S-acts, coherency and stability

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Monoids and acts

A **(right)** S-**act** is a set A together with a map

$$A \times S \to A, \ (a, s) \mapsto as$$

such that for all $a \in A, s, t \in S$

$$a1 = a \text{ and } (as)t = a(st).$$

Right ideals (including S) are S-acts.

*Let A be an S-***act.**

For any $s \in S$, we have a unary operation ρ_s on A given by $a \mapsto as$ and a morphism $\phi : S \to \mathcal{T}_A$ given by $s \mapsto \rho_s$.

Conversely, if $\phi : S \to \mathcal{T}_B$ is a morphism for a set B, then B is an S-act with $bs = b(s\phi)$.

Consequently, S-acts are **representations of monoids by mappings of sets.**

Elementary observations for S-acts

- An S-**morphism** from A to B is a map $\alpha : A \to B$ with $(as)\alpha = (a\alpha)s$ for all $a \in A, s \in S$.
- S-acts and S-morphisms form a category - products are products, coproducts are disjoint unions
- We have usual definitions of **free, projective, injective**, etc. including variations on **flat**.
- Free S-acts are disjoint unions of copies of S.

Elementary observations for S-acts cont.

- A **congruence** ρ on A is an equivalence relation such that

$$a \,\rho\, b \Rightarrow as \,\rho\, bs$$

  for all $a, b \in A$ and $s \in S$.
- ρ is **finitely generated** if ρ is the smallest congruence containing a finite set $H \subseteq A \times A$.
- If ρ is a congruence on A then A/ρ is an S-act; all monogenic S-acts are of the form S/ρ.
- An S-act A is **finitely generated** if

$$A = a_1 S \cup \ldots \cup a_n S$$

  for some $a_i \in A$ and **finitely presented** if

$$A \cong F_n/\rho$$

  for some finitely generated free F_n and finitely generated congruence ρ.

First order languages and L_S

A (first order) language L has alphabet:
variables, connectives (e.g. $\neg, \vee, \wedge, \rightarrow$ etc.), quantifiers ($\forall, \exists$), $=$,
brackets, commas
and some/all of symbols for constants, functions and relations.

There are rules for forming well formed formulae (wff); a **sentence** is a wff with no free variables (i.e. all variables are governed by quantifiers).

The language L_S has:

no constant or relational symbols (other than $=$)

for each $s \in S$, a unary function symbol ρ_s.

A point of convenience Let us agree to abbreviate $x\rho_s$ in wffs of L_S by xs.

L_S-structures and S-acts

Examples

$\neg(xs = xt)$ is a wff but not a sentence,

$(\forall x)\big(\neg(xs = xt)\big)$ is a sentence,

$(\exists \vee xsx$ is not a wff.

An L-**structure** is a set D equipped with enough distinguished elements (constants), functions and relations to 'interpret' the abstract symbols of L.

An L_S-structure is simply a set with a unary operation for each $s \in S$.

Clearly an S-act A is an L_S-structure where we interpret ρ_s by the map $x \mapsto xs$.

Model theory

A **theory** is a set of sentences in a first order language.

Model theory provides a range of techniques to study algebraic and relational structures etc. via properties of their associated **languages** and **theories**.

Model theory of R-modules is a well developed subject area.

Model theory of S-acts - much less is known - authors include **Ivanov, Mustafin, Stepanova** .

Stability is an area within model theory, introduced by **Morley 62**. Much of the development of the subject is due to **Shelah**; the definitive reference is **Shelah 90** though (quote from Wiki) *it is notoriously hard even for experts to read*.

Model theory lite: axiomatisability

Definition A class $\mathcal{A}$ of L_S-structures is **axiomatisable** if there is a theory Σ such that for any L_S-structure A, we have $A \in \mathcal{A}$ if and only if every sentence of Σ is true in A, i.e. A is a *model* of Σ.

