \times B$$

are connected via a double ordered tossing with double ordered skeleton $\mathcal{S}$, so that $\gamma_{\mathcal{S}}(b, b')$ holds. From considering the left hand side of the double ordered tossing, we have that $\delta_{\mathcal{S}}(a\mu, a'\mu)$ is true in A'. By assumption there is an S-embedding $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$ in $\mathcal{C}$ and $u_{\mathcal{S}}, u'_{\mathcal{S}} \in W_{\mathcal{S}}$ such that $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$, and an S-pomorphism $\nu : W'_{\mathcal{S}} \to A'$ such that $u_{\mathcal{S}}\,\tau_{\mathcal{S}}\nu = a\mu$, $u'_{\mathcal{S}}\,\tau_{\mathcal{S}}\nu = a'\mu$ and $W_{\mathcal{S}}\tau_{\mathcal{S}}\nu \subseteq A\mu$. Since $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$, there is a double ordered tossing from $(u_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ to $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ over $W'_{\mathcal{S}}$ and B, with double ordered skeleton $\mathcal{S}$. From (iii), it follows that $(u_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}, b')$ are connected via a double ordered tossing over $W_{\mathcal{S}}$ and B with double ordered skeleton $\mathcal{T}$ say. It follows that $\delta_{\mathcal{T}}(u_{\mathcal{S}}, u'_{\mathcal{S}})$ is true in $W_{\mathcal{S}}$ and so $\delta_{\mathcal{T}}(u_{\mathcal{S}}\tau_{\mathcal{S}}\nu, u'_{\mathcal{S}}\tau_{\mathcal{S}}\nu)$, that is, $\delta_{\mathcal{T}}(a\mu, a'\mu)$ is true in $A\mu$. Since μ is an ordered S-embedding we deduce that $\delta_{\mathcal{T}}(a, a')$ is true in A and consequently, (a, b) and (a', b') are connected via a double ordered tossing with double ordered skeleton $\mathcal{T}$ over A and B. Hence B is $\mathcal{C}$-flat as required. □

Our next aim is to show that the class of all S-embeddings of right S-posets has Condition (Free). To this end we present a 'Finitely Presented Flatness Lemma' for S-posets. First, a *po-congruence* on a left S-poset B is an equivalence relation ρ which is compatible with the action on S, such that in addition B/ρ may be a partially ordered in a way that the natural map $B \to B/\rho$ is S-pomorphism. For further details concerning congruences on ordered algebras, we refer the reader to [6] and for the specific case of S-posets, to [22]. Given a subset R of $B \times B$, it is possible to construct a po-congruence $\equiv_R$ on B such that $[a] \preceq_R [b]$ for every $(a, b) \in R$, where $\preceq_R$ is the ordering in $B/\equiv_R$, and is such that if $\alpha : B \to C$ is an S-pomorphism from B to any left S-poset C with $a\alpha \leq b\alpha$ for all $(a, b) \in R$, then there exists an S-pomorphism $\beta : B/\equiv_R \to C$ such that $[b]\beta = b\alpha$, for all $b \in B$.

For a double ordered skeleton $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ where

$$\mathcal{S}_1 = (s_1, t_1, \cdots, s_m, t_m) \text{ and } \mathcal{S}_2 = (u_1, v_1, \cdots, u_n, v_n),$$

we let F^{m+n} be the free right S-poset

$$xS \,\dot\cup\, x_2S \,\dot\cup\, \ldots x_mS \,\dot\cup\, y_2S \,\dot\cup\, y_3S \ldots\; y_nS \,\dot\cup\, x'S$$

and let $R_{\mathcal{S}}$ be the set

$$\big\{(xs_1, x_2t_1), (x_2s_2, x_3t_2), \ldots, (x_mt_{m-1}), (x_ms_m, x't_m),$$

$$(x'u_1, y_2v_1), (y_2u_2, y_3v_2), \ldots, (y_nu_n, xv_n)\big\}.$$

Let us abbreviate by $\equiv_{\mathcal{S}}$ the S-poset congruence $\equiv_{R_{\mathcal{S}}}$ induced by $R_{\mathcal{S}}$. We abbreviate the order $\preceq_{R_{\mathcal{S}}}$ on $F^{m+n}/\equiv_{\mathcal{S}}$ by $\preceq_{\mathcal{S}}$.

If B is a left S-poset and b, b_1, $,\cdots, b_m$, d_1, d_2, $,\cdots, d_n$, $b' \in B$ are such that

$$\theta_{\mathcal{S}_1}(b, b_1, \ldots, b_m, b') \text{ and } \theta_{\mathcal{S}_2}(b', d_1, \ldots, d_n, b)$$

hold, then the double ordered tossing

$$\begin{array}{rclcrcl} & & & & b & \le & s_1b_1 \\ [x]s_1 & \le & [x_2]t_1 & & t_1b_1 & \le & s_2b_2 \\ [x_2]s_2 & \le & [x_3]t_2 & & t_2b_2 & \le & s_3b_3 \\ & \vdots & & & & \vdots & \\ [x_m]s_m & \le & [x']t_m & & t_mb_m & \le & b' \end{array}$$

$$\begin{array}{rclcrcl} & & & & b' & \le & u_1d_1 \\ [x']u_1 & \le & [y_2]v_1 & & v_1d_1 & \le & u_2d_2 \\ [y_2]u_2 & \le & [y_3]v_2 & & v_2d_2 & \le & u_3d_3 \\ & \vdots & & & & \vdots & \\ [y_n]u_n & \le & [x]v_n & & v_nd_n & \le & b \end{array}$$

over $F^{m+n}/\equiv_{\mathcal{S}}$ and B is called a *double ordered standard tossing*; clearly it has double ordered skeleton $\mathcal{S}$.

It is clear that (by considering a trivial left S-poset B), the set of all double ordered skeletons $\mathbb{DOS}$ is the set of all finite even length sequences of elements of S, of length at least 4.

Lemma 4.4. *The following conditions are equivalent for a left S-poset B:*

(i) B is flat;

(ii) $-\otimes B$ maps S-embeddings of $[x]S \cup [x']S$ into $F^{m+n}/\equiv_{\mathcal{S}}$ in the category **Pos-S** *to monomorphisms in the category* **Pos**, *for every double ordered skeleton $\mathcal{S}$;*

(iii) if $\big([x], b\big)$ and $\big([x'], b'\big)$ are connected by a double ordered standard tossing over $F^{m+n}/\equiv_{\mathcal{S}}$ and B (with double ordered skeleton $\mathcal{S}$), then they are connected by a double ordered tossing over $[x]S \cup [x']S$ and B.

Proof. We will prove here only $(iii) \Rightarrow (i)$. Suppose that B satisfies condition (iii), let a, a' belongs to any right S-poset A, let $b, b' \in B$, and suppose that $a \otimes b = a' \otimes b'$ in $A \otimes B$ via a double ordered tossing with double ordered skeleton $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$, where $\mathcal{S}_1, \mathcal{S}_2$ have lengths m and n, respectively. By Remark 4.1, $\delta_{\mathcal{S}}(a, a')$ is true in A and $\gamma_{\mathcal{S}}(b, b')$ is true in B. Since $\delta_{\mathcal{S}}([x], [x'])$ holds in $F^{m+n}/\equiv_{\mathcal{S}}$, we have that $\big([x], b\big)$ and $\big([x'], b'\big)$ are connected by a double ordered standard tossing over $F^{m+n}/\equiv_{\mathcal{S}}$ and B. By the given hypothesis we have that $\big([x], b\big)$ and $\big([x'], b'\big)$ connected via a double ordered tossing in $\big([x]S \cup [x']S\big) \otimes B$, say with double ordered skeleton $\mathcal{U}$.

