Commutative orders

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Abstract

A subsemigroup S of a semigroup Q is a *left (right) order* in Q if every $q \in Q$ can be written as $q = a^*b (q = ba^*)$ for some $a, b \in S$, where a^* denotes the inverse of a in a subgroup of Q and if, in addition, every square-cancellable element of S lies in a subgroup of Q. If S is both a left order and a right order in Q we say that S is an *order* in Q. We show that if S is a left order in Q and S satisfies a permutation identity $x_1 \ldots x_n = x_{1\pi} \ldots x_{n\pi}$ where $1 < 1\pi$ and $n\pi < n$, then S and Q are commutative. We give a characterisation of commutative orders and decide the question of when one semigroup of quotients of a commutative semigroup is a homomorphic image of another. This enables us to show that certain semigroups have maximum and minimum semigroups of quotients. We give examples to show that this is not true in general.

1 Introduction

It is well known that a semigroup S has a group of left quotients if and only if S is cancellative and right reversible, that is, $Sa \cap Sb \neq \emptyset$ for all $a, b \in S$ [CP, Theorem 1.24]. Thus a commutative semigroup S has a group G of left quotients if and only if S is cancellative; in this case it is easy to see that G is also commutative. We are concerned in this paper with the more general notion of a *semigroup* of left quotients. The concept we use is that introduced by Fountain and Petrich in [3]; the main idea is that we consider inverses of elements in *any* subgroup of a semigroup, and not just the group of units.

Let S be a subsemigroup of a semigroup Q. Then Q is a *semigroup of left quotients* of S if every $q \in Q$ can be written as $q = a^*b$ for some $a, b \in S$, where a^* denotes the inverse of a in a subgroup of Q and if, in addition, every element of S satisfying a weak cancellation condition known as square-cancellability lies in a subgroup of Q. Clearly if S has a group of left quotients G then G is also a semigroup of left quotients of S. If Q is a semigroup of left quotients of S we also say that S is a *left order* in Q. *Semigroups of right quotients* and *right orders* are defined dually. If S is both a left order and a right order in Q then S is an *order* in Q and Q is a *semigroup of quotients* of S.

It is natural to hope that if a commutative semigroup S is a left order in Q, then Q is also commutative so that S is an order in Q. This is true; indeed if S satisfies any permutation identity $x_1 ... x_n = x_{1\pi} ... x_{n\pi}$ where $1 < 1\pi$ and $n\pi < n$, then Q is commutative, as we show in Theorem 3.1. Semigroups satisfying a permutation identity have attracted interest from a number of authors; see, for example, [10] and [7]. We remark that it is shown in [7] that if a semigroup S satisfies a permutation identity of the aforementioned kind and $S^2 = S$, that is, S is *globally idempotent*, then S is commutative. Our left orders, however, are not necessarily globally idempotent.

A general description of semigroups that are left orders is not known and would undoubtedly be unwieldy. Authors have therefore concentrated on studying semigroups that are (left) orders in semigroups in a particular class, for example, orders in completely 0-simple semigroups are characterised in [3]. Surprisingly, orders in commutative semigroups have not been studied, a situation we hope to amend in this paper. In particular we give a description of orders in commutative semigroups. In view of Theorem 3.1, this is the class of commutative orders.

Section 2 contains definitions and preliminary remarks on orders. Section 3 concentrates on proving the above mentioned result that left orders satisfying certain permutation identities, and their semigroups of (left) quotients, are commutative. Then in Section 4 we give the promised description of commutative orders. Theorems 4.2 and 4.3 show that if S is a commutative semigroup then the existence of a semigroup of quotients of S is dependent upon the existence of a preorder on S satisfying certain conditions. The versatility of this result is illustrated in Examples 7.4 and 7.5, where it is used to construct semigroups of quotients having various prescribed properties.

An example is given in [4] of a commutative semigroup having non-isomorphic semigroups of quotients. With this in mind we determine in Section 5 when one semigroup of quotients of a commutative semigroup S is a homomorphic image of another. As a corollary, we can decide when two semigroups of quotients of S are isomorphic. We also show that for certain commutative orders S, namely those in which all elements are square-cancellative, the set of semigroups of quotients of S forms a complete lattice under a natural partial order. Such orders S have a *maximum* and a *minimum* semigroup of quotients.

In Section 6 we study the situation where a commutative semigroup S is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$, where S_α is an order in Q_α for each $\alpha \in Y$. We give necessary and sufficient conditions for S to be an order in Q, where Q is a semilattice Y of semigroups $Q_\alpha, \alpha \in Y$. As a consequence of this we can show that if S is a semilattice Y of commutative cancellative semigroup $S_\alpha, \alpha \in Y$, then S is an order in a commutative regular semigroup. This corollary is also known from [4], which studies left orders in regular semigroups with central idempotents.

In our final section we give a number of examples to illustrate our results. Examples 7.1 and 7.2 are examples of commutative orders where not all elements are square-cancellable, but such that the set of semigroups of quotients of each forms a complete lattice. They also show that the preorder in Theorem 4.2 cannot always be replaced by the preorder $\leq_{\mathcal{H}^*}$ (unlike the case where all elements of an order are square-cancellable). Examples 7.4 (7.5) are, respectively, commutative orders having a maximum but no minimum (a minimum but no maximum) semigroup of quotients.