Example Let Σ_S be the theory

$$\Sigma_S = \big\{(\forall x)((xs)t = x(st)) : s, t \in S\big\} \cup \{(\forall x)(x\,1 = x)\}.$$

Then Σ_S axiomatises the class of S-acts (within all L_S-structures).

Example Let Π_S be the theory

$$\Sigma_S \cup \big\{(\exists x)\big(\neg(xs = xt)\big) : s, t \in S, s \neq t\}.$$

Then Π_S axiomatises the faithful S-acts.

Model theory of S-acts

Mustafin 88 There exists a monoid S and an S-act A such that $\mathrm{Th}(A)$ is not stable. *This contrasts with the situation for modules*.

Our aim today To look at finitary properties for monoids arising from existence and stability of the model companion of Σ_S.

A **finitary property** for a monoid is one held by finite monoids, e.g. S is **weakly right noetherian** if S has the ascending chain condition on right ideals.

Some classes of S-acts (such as the projectives and injectives) are axiomatisable iff S satisfies some finitary conditions.

Existentially closed S-acts

Let A be an S-act. An **equation** over A has the form

$$xs = xt,\ xs = yt \text{ or } xs = a$$

where x, y are variables, $s, t \in S$ and $a \in A$. **Inequations** look like

$$xs \neq xt,\ xs \neq yt \text{ or } xs \neq a.$$

A set of equations and inequations is **consistent** if it has a solution in some S-act $B \supseteq A$.

Definition A is **existentially closed** if every finite consistent set of equations and inequations over A has a solution in A.

Let $\mathcal{E}$ denote the class of existentially closed S-acts.

Question When is $\mathcal{E}$ axiomatisable?

Model companions

Definition Let T, T^* be theories in a first order language L. Then T^* is a **model companion** of T if every model of T embeds into a model of T^* and vice versa, and embeddings between models of T^* are elementary embeddings.

Theorem **Wheeler 76** Σ_S has a model companion Σ_S^* precisely when $\mathcal{E}$ (the class of existentially closed S-acts) is axiomatisable and in this case, Σ_S^* axiomatises $\mathcal{E}$.

Question When does Σ_S^* exist? i.e. When is $\mathcal{E}$ axiomatisable?

Right coherent monoids

Definition S is **right coherent** if every finitely generated S-subact of every finitely presented S-act is finitely presented.

Let A be an S-act and let $z \in A$. Put

$$\mathbf{r}(z) = \{(u, v) \in S \times S : zu = zv\}$$

and notice that $\mathbf{r}(z)$ is a right congruence on S.

Theorem **Wheeler 76, G 87, 92, Ivanov 92** The f.a.e. for S:

1. Σ_S^* exists;
2. S is right coherent;
3. every finitely generated S-subact of every S/ρ, where ρ is finitely generated, is finitely presented;
4. for every finitely generated right congruence ρ on S and every $a, b \in S$ we have $\mathbf{r}([a])$ is finitely generated, and $[a]S \cap [b]S$ is finitely generated.

Right coherent and right noetherian monoids

Definition S is **right noetherian** if every right congruence is finitely generated.

Fact If S is right noetherian, it is weakly right noetherian.

Theorem **Normak 77** If S is right noetherian, it is right coherent.

Example **Fountain 92** There exists a weakly right noetherian S which is not right coherent.

Right coherent and right noetherian monoids

Definitions S is

1. **regular** if $\forall a \in S \exists x \in S, a = axa$;
2. **inverse** if S is regular and $ef = fe$ for all $e = e^2, f = f^2 \in S$;
3. **Clifford** if S is regular and $ae = ea$ for all $a, e = e^2 \in S$.

Right coherent and right noetherian monoids: Examples

Results variously due to Gould, Hartmann, Ruskuc, Yang

1. **1992** The free commutative monoid $\mathcal{FC}(X)$ on X; *this follows from Rédei's theorem that if X is finite, then $\mathcal{FC}(X)$ is noetherian, so coherent from Normak; an easy argument the gives the infinite case*
2. **1992** Clifford monoids;
3. **2012** the free monoid X^*;
4. **2005, 2011** weakly right noetherian regular monoids so the **bicyclic monoid** and $BR(G, \theta)$;
5. **2012** the *double Bruck-Reilly* with identity adjoined;
6. **2013** primitive inverse monoids, $\mathcal{B}^0(M, I)$ where M is right coherent;
7. **2012** Free left ample
8. **2012** Free inverse monoids are NOT right coherent.