Since $\delta_{\mathcal{S}}(a, a')$ is true in A, there are elements $a_2, \ldots, a_m, c_2, \ldots, c_n \in A$ such that

$$\epsilon_{\mathcal{S}_1}(a, a_2, \ldots, a_m, a') \text{ and } \epsilon_{\mathcal{S}_2}(a', c_2, \ldots, c_n, a)$$

hold in A. Let $\phi : F^{m+n} \to A$ be the S-pomorphism which is defined by $x\phi = a$, $x_i\phi = a_i$ $(2 \le i \le m)$, $x'\phi = a'$ and $y_j\phi = c_j$ $(2 \le j \le n)$. Since $u\phi \le u'\phi$ for all $(u, u') \in R_{\mathcal{S}}$, we have that $\overline{\phi} : F^{m+n}/\equiv_{\mathcal{S}} \to A$ given by $[z]\overline{\phi} = z\phi$ is a well defined S-pomorphism. We have that $\delta_{\mathcal{U}}([x], [x'])$ holds in $[x]S \cup [x']S$, so that by Remark 4.1, $\delta_{\mathcal{U}}(a, a')$ holds in $aS \cup a'S$. Since also $\gamma_{\mathcal{U}}(b, b')$ holds in B, we have that (a, a') and (b, b') are connected by a double ordered tossing over $aS \cup a'S$ and B, so that $a \otimes b = a' \otimes b'$ in $(aS \cup a'S) \otimes B$. Thus B is flat, as required. □

With a similar argument, we prove the following.

Lemma 4.5. *The class Pos-S of all right S-posets has Condition (Free).*

Proof. Let $\mathcal{S}$ be a double ordered skeleton of length $m+n$, let $W'_{\mathcal{S}} = F^{m+n}/\equiv_{\mathcal{S}}$, $W_{\mathcal{S}} = [x]S \cup [x']S$ and let $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$ denote inclusion. Then $[x], [x'] \in W_{\mathcal{S}}$ and $\delta_{\mathcal{S}}([x]\tau_{\mathcal{S}}, [x']\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$.

Let $\mu : A \to A'$ be any right S-poset embedding such that $\delta_{\mathcal{S}}(a\mu, a'\mu)$ holds in A', for some $a, a' \in A$. As in Lemma 4.4, there is as a consequence an S-pomorphism $\nu : W'_{\mathcal{S}} \to A'$ such that $[x]\tau_{\mathcal{S}}\nu = a\mu$ and $[x']\tau_{\mathcal{S}}\nu = a'\mu$. Clearly

$$W_{\mathcal{S}}\tau_{\mathcal{S}}\nu = ([x]S \cup [x']S)\tau_{\mathcal{S}}\nu = [x]\tau_{\mathcal{S}}\nu S \cup [x']\tau_{\mathcal{S}}\nu S = a\mu S \cup a'\mu S = (aS \cup a'S)\mu \subseteq A\mu.$$

Thus, with $u_{\mathcal{S}} = [x]$ and $u'_{\mathcal{S}} = [x']$, we see that Condition (Free) holds. $\square$

Let $\mathcal{C}$ be a class of S-embeddings of right S-posets. Let $\overline{\mathcal{C}}$ be the set of products of S-pomorphisms in $\mathcal{C}$ (with the obvious definition and pointwise ordering).

Lemma 4.6. *Let $\mathcal{C}$ be a class of S-embeddings of right S-posets, satisfying Condition (Free). If a left S-poset B is $\mathcal{C}$-flat, then it is $\overline{\mathcal{C}}$-flat.*

Proof. Let I be an indexing set and let $\gamma_i : A_i \to A'_i \in \mathcal{C}$ for all $i \in I$. Let $A = \prod_{i\in I} A_i$ and $A' = \prod_{i\in I} A'_i$, and let $\gamma : A \to A'$ be the canonical embedding, so that $(a_i)\gamma = (a_i\gamma_i)$.

Suppose B is a $\mathcal{C}$-flat left S-poset. Let $\underline{a} = (a_i), \underline{a}' = (a'_i) \in A$ and $b, b' \in B$ be such that $\underline{a}\gamma \otimes b = \underline{a}'\gamma \otimes b$ in $A' \otimes B$. Then for some double ordered skeleton $\mathcal{S}$,

$$A' \models \delta_{\mathcal{S}}(\underline{a}\gamma, \underline{a}'\gamma) \text{ and } B \models \gamma_{\mathcal{S}}(b, b').$$

It follows that for each $i \in I$,

$$A_i \models \delta_{\mathcal{S}}(a_i\gamma_i, a'_i\gamma_i).$$

By assumption that $\mathcal{C}$ has Condition (Free), there exist $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}} \in \mathcal{C}$ and $u_{\mathcal{S}}, u'_{\mathcal{S}} \in W_{\mathcal{S}}$ such that $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$. Further, for each $i \in I$, as $\delta_{\mathcal{S}}(a_i\gamma_i, a'_i\gamma_i)$ is true in A'_i, there exists an S-pomorphism $\nu_i : W'_{\mathcal{S}} \to A'_i$ such that $u_{\mathcal{S}}\,\tau_{\mathcal{S}}\nu_i = a_i\gamma_i$, $u'_{\mathcal{S}}\tau_{\mathcal{S}}\nu_i = a'_i\gamma_i$ and $W_{\mathcal{S}}\tau_{\mathcal{S}}\nu_i \subseteq A_i\gamma_i$.

We have $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$ and $\gamma_{\mathcal{S}}(b, b')$ is true in B, giving that $u_{\mathcal{S}}\tau_{\mathcal{S}} \otimes b = u'_{\mathcal{S}}\tau_{\mathcal{S}} \otimes b'$ in $W'_{\mathcal{S}} \otimes B$. As B is a $\mathcal{C}$-flat left S-poset and $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}} \in \mathcal{C}$, we have that $u_{\mathcal{S}} \otimes b = u'_{\mathcal{S}} \otimes b'$ in $W_{\mathcal{S}} \otimes B$, say via a double ordered tossing with double ordered skeleton $\mathcal{U}$. It follows that

$$W_{\mathcal{S}} \models \delta_{\mathcal{U}}(u_{\mathcal{S}}, u'_{\mathcal{S}}) \text{ and } B \models \gamma_{\mathcal{U}}(b, b').$$

By (ii) of Remark 4.1, we have that

$$A_i\gamma_i \models \delta_{\mathcal{U}}(u_{\mathcal{S}}\tau_{\mathcal{S}}\nu_i, u'_{\mathcal{S}}\tau_{\mathcal{S}}\nu_i),$$

that is,

$$A_i\gamma_i \models \delta_{\mathcal{U}}(a_i\gamma_i, a'_i\gamma_i).$$

Writing $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2)$ where $\mathcal{U}_1$ has length h and $\mathcal{U}_2$ has length k, we have that there are elements $w_{i,2}, \dots, w_{i,h}, z_{i,2}, \dots, z_{i,k} \in A_i$ such that

$$\epsilon_{\mathcal{S}_1}(a_i\gamma_i, w_{i,2}\gamma_i, \dots, w_{i,h}\gamma_i, a'_i\gamma_i) \text{ and } \epsilon_{\mathcal{S}_2}(a'_i\gamma_i, z_{i,2}\gamma_i, \dots, z_{i,k}\gamma_i, a_i\gamma_i)$$

are true. But γ_i is an *S-embedding*, so that

$$\epsilon_{\mathcal{S}_1}(a_i, w_{i,2}, \dots, w_{i,h}, a'_i) \text{ and } \epsilon_{\mathcal{S}_2}(a'_i, z_{i,2}, \dots, z_{i,k}, a_i)$$

hold in A_i. Hence $\delta_{\mathcal{U}}(a_i, a'_i)$ is true in each A_i and so $\delta_{\mathcal{U}}(\underline{a}, \underline{a}')$ holds in A. Together with $\gamma_{\mathcal{U}}(b, b')$ being true in B, we deduce that $\underline{a} \otimes b = \underline{a}' \otimes b'$ in $A \otimes B$, as required. $\square$

We now come to our first main result. The technique used is inspired by that of [1], but there are some differences: first we are working in a more general context and second, we are dealing with orderings.