Stability: a bit of background

Stability properties for theories - **stable, superstable and totally transcendental** - arose from the question of how many models a theory has of any given cardinality.

Shelah 78 Showed that a non-superstable theory, has 2^λ models of cardinality λ for any $\lambda > \max\{\aleph_0, |T|\}$.

The philosophy then is that, in these cases, there are too many models to attempt to classify (e.g. by means of a sensible structure theorem).

It is reasonable therefore for the algebraist to consider for a given (axiomatisable) class of algebras 'how stable' is the theory associated with it.

A quick view of stability properties

We will be looking at stability properties of Σ_S^*.

We have ascertained that for Σ_S^* to exist, S must be right coherent.

Assume now that S is right coherent.

Is Σ_S^* **stable**? If so, when is Σ_S^* **superstable**?

...and when is Σ_S^* **totally transcendental**?

Types

Let A be an S-act. Then $L_S(A)$ is the language L_S augmented with a symbols representing the elements of A.

A is an S-subact of $\mathbf{M}$ where $\mathbf{M}$ is the 'monster model' of Σ_S^*. Let $c \in \mathbf{M}$. Then

$$\text{tp}(c/A) = \{\phi(x) \in L_S(A) : \mathbf{M} \models \phi(c)\}$$

is a **type over** A.

The **Stone space** $S(A)$ of A is the collection of all types over A equipped with the topology with basis

$$\langle \phi(x) \rangle = \{p \in S(A) : \phi(x) \in p\}.$$

Stability conditions

Definition For an infinite cardinal κ we say Σ_S^* is:

- κ-**stable** if for all S-acts A with $|A| \leq \kappa$ we have $|S(A)| \leq \kappa$;
- **stable** if Σ_S^* is κ-stable for *some* κ;
- **superstable** if Σ_S^* is κ-stable for all $\kappa \geq 2^{|T|}$ where $T = \Sigma_S^*$.

Fact **Lascar, 76** A complete theory is superstable if and only if every type p has U-rank $\mathrm{U}(p) < \infty$.

Definition A complete theory is **totally transcendental** if and only if every type p has Morley rank $\mathrm{M}(p) < \infty$.

Fact For any type p over a theory T we have $\mathrm{U}(p) \leq \mathrm{M}(p) \leq \infty$.

Stability of Σ_S^*

Let $\mathcal{RI}$ and $\mathcal{RC}$ denote the lattices of right ideals and right congruences on S.

Let A be an S-act. An A-triple (I, ρ, f) is a triple such that:

$$I \in \mathcal{RI},\ \rho \in \mathcal{RC},\ I \text{ is } \rho\text{-saturated}$$

and

$$f : I \to A \text{ is an } S\text{-morphism with } \operatorname{Ker} f = \rho \cap (I \times I).$$

Let $T(A)$ denote the set of all A-triples.

Theorem **Fountain, G 87/08** $S(A)$ is in bijective correspondence with $T(A)$ under $p \mapsto (I_p, \rho_p, f_p)$.

Stability and superstability of Σ_S^*

Theorem **Fountain, G 87/08, Ivanov 92**

1. Σ_S^* is stable.
2. Σ_S^* is superstable if and only if S is weakly right noetherian.

We prove this by using a notion of rank on ρ-saturated right ideals for a right congruence ρ so get a value of U-rank of a type in algebraic terms.

Weakly right noetherian (wrn)

1. Groups are wrn.
2. Semilattices are wrn iff they have the ascending chain condition (as posets) and no infinite antichains (folklore).
3. Bicyclic monoid $B = \mathbb{N}^0 \times \mathbb{N}^0$ with binary operation

$$(a, b)(c, d) = (a - b + t, d - c + t) \text{ where } t = \max\{b, c\}$$

   is inverse and wrn.
4. Free inverse monoids on non trivial X are not wrn.
5. S regular, wrn implies S right coherent.
6. Brandt semigroups with identity adjoined are right coherent but need not be wrn.

Total transcendence

A theory T is **totally transcendental** if and only every type p has Morley rank $\mathrm{M}(p)$. These are the best theories in terms of stability.

A totally transcendental theory is superstable.

Fountain, G 87/08 characterised those (right coherent) S such that for every type p, we have $\mathrm{U}(p) = \mathrm{M}(p) < \infty$.

Note that if $\mathrm{M}(p) < \infty$ for a type over a theory of modules, then $\mathrm{U}(p) = \mathrm{M}(p)$ - is this true for S-acts?

The finite type topology in $\mathcal{RC}$

We define a topology on $\mathcal{RC}$ by means of a basis.

Let $\nu \in \mathcal{RC}$ be finitely generated and let $K \subseteq (S \times S) \setminus \nu$ be finite. Put

$$[\nu, K] = \{\rho \in \mathcal{RC} : \nu \subseteq \rho \subseteq (S \times S) \setminus K\}.$$

The **finite type topology** has sets $[\nu, K]$ as a basis.

The **Cantor-Bendixon** rank of a point in topological space measures how far a point is from being isolated.

The **S-rank** of $\rho \in \mathcal{RC}$ is the Cantor-Bendixon rank of ρ with respect to the finite type topology.

S-rank

We make this explicit by defining subsets $\mathcal{C}^\alpha$ of $\mathcal{RC}$ for each ordinal α, as follows:
(I) $\mathcal{C}^0 = \mathcal{C}$;
(II) if α is a limit ordinal, then

$$\mathcal{C}^\alpha = \bigcap\{\mathcal{C}^\beta : \beta < \alpha\};$$

(III) $\rho \in \mathcal{C}^{\alpha+1}$ if and only if $\rho \in \mathcal{C}^\alpha$ and for all subsets of finite type $[\nu, K]$ with $\rho \in [\nu, K]$, there exists $\theta \in \mathcal{C}^\alpha$ with

$$\theta \in [\nu, K], \theta \neq \rho.$$

The **S-rank** $\mathrm{S}(\rho)$ of $\rho \in \mathcal{RC}$ is ∞ if $\rho \in \mathcal{C}^\alpha$ for all α, and otherwise $\mathrm{S}(\rho) = \alpha$ where $\rho \in \mathcal{C}^\alpha \setminus \mathcal{C}^{\alpha+1}$. If $\mathrm{S}(\rho) < \infty$, then we say that ρ **has S-rank**.

S is **ranked** if every $\rho \in \mathcal{RC}$ has S-rank.

Ranked semigroups

Theorem **G 05** Σ_S^* is totally transcendental if and only if S is wrn and every $\rho \in \mathcal{RC}$ has S-rank.

1. If S is right noetherian, then S is ranked.
2. If S inverse and ranked, then S is wrn.
3. Let S be ranked. Then any maximal subgroup and monoid $\mathcal{J}$-class is ranked.
4. A ranked monoid cannot contain a bicyclic $\mathcal{J}$-class.
5. The chain C

$$e_0 > e_1 > \dots$$

   is right coherent, weakly right noetherian but not ranked, so Σ_C^* is superstable but not totally transcendental.
6. $\mathbb{Z}$ under $+$ is (right) noetherian, right coherent and ranked so $\Sigma_{\mathbb{Z}}^*$ is t.t., but there is a type p with $\mathrm{M}(p) \neq \mathrm{U}(p)$.

Questions

1. For an arbitrary S, does ranked imply wrn?
2. Connections of right coherency to products of (weakly, strongly) flat *left* S-acts?
3. Ranked groups?
4. Structure of S-acts where S satisfies suitable finitary properties.