Theorem 4.7. *Let $\mathcal{C}$ be a class of ordered embeddings of right S-posets satisfying Condition (Free). Then the following conditions are equivalent for a pomonoid S:*

(1) the class $\mathcal{CF}$ is axiomatisable;

(2) the class $\mathcal{CF}$ is closed under formation of ultraproducts;

(3) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exist finitely many double ordered replacement skeletons $\mathcal{S}_1, \ldots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any embedding $\gamma : A \to A'$ in $\mathcal{C}$ and any $\mathcal{C}$-flat left S-poset B, if $(a\gamma, b)$, $(a'\gamma, b') \in A' \times B$ are connected by a double ordered tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a double ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;

(4) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exists finitely many double ordered replacement skeletons $\mathcal{S}_1, \ldots, \mathcal{S}_{\beta(\mathcal{S})}$ such that, for any $\mathcal{C}$-flat left S-poset B, if $(u_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ are connected by the double ordered tossing $\mathcal{T}$ over $W'_{\mathcal{S}}$ and B (with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$), then $(u_{\mathcal{S}}, b)$, and $(u'_{\mathcal{S}}, b')$ are connected by a double ordered tossing $\mathcal{T}'$ over $W_{\mathcal{S}}$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$.

Proof. The implication (1) implies (2) is clear from Łos's Theorem.

To prove (2) $\Rightarrow$ (3), we suppose that $\mathcal{CF}$, the class of $\mathcal{C}$-flat left S-posets, is closed under formation of ultraproducts and that (3) is false. Let J be the family of finite subsets of $\mathbb{DOS}$. We suppose that there exists a double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ such that for every subset f of J, there exists an S-embedding $\gamma_f : A_f \to A'_f \in \mathcal{C}$, a $\mathcal{C}$-flat left S-poset B_f, and pairs $(a_f\gamma_f, b_f)$, $(a'_f\gamma_f, b'_f) \in A'_f \times B_f$ such that $(a_f\gamma_f, b_f)$ and $(a'_f\gamma_f, b'_f)$ are connected over A'_f and B_f by a double ordered tossing $\mathcal{T}_f$ with double ordered skeleton $\mathcal{S}$, but such that no double ordered replacement tossing over A_f and B_f connecting (a_f, b_f) and (a'_f, b'_f) has a double ordered skeleton belonging to the set f.

Let $J_{\mathcal{S}} = \{f \in J : \mathcal{S} \in f\}$ for each $\mathcal{S} \in \mathbb{DOS}$. Then there exists an ultrafilter Φ on J containing each $J_{\mathcal{S}}$, as each intersection of finitely many of the sets $J_{\mathcal{S}}$ is non-empty.

We now define $A' = \prod_{f\in J} A'_f$, $A = \prod_{f\in J} A_f$ and $B = \prod_{f\in J} B_f$. Let $\gamma : A \to A'$ be the S-embedding given by $(a_f)\gamma = (a_f\gamma_f)$. We note here that $\underline{a}\gamma \otimes \underline{b} = \underline{a}'\gamma \otimes \underline{b}'$ in $A' \otimes B$, where $\underline{a} = (a_f)$, $\underline{a}' = (a'_f)$, $\underline{b} = (b_f)$ and $\underline{b}' = (b'_f)$ and that this equality is determined by a double ordered tossing over A' and B (the " product" of the double ordered tossings $\mathcal{T}_f$) having double ordered skeleton $\mathcal{S}$. It follows that the equality for $\underline{a}\gamma \otimes \underline{b}_\Phi = \underline{a}'\gamma \otimes \underline{b}'_\Phi$ holds also in $A' \otimes \mathcal{U}$ where $\mathcal{U} = (\prod_{f\in J} B_f)/\Phi$, and can be determined by a double ordered tossing over A' and $\mathcal{U}$ with double ordered skeleton $\mathcal{S}$.

By assumption, $\mathcal{U}$ is $\mathcal{C}$-flat, and by Lemma 4.6 above, $\underline{a} \otimes \underline{b}_\Phi = \underline{a}' \otimes \underline{b}'_\Phi$ in $A \otimes \mathcal{U}$, say via a double ordered tossing with double ordered skeleton $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)$ of length $h + k$, say

$$\mathcal{V}_1 = (d_1, e_1, \ldots, d_h, e_h) \text{ and } \mathcal{V}_2 = (g_1, \ell_1, \ldots, g_k, \ell_k).$$

Hence

$$A \models \delta_{\mathcal{V}}(\underline{a}, \underline{a}') \text{ and } \mathcal{U} \models \gamma_{\mathcal{V}}(\underline{b}_\Phi, \underline{b}'_\Phi).$$

Certainly $A_f \models \delta_{\mathcal{V}}(a_f, a'_f)$ for every f. Considering now the truth of $\gamma_{\mathcal{V}}(\underline{b}_\Phi, \underline{b}'_\Phi)$, there exist

$$(b_{1,f})_\Phi, \ldots, (b_{h,f})_\Phi, (c_{1,f})_\Phi, \ldots, (c_{k,f})_\Phi \in \mathcal{U}$$

such that

$$\begin{array}{rclrcl} \underline{b}_\Phi & \leq & d_1(b_{1,f})_\Phi & \underline{b}'_\Phi & \leq & g_1(c_{1,f})_\Phi \\ e_1(b_{1,f})_\Phi & \leq & d_2(b_{2,f})_\Phi & \ell_1(c_{1,f})_\Phi & \leq & g_2(c_{2,f})_\Phi \\ & \vdots & & & \vdots & \\ e_h(b_{h,f})_\Phi & \leq & \underline{b}'_\Phi & \ell_k(c_{k,f})_\Phi & \leq & \underline{b}_\Phi. \end{array}$$

As Φ is closed under finite intersections, there exists $D \in \Phi$ such that

$$\begin{array}{rclrcl} b_f & \leq & d_1 b_{1,f} & b'_f & \leq & g_1 c_{1,f} \\ e_1 b_{1,f} & \leq & d_2 b_{2,f} & \ell_1 c_{1,f} & \leq & g_2 c_{2,f} \\ & \vdots & & & \vdots & \\ e_h b_{h,f} & \leq & b'_f & \ell_k c_{k,f} & \leq & b_f \end{array}$$

for all $f \in D$.

Now suppose that $f \in D \cap J_{\mathcal{V}}$, then from the double ordered tossing just considered, we see that $\mathcal{V}$ is the double ordered skeleton of a double ordered tossing over A_f and B_f connecting the pairs (a_f, b_f) and (a'_f, b'_f); that is, $\mathcal{V}$ a double ordered replacement skeleton for the double ordered skeleton $\mathcal{S}$ of the double ordered tossing $\mathcal{T}_f$. But $\mathcal{V}$ belongs to f, a contradiction. This completes the proof that (2) implies that (3).

It is clear that (3) implies that (4).

Now we want to prove that (4) $\Rightarrow$ (1). We assume that (4) holds. We aim to use this condition to construct a set of axioms for $\mathcal{CF}$.

Let $\mathbb{S}_1$ denote the set of all elements of $\mathbb{DOS}$ such that if $\mathcal{S} \in \mathbb{S}_1$, then there is no $\mathcal{C}$-flat left S-poset B such that $\gamma_{\mathcal{S}}(b, b')$ holds in B for any $b, b' \in B$. For $\mathcal{S} \in \mathbb{S}_1$ we put

$$\psi_{\mathcal{S}} : (\forall x)(\forall x')\neg\gamma_{\mathcal{S}}(x, x')$$

For $\mathcal{S} \in \mathbb{S}_2 = \mathbb{DOS} \setminus \mathbb{S}_1$, there must be a $B \in \mathcal{CF}$ and $b, b' \in B$ such that $\gamma_{\mathcal{S}}(b, b')$ is true in B, whence there is a double ordered tossing from $(u_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ to $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ over $W'_{\mathcal{S}}$ and B with double ordered skeleton $\mathcal{S}$.

Let $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ be a minimum set of double ordered replacement skeletons for double ordered tossings with double ordered skeleton $\mathcal{S}$ connecting pairs of the form $(u_{\mathcal{S}}\tau_{\mathcal{S}}, c)$ to $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, c')$ where $c, c' \in C$ and C ranges over $\mathcal{CF}$. Hence for each k in $\{1, \cdots, \beta(\mathcal{S})\}$, there exists a $\mathcal{C}$-flat left S-poset C_k, elements $c_k, c'_k \in C_k$ such that

$$W_{\mathcal{S}} \models \delta_{\mathcal{S}_k}(u_{\mathcal{S}}, u'_{\mathcal{S}}) \text{ and } C_k \models \gamma_{\mathcal{S}_k}(c_k, c'_k).$$

We define $\phi_{\mathcal{S}}$ to be the sentence

$$\phi_{\mathcal{S}} := (\forall y)(\forall y')\big(\gamma_{\mathcal{S}}(y, y') \to \gamma_{\mathcal{S}_1}(y, y') \vee \ldots \vee \gamma_{\mathcal{S}_{\beta(\mathcal{S})}}(y, y')\big).$$

Let

$$\Sigma_{\mathcal{CF}} = \{\psi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_1\} \cup \{\phi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_2\}.$$

We claim that $\Sigma_{\mathcal{CF}}$ axiomatises $\mathcal{CF}$.

Suppose first that D is any $\mathcal{C}$-flat left S-poset. By choice of $\mathbb{S}_1$, it is clear that $D \models \psi_{\mathcal{S}}$ for any $\mathcal{S} \in \mathbb{S}_1$.

Now take any $\mathcal{S} \in \mathbb{S}_2$, and suppose that $d, d' \in D$ are such that D satisfies $\gamma_{\mathcal{S}}(d, d')$. Then, as noted earlier, $(u_{\mathcal{S}}\tau_{\mathcal{S}}, d)$ and $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, d')$ are joined over $W'_{\mathcal{S}}$ and D by a double ordered tossing with double ordered skeleton $\mathcal{S}$, and therefore, by assumption, there is a double ordered tossing over $W_{\mathcal{S}}$ and D joining $(u_{\mathcal{S}}, d)$ and $(u'_{\mathcal{S}}, d')$ with double ordered skeleton $\mathcal{S}_k$ for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$. It is now clear that $\gamma_{\mathcal{S}_k}(d, d')$ holds in D, as required. We have now shown that $D \models \Sigma_{\mathcal{CF}}$.

Finally we show that a left S-poset C that satisfies $\Sigma_{\mathcal{CF}}$ must be $\mathcal{C}$-flat. We need to show that Condition (3) of Lemma 4.3 holds for C. Let $\mathcal{S} \in \mathbb{DOS}$ and suppose we have a double ordered tossing with double ordered skeleton $\mathcal{S}$ connecting $(u_{\mathcal{S}}\tau_{\mathcal{S}}, c)$ and $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, c')$ over $W'_{\mathcal{S}}$ and C. Then

$$W'_{\mathcal{S}} \models \delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}}) \text{ and } C \models \gamma_{\mathcal{S}}(c, c').$$

If $\mathcal{S}$ belonged to $\mathbb{S}_1$, then C would satisfy the sentence $(\forall y)(\forall y')\neg\gamma_{\mathcal{S}}(y, y')$ and so $\neg\gamma_{\mathcal{S}}(c, c')$ would hold, which is would be a contradiction. Therefore we conclude that $\mathcal{S}$ belongs to $\mathbb{S}_2$. Because C satisfies $\phi_{\mathcal{S}}$ and because $\gamma_{\mathcal{S}}(c, c')$ holds, it follows that $\gamma_{\mathcal{S}_k}(c, c')$ holds for some $k \in \{1, 2, \cdots, \beta(\mathcal{S})\}$. But $W_{\mathcal{S}} \models \delta_{\mathcal{S}_k}(u_{\mathcal{S}}, u'_{\mathcal{S}})$, whence $(u_{\mathcal{S}}, c)$ and $(u'_{\mathcal{S}}, c')$ are connected via a double ordered tossing over $W_{\mathcal{S}}$ and C with double ordered skeleton $\mathcal{S}_k$, showing that C is $\mathcal{C}$-flat. $\square$

We recall that the definition of a flat left S-poset is that it is $\mathcal{C}$-flat where $\mathcal{C}$ is the class of *all* S-embeddings of right S-posets. The class of all flat left S-posets is denoted by $\mathcal{F}$.

By Lemma 4.5, the class of all right S-posets has Condition (Free), so from Theorem 4.7, we immediately have the following corollary.

Corollary 4.8. *The following conditions are equivalent for a pomonoid S:*

(1) the class $\mathcal{F}$ is axiomatisable;

(2) the class $\mathcal{F}$ is closed under formation of ultraproducts;

(3) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exist finitely many double ordered replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any right S-poset ordered embedding $\gamma : A \to A'$, and any flat left S-poset B, if $(a\gamma, b)$, $(a'\gamma, b') \in A' \times B$ are connected by a double ordered tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a double ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;

(4) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exist finitely many double ordered replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any right S-poset A and any flat left S-poset B, if (a, b), $(a', b') \in A \times B$ are connected by a double ordered tossing $\mathcal{T}$ over A and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a double ordered tossing $\mathcal{T}'$ over $aS \cup a'S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;

(5) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exists finitely many double ordered replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ such that, for any flat left S-poset B, if $([x], b)$ and $([x'], b')$ are connected by a double ordered tossing $\mathcal{T}$ over $F^{m+n}/\equiv_{\mathcal{S}}$ and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $([x], b)$ and $([x'], b')$ are connected by a double ordered tossing $\mathcal{T}'$ over $[x]S \cup [x']S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$.

4.2. **Axiomatisability of $\mathcal{CF}$ in the general case.** We continue to consider a class $\mathcal{C}$ of ordered embeddings of right S-posets , but now drop our assumption that Condition (Free) holds. The results and proofs of this section are analogous to those for weakly flat S-acts in [1]. Given that we have demonstrated in the previous subsection how to adapt to the ordered case, we omit proofs. The reader wishing to find full details can consult [18]. Note that the conditions in (3) below appear weaker than those in Theorem 4.7, as we are only asking that for chosen elements a, a' and double ordered skeleton $\mathcal{S}$, there are finitely many double ordered replacement skeletons, in the sense made specific below.

Theorem 4.9. *Let $\mathcal{C}$ be a class of S-embeddings of right S-posets.*

The following conditions are equivalent:

(1) the class $\mathcal{CF}$ is axiomatisable;

(2) the class $\mathcal{CF}$ is closed under ultraproducts;

(3) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ and $a, a' \in A$, where $\mu : A \to A'$ is in $\mathcal{C}$, there exist finitely many double ordered skeleton $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(a,\mathcal{S},a',\mu)}$, such that for any $\mathcal{C}$-flat left S-poset B, if $(a\mu, b)$, $(a'\mu, b')$ are connected by a double ordered tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a double ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(a, \mathcal{S}, a', \mu)\}$.

We now apply Theorem 4.9 to the class of all S-embeddings of right ideals into S, and the class of all S-embeddings of principal right ideals into S. In these corollaries we do not need to mention the S-embeddings μ, since they are all inclusion maps of right ideals into S.

Corollary 4.10. *The following are equivalent for a pomonoid S:*

(i) the class $\mathcal{WF}$ is axiomatisable;

(ii) the class $\mathcal{WF}$ is closed under ultraproducts;

(iii) for every double ordered skeleton $\mathcal{S}$ and $a, a' \in S$ there exists finitely many double ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(a,\mathcal{S},a')}$ such that for any weakly flat left S-poset B, if (a, b), $(a', b') \in S \times B$ are connected by a double ordered tossing $\mathcal{T}$ over S and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$ then (a, b) and (a', b') are connected by a double ordered tossing $\mathcal{T}'$ over $aS \cup a'S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$ for some $k \in \{1, \cdots, \beta(a, \mathcal{S}, a')\}$.

We end this section by considering the axiomatisability of principally weakly flat S-posets. We first remark that if aS is a principal right ideals of S and B is a left S-poset, then

$$au \otimes b = av \otimes b' \text{ in } aS \otimes B \text{ if and only if } a \otimes ub = a \otimes vb' \text{ in } aS \otimes B$$

with a similar statement for $S \otimes B$. Thus B is principally weakly flat if and only if for all $a \in S$, if $a \otimes b = a \otimes b'$ in $S \otimes B$, then $a \otimes b = a \otimes b'$ in $aS \cup B$. From Theorem 4.9 and its proof we have the following result for $\mathcal{PWF}$.

Corollary 4.11. *The following conditions are equivalent for a pomonoid $\mathcal{S}$:*
(i) the class $\mathcal{PWF}$ is axiomatisable;
(ii) the class $\mathcal{PWF}$ is closed under ultraproducts;
(iii) for every double ordered skeleton $\mathcal{S}$ over S and $a \in S$ there exists finitely many double ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\gamma(a,\mathcal{S})}$ over S, such that for any principally weakly flat left S-poset B, if (a, b), $(a, b') \in S \otimes B$ are connected by a double ordered tossing $\mathcal{T}$ over S and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a, b') are connected by a double ordered tossing $\mathcal{T}'$ over aS and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \gamma(a, \mathcal{S})\}$.

5. Axiomatisability of $\mathcal{CPF}$

In this section we briefly explain how the methods and results of Section 4 may be adapted to the case when $- \otimes B$ preserves embeddings, rather than merely taking embeddings to monomorphisms. We omit proofs, as they follow now established patterns. Further details may be found in [18].

We introduce a condition on a class $\mathcal{C}$ of embeddings of right S-posets called Condition (Free)$^{\leq}$.

Let

$$\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m)$$

be an ordered skeleton of length m. We put

$$\delta^{\leq}_{\mathcal{S}}(x, x') := (\exists x_2 \ldots \exists x_m)\epsilon_{\mathcal{S}}(x, x_2, \ldots, x_m, x')$$

and

$$\gamma^{\leq}_{\mathcal{S}}(x, x') := (\exists x_1 \ldots \exists x_m)\theta_{\mathcal{S}}(x, x_1, \ldots, x_m, x')$$

where ϵ and θ are defined as in Section 4. Notice that similar comments to those in Remark 4.1 hold, in particular, if A is a right and B a left S-poset, then the pair $(a, b) \in A \times B$ is connected to the pair $(a', b') \in A \times B$ via an ordered tossing with ordered skeleton $\mathcal{S}$ if and only if $\delta^{\leq}_{\mathcal{S}}(a, a')$ is true in A and $\gamma^{\leq}_{\mathcal{S}}(b, b')$ is true in B.

Definition 5.1. We say that $\mathcal{C}$ satisfies Condition (Free)$^{\leq}$ if for each ordered skeleton $\mathcal{S}$ there is an S-embedding $\kappa_{\mathcal{S}} : V_{\mathcal{S}} \to V'_{\mathcal{S}}$ in $\mathcal{C}$ and $v_{\mathcal{S}}, v'_{\mathcal{S}} \in V_{\mathcal{S}}$ such that $\delta^{\leq}_{\mathcal{S}}(v_{\mathcal{S}}\kappa_{\mathcal{S}}, v'_{\mathcal{S}}\kappa_{\mathcal{S}})$ is true in $V'_{\mathcal{S}}$ and further, for any S-embedding $\mu : A \to A' \in \mathcal{C}$ and any $a, a' \in A$ such that $\delta^{\leq}_{\mathcal{S}}(a\mu, a'\mu)$ is true in A', there is a morphism $\nu : V'_{\mathcal{S}} \to A'$ such that $u_{\mathcal{S}}\kappa_{\mathcal{S}}\nu = a\mu$, $u'_{\mathcal{S}}\kappa_{\mathcal{S}}\nu = a'\mu$ and $V_{\mathcal{S}}\kappa_{\mathcal{S}}\nu \subseteq A\mu$.

As in Lemma 4.3, we can show that if $\mathcal{C}$ be a class of embeddings of right S-posets satisfying Condition (Free)$^{\leq}$, then to show that a left S-poset B is in $\mathcal{CPF}$, that is, B is $\mathcal{C}$-po-flat, it is enough to show that for any ordered skeleton $\mathcal{S}$, if $(v_{\mathcal{S}}\kappa_{\mathcal{S}}, b)$ and $(v'_{\mathcal{S}}\kappa_{\mathcal{S}}, b')$ are connected by an ordered tossing over $V'_{\mathcal{S}}$ and B with ordered skeleton $\mathcal{S}$, then $(v_{\mathcal{S}}, b)$ and $(v'_{\mathcal{S}}, b')$ are connected by an ordered tossing over $V_{\mathcal{S}}$ and B. Moreover, if $B \in \mathcal{CPF}$, then $B \in \overline{\mathcal{C}}\mathcal{PF}$.

Everything is then in place to prove the next result.

Theorem 5.2. *Let $\mathcal{C}$ be a class of S-embeddings of right S-posets satisfying Condition (Free)$^{\leq}$. Then the following conditions are equivalent for a pomonoid S:*
(i) the class $\mathcal{CPF}$ is axiomatisable;
(ii) the class $\mathcal{CPF}$ is closed under formation of ultraproducts;

(iii) for every ordered skeleton $\mathcal{S}$ there exist finitely many replacement ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any embedding $\gamma : A \to A'$ in $\mathcal{C}$ and any $\mathcal{C}$-po-flat left S-poset B, if $a\gamma \otimes b \leq a'\gamma \otimes b' \in A' \otimes B$ by an ordered tossing $\mathcal{T}$ with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $a \otimes b \leq a' \otimes b'$ by an ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;

(iv) for every ordered skeleton $\mathcal{S}$ there exists finitely many replacement ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ such that, for any $\mathcal{C}$-po-flat left S-poset B, if $(v_{\mathcal{S}}\kappa_{\mathcal{S}}, b)$ and $(v'_{\mathcal{S}}\kappa_{\mathcal{S}}, b')$ are such that $v_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes b \leq v'_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes b'$ by an ordered tossing $\mathcal{T}$ over $V'_{\mathcal{S}}$ and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $v_{\mathcal{S}} \otimes b \leq v'_{\mathcal{S}} \otimes b'$ are connected by an ordered tossing $\mathcal{T}'$ over $V_{\mathcal{S}}$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$.

To show that the class of all embeddings of right S-posets has Condition (Free)$^{\leq}$, for an ordered skeleton

$$\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m)$$

we let F^m be the free right S-poset

$$xS \,\dot\cup\, x_2S \,\dot\cup\, \ldots x_mS \,\dot\cup\, x'S$$

and put

$$T_{\mathcal{S}} = \big\{(xs_1, x_2t_1), (x_2s_2, x_3t_2), \ldots, (x_ms_m, x't_m)\big\}.$$

Let $=_{\mathcal{S}}$ be $\equiv_{T_{\mathcal{S}}}$, the S-poset congruence which is induced by $T_{\mathcal{S}}$. Abbreviate the order $\preceq_{T_{\mathcal{S}}}$ by $\leq_{\mathcal{S}}$ so that $[a] \leq_{\mathcal{S}} [b]$ for all $(a, b) \in T_{\mathcal{S}}$. We defined an *ordered standard tossing* from $([x], b)$ to $([x'], b')$ where $b, b' \in B$ for a left S-poset B in the analogous way to a double ordered standard tossing.

The proof of the next lemma follows that of Lemma 4.4.

Lemma 5.3. *The following conditions are equivalent for a left S-poset B:*

(i) B is po-flat;

(ii) $-\otimes B$ maps S-embeddings of $[x]S \cup [x']S$ into $F^m/=_{\mathcal{S}}$ in the category **Pos-S** *to embeddings in the category of* **Pos***, for every ordered skeleton $\mathcal{S}$;*

(iii) if the inequality $[x] \otimes b \leq [x'] \otimes b'$ holds by an ordered standard tossing over $F^m/=_{\mathcal{S}}$ and B with ordered skeleton $\mathcal{S}$, then $[x] \otimes b \leq [x'] \otimes b'$ holds by an ordered tossing over $[x]S \cup [x']S$ and B.

As in Lemma 4.5 we then have:

Lemma 5.4. *The class Pos-S of all right S-posets has Condition (Free)$^{\leq}$.*

We can now deduce the following corollary, which appears without proof in [16]. The reader should note that in that article, (weakly) po-flat S-posets are referred to as being (weakly) flat.

Corollary 5.5. [16] *The following conditions are equivalent for a pomonoid S:*

(i) the class $\mathcal{PF}$ is axiomatisable;

(ii)the class $\mathcal{PF}$ is closed under formation of ultraproducts;

(iii) for every ordered skeleton $\mathcal{S}$ there exist finitely many replacement ordered skeletons $\mathcal{S}_1, \ldots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any right S-poset A and any po-flat left S-poset B, if $a \otimes b \leq a' \otimes b'$ exists in $A \otimes B$ by a ordered tossing $\mathcal{T}$ with ordered skeleton $\mathcal{S}$, then $a \otimes b \leq a' \otimes b'$ also exists in $(aS \cup a'S) \otimes B$ by a replacement ordered tossing $\mathcal{T}'$, such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$.

We now drop our assumption that Condition (Free)$^{\leq}$ holds. The proof of the next result follows that of Theorem 4.9.

Theorem 5.6. *Let $\mathcal{C}$ be a class of embeddings of right S-posets over a pomonoid S. Then the following conditions are equivalent for a pomonoid S:*

(1) the class $\mathcal{CPF}$ is axiomatisable;

(2) the class $\mathcal{CPF}$ is closed under ultraproducts;

(3) for every ordered skeleton $\mathcal{S}$ over S and $a, a' \in A$, where $\mu : A \to A'$ is in $\mathcal{C}$, there exist finitely many ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(a,\mathcal{S},a')}$, such that for any $\mathcal{C}$-po-flat left S-act B, if $a\mu \otimes b \leq a'\mu \otimes b'$ by an ordered tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $a \otimes b \leq a' \otimes b'$ by an ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(a, \mathcal{S}, a')\}$.

Theorem 5.6 can be specialised to the cases where $\mathcal{C}$ consists of all inclusions of (principal) right ideals of S into S, thus giving necessary and sufficient conditions on S such that $\mathcal{WPF}$ (a result also found in [16]) ($\mathcal{PWPF}$) is axiomatisable. The statements of these results are obtained from those of Corollaries 4.10 and 4.11, with the word 'double' omitted and 'flat' replaced by 'po-flat'. Further details may be found in [18].

6. Axiomatisability of Projective and Free S-posets

The axiomatisability problems for $\mathcal{P}r$ and $\mathcal{F}r$ are easily solved from the results of Section 3 and the answers to the corresponding questions in the S-act case. We note that the axiomatisability of the class of S-posets that are *free over posets* (a concept we have not used here) is considered in [16, 17].

6.1. **Axiomatisability of $\mathcal{P}r$.** The question of the axiomatisability of $\mathcal{P}r$ was addressed in [16]. Without giving much detail, Pervukhin and Stepanova indicate that if every ultrapower of a pomonoid S is projective as a left S-poset, then it can be argued, following the corresponding proofs for S-acts, that S is poperfect, which here can be taken to mean $\mathcal{SF} = \mathcal{P}r$ in the class of left S-posets. In [16] this is then utilised to show that $\mathcal{P}r$ is axiomatisable if and only if $\mathcal{SF}$ is axiomatisable and $\mathcal{SF} = \mathcal{P}r$. Notice that in [16], the classes $\mathcal{SF}$ and $\mathcal{P}r$ are denoted by $\mathcal{SF}^{<}$ and $\mathcal{P}^{<}$, to distinguish them from the classes of strongly flat and projective left S-*acts*, a convention we have not followed here.

The current authors have shown that a pomonoid S is left perfect as a *monoid* if and only if it is left perfect as a *pomonoid* [14]. With this in mind we can give a short and direct proof of the result of [16].

Theorem 6.1. [16] *The following are equivalent for a pomonoid S:*

(1) the class $\mathcal{P}r$ is axiomatisable;

(2) every ultrapower of S is projective as a left S-poset;

(3) the class $\mathcal{SF}$ is axiomatisable and $\mathcal{SF} = \mathcal{P}r$.

Proof. Clearly we need only prove that (2) implies (3); suppose that (2) holds.

Let $\mathcal{U}$ be an ultrapower of S *as a left S-act* with respect to an ultrafilter Φ, then

$$\mathcal{U} = \prod_{\gamma \in \Lambda} S^{\gamma} / \equiv_{\Phi}$$

where each S^{γ} is a copy of S,

$$(a_i) \equiv_{\Phi} (b_i) \Leftrightarrow \{i : a_i = b_i\} \in \Phi$$

and

$$s(a_i)_{\Phi} = (sa_i)_{\Phi} \text{ is a well-defined } S\text{-action.}$$

Consider the corresponding ultrapower of S as a left S-poset, that is,

$$\mathcal{U}' = \prod_{\gamma \in \Lambda} S^{\gamma} / \equiv_{\Phi}$$

where $\equiv_{\Phi}$ and the S-action are defined as before and

$$(a_i)_{\Phi} \leq (b_i)_{\Phi} \Leftrightarrow \{i : a_i \leq b_i\} \in \Phi \qquad (*).$$

In other words $\mathcal{U}$ is $\mathcal{U}'$ equipped with the partial order defined as in $(*)$.

We are supposing $\mathcal{U}'$ is projective as a left S-poset, that is, there exists a disjoint union $\bigcup_{i\in I} Se_i$ where e_is are idempotents, and an S-po-isomorphism $\theta : \mathcal{U}' \to \bigcup_{i\in I} Se_i$. Regarding $\bigcup_{i\in I} Se_i$ as an S-act, $\theta : \mathcal{U} \to \bigcup_{i\in I} Se_i$ is certainly an S-act isomorphism. We can conclude that every ultrapower of S as a left S-act is projective. From [13, Theorem 8.6], S is left perfect, so from [14, Theorem 6.3], S is left poperfect. Hence $\mathcal{SF} = \mathcal{P}r$. From the remark following Theorem 3.1, $\mathcal{SF}$ is axiomatisable. □

6.2. **Axiomatisability of $\mathcal{F}r$.** To explain our result we need to recall the following definition from [12]. Let $e \in E(S)$, where $E(S)$ is the set of idempotents of a monoid S, and let $a \in S$. We say that $a = xy$ is an *e-good factorisation of a through x* if $y \neq wz$ for any w, z with $e = xw$ and $e\,\mathcal{L}\,w$ (see [12]).

Theorem 6.2. *The following conditions are equivalent for a pomonoid S:*

(1) every ultrapower of the left S-poset S is free;

(2) $\mathcal{P}r$ is axiomatisable and S satisfies $(*)$: for all $e \in E(S) \setminus \{1\}$, there exists a finite set $f \subseteq S$ such that any $a \in S$ has an e-good factorization through x, for some $x \in f$;

(3) the class $\mathcal{F}r$ is axiomatisable.

Proof. (1) ⇒ (2) Since every ultrapower of S is free as a left S-poset, it is free as a left S-act with the same argument as in Theorem 6.1. By [12, Theorem 5.3], S satisfies $(*)$. Also by Theorem 6.1, $\mathcal{P}r$ is axiomatisable.

(2) ⇒ (3) If $\mathcal{P}r$ is axiomatisable, then every ultrapower of copies of S is projective as a left S-poset, and hence as a left S-act. From [13, Lemma 8.4], it follows that for any $e \in E(S)$ and $u \in S$, there are only finitely many $x \in S$ such that $e = ux$. This permits us to define the sentences φ_e as in [13]. Let $\sum_{\mathcal{P}r}$ be the set of sentences axiomatising the projective left S-posets. Then, as in [13, Theorem 9.1],

$$\sum_{\mathcal{P}r} \cup \big\{\varphi_e : e \in E(S) \setminus \{1\}\big\}.$$

axiomatises $\mathcal{F}r$.

□

7. Examples

We present some examples of a transparent nature to illustrate conditions arising in the axiomatisability results of the previous sections. For further examples we refer the reader to the forthcoming [18].

Before proceeding we note the following useful fact.

Lemma 7.1. *Let S be a pomonoid and let $\mathcal{C}, \mathcal{D}$ be classes of left S-posets such that $S \in \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{D}$. Suppose that $\mathcal{D}$ is axiomatisable if and only if every ultrapower of S lies in $\mathcal{D}$. Then if $\mathcal{C}$ is axiomatisable, so is $\mathcal{D}$.*

Let $S = \mathbb{N} \cup \{\epsilon\}$ be the natural numbers under the minimum operation, with an identity ϵ adjoined. Certainly S is a semilattice and as such may immediately be equipped with two partial orders: equality and the natural partial order. We denote the resulting pomonoids by $S^{=}$ and $S^{\leq}$; to simplify notation we continue to use S for the underlying monoid. We say that a pomonoid U is *left absolutely (po)-flat* if every left S-poset is (po)-flat.

Lemma 7.2. *For the pomonoid $S^{=}$:*

(i) $\mathbf{R}^{\leq}(\epsilon, \epsilon) = \mathbf{R}(\epsilon, \epsilon) = (\epsilon, \epsilon)S$;

(ii) for $s, t \in S$ with $s < t$ (under the natural partial order!) we have $\mathbf{R}^{\leq}(s, t) = \mathbf{R}(s, t) = (\epsilon, s)S$;

(iii) for $s \in S$ with $s \neq \epsilon$, $\mathbf{R}^{\leq}(s, s) = \mathbf{R}(s, s)$ is not contained in any finitely generated S-subposet of $S \times S$;

(iv) for any $s,t \in S$, $\mathbf{r}^{\leq}(s,t) = \mathbf{r}(s,t)$ is finitely generated (but notice that $\mathbb{N}$ is a non-finitely generated (right) ideal of S);
(v) $\mathcal{PWP}_w$ is not axiomatisable;
(vi) $\mathcal{E}, \mathcal{EP}$ are axiomatisable;
(vii) every left S-poset is weakly flat, so that $\mathcal{WF}$ is axiomatisable;
(viii) $S^{=}$ is not left absolutely flat.

Proof. (i) is clear; for (ii), it is certainly true that $(\epsilon, s) \in \mathbf{R}(s,t)$ and if $su = tv$, then as $s < t$ we have $su = v$ and $(u,v) = (\epsilon, s)u$.

(iii) Suppose that $\mathbf{R}(s,s) \subseteq T$ where T is an S-subposet of $S \times S$. Let X be a set of generators for T. For any $t \geq s$ we have that $(\epsilon, t) \in T$ so that $(\epsilon, t) = (u,v)p$ for some $(u,v) \in X$ and $p \in S$. Then $u = p = \epsilon$ and so $t = v$. Thus $(\epsilon, t) \in X$ and X cannot be finitely generated.

(iv) For any $s \in S$ we have $\mathbf{r}(s,s) = S = \epsilon S$. If $s \neq t$ then, supposing that $s < t$, we have that $su = tu$ if and only if $su = u$, so that $\mathbf{r}(s,t) = sS$.

(v) This follows from (iii) and the remark following Theorem 3.6.

(vi) It is immediate from Theorem 3.1, Lemma 3.3, (ii) and (iv) that $\mathcal{E}$ and $\mathcal{EP}$ are axiomatisable.

(vii) We claim that every left $S^{=}$-poset lies in $\mathcal{WF}$. To see this, let B be a left $S^{=}$-poset and suppose that $a \otimes b = a' \otimes b'$ in $S \otimes B$. Then $a \otimes b \leq a' \otimes b'$ and so by Lemma 2.2 there exists $a_2, a_3, \dots, a_m \in S$ and $b_1, b_2, \dots, b_m \in B$ such that

$$\begin{array}{rclrcl} & & & b & \leq & s_1 b_1 \\ a s_1 & = & a_2 t_1 & t_1 b_1 & \leq & s_2 b_2 \\ a_2 s_2 & = & a_3 t_2 & t_2 b_2 & \leq & s_3 b_3 \\ & \vdots & & & \vdots & \\ a_m s_m & = & a' t_m & t_m b_m & \leq & b'. \end{array}$$

Let u be the greater of a and a' under the natural partial order. Then we have an ordered tossing

$$\begin{array}{rclrcl} & & & b & \leq & s_1 b_1 \\ a s_1 & = & (u a_2) t_1 & t_1 b_1 & \leq & s_2 b_2 \\ (u a_2) s_2 & = & (u a_3) t_2 & t_2 b_2 & \leq & s_3 b_3 \\ & \vdots & & & \vdots & \\ (u a_m) s_m & = & a' t_m & t_m b_m & \leq & b' \end{array}$$

over $aS \cup a'S$ and B, so that $a \otimes b \leq a' \otimes b'$ in $aS \cup a'S \otimes B$. Together with the dual we obtain that $a \otimes b = a' \otimes b'$ in $aS \cup a'S \otimes B$, as required.

$(viii)$ We make use of [4]; by Example 1 of that article, $T = \{1, 2, \epsilon\}$ (with order equality) is not left absolutely flat. Let A be a right $T^{=}$-poset; we make A into a right $S^{=}$-poset, extending the action of T by putting $au = a2$ for all $u \in \{3, 4, \dots\}$. We can perform the dual procedure for a left T-poset B.

Suppose that $S^{=}$ is left absolutely flat. Consider a right T-poset A, a left T-poset B and elements $a, a' \in A$, $b, b' \in B$ such that $a \otimes b = a' \otimes b'$ in $A \otimes_T B$, where the subscript T means the tensor product is over T. Certainly then $a \otimes b = a' \otimes b'$ in $A \otimes_S B$, so that as $S^{=}$ is left absolutely flat, $a \otimes b = a' \otimes b'$ in $aS \cup a'S \otimes_S B$, say with double ordered tossing having double ordered skeleton $\mathcal{S}$. Now $aS = aT, a'S = a'T$ and as $us = u2$ and $sv = 2v$ for any $u \in A$, $v \in B$ and $s \in S \setminus T$, we can replace every occurrence of elements of $S \setminus T$ in $\mathcal{S}$ by 2. Hence $a \otimes b = a' \otimes b'$ in $aT \cup a'T \otimes_T B$. Thus T is left absolutely flat, contradicting Example 1 of [4]. $\square$

Notice that as $\mathcal{PWP}_w$ is not axiomatisable, Lemma 7.1 tells us that neither are any of the classes above and to the left of $\mathcal{PWP}_w$ in the table of Remark 2.4.

We now consider $S^{\leq}$, where we see a slightly different pattern.

Lemma 7.3. *For the pomonoid $S^{\leq}$:*
(i) $\mathbf{R}^{\leq}(s,s)$ is not finitely generated for any $s \in S$;
(ii) for $s,t \in S$ with $s < t$, $\mathbf{R}^{\leq}(s,t)$ is not finitely generated;
(iii) for $s,t \in S$ with $s > t$, $\mathbf{R}^{\leq}(s,t) = \bigcup_{r \leq t}(r,\epsilon)S$;
(iv) for any $s,t \in S$, $\mathbf{r}^{\leq}(s,t)$ is finitely generated (but notice that $\mathbb{N}$ is a non-finitely generated (right) ideal of S);
(v) $\mathcal{PWP}$ is not axiomatisable;
(vi) $\mathcal{P}_w, \mathcal{E}$ and $\mathcal{EP}$ are axiomatisable;
(vii) every left S-poset is weakly flat, so that $\mathcal{WF}$ is axiomatisable;
$(viii)$ $S^{\leq}$ is not left absolutely flat.

Proof. (i) Suppose that X were a set of generators for $\mathbf{R}^{\leq}(\epsilon,\epsilon)$. For any $s \in S$, we have that $(s,\epsilon) \in \mathbf{R}^{\leq}(\epsilon,\epsilon)$ and so $(s,\epsilon) = (x_s,y_s)p_s$ for some $(x_s,y_s) \in X$ and $p_s \in S$. Then $y_s = p_s = \epsilon$ and $s = x_s$, so that $(x_s,y_s) = (s,\epsilon)$ and X cannot be finite.

That $\mathbf{R}^{\leq}(s,s)$ is not finitely generated for any $s \neq \epsilon$, follows from (iii) of Lemma 7.2.

(ii) If $s < t$, then $(r,\epsilon) \in \mathbf{R}^{\leq}(s,t)$ for any $r \geq s$; as in (i) it follows that $\mathbf{R}^{\leq}(s,t)$ is not finitely generated.

(iii) Suppose that $s > t$. Certainly for any $r \in \{1,\dots,t\}$ we have that $(r,\epsilon) \in \mathbf{R}^{\leq}(s,t)$. On the other hand, if $su \leq tv$, then we must have that $su = u \leq tv \leq t, v$, and it follows that $(u,v) = (u,\epsilon)v$.

(iv) Clearly, for $s \leq t$ we have that $\mathbf{r}^{\leq}(s,t) = S = \epsilon S$. On the other hand, if $s > t$, it is easy to see that $\mathbf{r}^{\leq}(s,t) = tS$.

(v) This follows from Theorem 3.4 and (i).

(vi) The class $\mathcal{E}$ is axiomatisable from Theorem 3.1 and (iv).

To show that $\mathcal{P}_w$ is axiomatisable, we must show that Condition (4) of Theorem 3.5 holds. Clearly we need only check the pairs $\rho = (s,t)$ with $\mathbf{R}^{\leq}(s,t)$ not finitely generated. If $\rho = (s,s)$, consider $(\epsilon,s) \in \mathbf{R}^{\leq}(s,s)$. Then if $sx \leq sy$, we have that $x \leq \epsilon x, sx \leq sy \leq y$.

Suppose now that $s \neq \epsilon$ and $\rho = (s,\epsilon)$. Again, $(\epsilon,s) \in \mathbf{R}^{\leq}(s,\epsilon)$. Then if $su \leq \epsilon v$, we have that $u \leq \epsilon u, su \leq \epsilon v \leq v$.

Finally, suppose that $s < t < \epsilon$. Let

$$f = \{(u,v) : 1 \leq u,v \leq t, su \leq tv\} \cup \{(\epsilon,s),(\epsilon,s+1),\dots,(\epsilon,t)\}.$$

It is clear that $f \subseteq \mathbf{R}^{\leq}(s,t)$. If $u,v \in S$ and $su \leq tv$, then if $u,v \leq t$ we have that $(u,v) \in f$. If $v \geq t$ then $u \leq \epsilon u, tu \leq t \leq v$. If $v \leq t \leq u$, then as $su \leq tv$ we have that $s \leq v \leq t$ and $u \leq \epsilon u, vu = v \leq v$.

It follows from Lemma 3.3 that for $\mathcal{EP}$ we need only consider the case where $s \neq t$ and $\mathbf{R}^{\leq}(s,t)$ is not finitely generated. This only happens when $s < t$ and here $(\epsilon,\epsilon) \in \mathbf{R}^{\leq}(s,t)$. If $sa \leq ta$ for some $a \in A \in \mathcal{EP}$, then we certainly have $(a,a) = (\epsilon,\epsilon)a$ as required.

$(vii), (viii)$ may be proven as in Lemma 7.2, making use of Example 5 of [4].

□

For an example of a different flavour, we consider $S = G^0$, where G is an infinite group and the only non-trivial pairs in $\leq$ are $0 \leq g$, for any $g \in G$. Now familiar arguments give that $\mathcal{P}$ (and hence, $\mathcal{SF}$) are not axiomatisable, $\mathcal{E}, \mathcal{EP}$ and $\mathcal{P}_w$ are axiomatisable, and $\mathcal{PWP}$ is not axiomatisable. Details of this example (and many others) appear in [18].

8. Some Open Problems

We aim to axiomatise the class of left S-posets satisfying Condition (WP), and (WP$_w$). The finitary conditions that arise in axiomatising classes of S-posets, as in the case for M-acts, are related to more standard finitary conditions such as chain conditions. We aim to investigate these connections, particularly in the context of inverse monoids equipped with the natural partial order. For examples in the unordered case, we refer the reader to [12] and for progress to date in the ordered case, to [18].

References

[1] S. Bulman-Fleming and V. Gould, 'Axiomatisability of weakly flat, flat and projective S-acts', *Communications in Algebra* **30** (2002), 5575-5593.

[2] S. Bulman-Fleming and V. Laan, 'Lazard's Theorem for S-posets', *Math. Nachr.* **278** (2005), 1743–1755.

[3] S. Bulman-Fleming, 'Flatness properties of S-posets: an overview', p. 28–40 in *Proceedings of the International Conference on Semigroups, Acts and Categories, with Applications to Graphs*, Estonian Mathematical Society, Tartu (2008).

[4] S. Bulman-Fleming and N. Sohail, 'Examples concerning absolute flatness and amalgamation in pomonoids', *Semigroup Forum*, to appear DOI 10.1007/s00233-009-9202-9.

[5] C.C. Chang and H.J. Keisler, '*Model Theory*' North-Holland Publishing Company-Amsterdam London, American Elsevier Publishing Company, Inc.-New York (1973).

[6] G. Czedli and A. Lenkehegyi, 'On classes of ordered algebras and quasiorder distributivity', *Acta Sci.Math.(Szeged)* **46** (1983), 41–54.

[7] S. M. Fahkruddin, 'Absolute flatness and amalgams in pomonoids', *Semigroup Forum* **33**, (1986) 15–22.

[8] S. M. Fahkruddin, 'On the category of S-posets', *Acta Sci. Math. (Szeged)* **52** (1988), 85–92.

[9] A. Golchin, 'On condition (EP)', *Int. Math. Forum* **2** (2007), 911–918.

[10] A. Golchin and P. Rezaei, 'Subpullbacks and flatness properties of S-posets', *Communications in Algebra* to appear.

[11] V. Gould 'Axiomatisability problems for S-systems', *J. London Math. Soc.* **35** (1987), 193–201.

[12] V. Gould, 'Axiomatisability of free, projective and flat acts', 41–56 in *Proceedings of the meeting 'Semigroups, Acts and Categories', Tartu, June 2007*, Mathematical Studies 3, Estonian Mathematical Society, Tartu (2008).

[13] V. Gould, A. Mikhalev, E. Palyutin and A. Stepanova, 'Model theoretic properties of free, projective and flat acts', *Fund. Appl. Math.*, **14** (2008), 63–110.

[14] V. Gould and L. Shaheen, 'Perfection for Pomonoids', *Semigroup Forum*, to appear.

[15] M. Kilp, U. Knauer, A.V. Mikhalev, '*Monoids, Acts, and Categories*', de Gruyter, Berlin (2000).

[16] M.A. Pervukhin and A.A. Stepanova, 'Axiomatisability and completeness of some classes of partially ordered polygons', *Algebra and Logic* **48** (2009), 54–71.

[17] M.A. Pervukhin and A.A. Stepanova, 'Axiomatizability of free S-posets', *Fundamentalnaya i prikladnaya matematika* (Russian) **15** (2009), 99–115.

[18] L. Shaheen, 'Axiomatisability Problems for S-acts and S-posets', *Thesis in preparation.*

[19] X. Shi, 'Strongly flat and po-flat S-posets', *Communications in Algebra* **33** (2005), 4515–4531.

[20] X. Shi, Z. Liu, F. Wang, S. Bulman-Fleming, 'Indecomposable, Projective, and Flat S-posets', *Communications in Algebra* **33** (2005), 235–251.

[21] A.A. Stepanova, 'Axiomatisability and completeness in some classes of S-polygons', *Algebra and Logic*, **30** (1991/1992), 379–388.

[22] X.-Y. Xie and X. Shi 'Order-Congruence on S-posets', *Comm. Korean Math. Soc* **20** (2005), 1–14.

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