2 Preliminaries

We assume the reader has some knowledge of algebraic semigroup theory, in particular the definitions and elementary facts concerning Green's relations. Any undefined notation or concepts may be found in the standard references [1] or [6]. We deviate from standard notation in denoting by a^* the group inverse, where it exists, of an element a of a semigroup Q. That is, a^* exists if and only if a lies in a subgroup of Q, and the inverse of a in this subgroup is a^*. By a famous result of Green [H, Theorem II 2.5], a^* exists if and only if a is related to its square by the relation $\mathcal{H}$. Moreover, where a^* exists it is unique. We write $\mathcal{H}(Q)$ for the union of the subgroups of Q. That is, $\mathcal{H}(Q) = \{a : a\mathcal{H}a^2\}$. In general, $\mathcal{H}(Q)$ will not be a subsemigroup.

To define square-cancellability we make use of a generalisation of Green's relations. Let a, b be elements of a semigroup S. The relation $\mathcal{L}^*$ is defined by the rule that $a\mathcal{L}^*b$ if and only if $a\mathcal{L}b$ in some oversemigroup of S. This is equivalent to the cancellation condition that

$$ax = ay \text{ if and only if } bx = by$$

for all $x, y \in S^1$ [2]. The relation $\mathcal{R}^*$ is defined dually and we put $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$. It is easy to see that $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence. Thus if S is commutative, $\mathcal{L}^* = \mathcal{R}^* = \mathcal{H}^*$ is a congruence on S.

An element a of a semigroup S is *square-cancellable* if $a\mathcal{H}^*a^2$. Square-cancellability is thus a necessary condition for an element to lie in a subgroup of an oversemigroup. The definition of a semigroup of (left) quotients insists that all such elements must lie in subgroups of any semigroup of (left) quotients. We denote by $\mathcal{S}(S)$ the set of square-cancellable elements of a semigroup S.

Let S be a subsemigroup of Q. Then S is a *weak left order* in Q if any $q \in Q$ can be written as $q = a^*b$ where $a, b \in S$. If in addition $\mathcal{H}(Q) \cap S = \mathcal{S}(S)$, then S is a *left order* in Q and Q is a *semigroup of left quotients* of S. The left-right dual and the two-sided notions are defined in the obvious way.

If S is a weak left order in Q, then by definition any $q \in Q$ may be written as $q = a^*b$ where $a, b \in S$. Hence $q = (a^2)^*ab$ and $abQ^1 \subseteq aQ^1 = a^2Q^1$. Thus any $q \in Q$ can be written as $q = c^*d$ where $c, d \in S$ and $d \leq_{\mathcal{R}} c$ in Q. Here $\leq_{\mathcal{R}}$ is the preorder given by

$$u \leq_{\mathcal{R}} v \text{ if and only if } uQ^1 \subseteq vQ^1$$

where $u, v \in Q$. The preorder $\leq_{\mathcal{L}}$ is defined dually; $\leq_{\mathcal{R}}$ ($\leq_{\mathcal{L}}$) is left (right) compatible with multiplication and has associated equivalence relation $\mathcal{R}$ ($\mathcal{L}$). Where Q is commutative $\leq_{\mathcal{R}} = \leq_{\mathcal{L}}$ and we denote this relation by $\leq_{\mathcal{H}}$. Also from the definition, we have that if S is a weak left order in Q then given any $q \in Q$, there is an $a \in \mathcal{S}(S)$ with $q \leq_{\mathcal{R}} a$ in Q; from this, $q = aa^*q$ and $Qq = Q^1q$. Further, if $q = a^*b$ where $a, b \in S$ and $b \leq_{\mathcal{R}} a$, then $q\mathcal{L}b$ in Q. Thus S has non-empty intersection with every $\mathcal{L}$-class of Q; if Q is commutative, $q\mathcal{H}b$ so that S has non-empty intersection with every $\mathcal{H}$-class of Q. We state these facts as a lemma, which we use repeatedly.

Lemma 2.1 *Let S be an order in a commutative semigroup Q. Then any $q \in Q$ can be written as $q = a^*b$ where $a, b \in S$, $b \leq_{\mathcal{H}} a$ and $q\mathcal{H}b$ in Q. In particular, S has non-empty intersection with every $\mathcal{H}$-class of Q.*

In Section 3 we show that if S is commutative and a weak left order in Q, then Q is commutative. The results of Sections 4 and 5 show that if S is a commutative order, then a semigroup of quotients Q is determined by the preorder $\leq_{\mathcal{H}} \cap (S \times S)$ (where $\leq_{\mathcal{H}}$ is the relation on Q). An obvious candidate for this preorder is $\leq_{\mathcal{H}^*}$, which is defined on a commutative semigroup S by the rule that $a \leq_{\mathcal{H}^*} b$ if

$$xb = yb \text{ implies that } xa = ya$$

for all $x, y \in S^1$. As in [8], if $a, b \in S$ then $a \leq_{\mathcal{H}^*} b$ if and only if $a \leq_{\mathcal{R}} b$ in T and $a \leq_{\mathcal{L}} b$ in U, for some oversemigroups T and U of S. Clearly $\leq_{\mathcal{H}^*}$ is a preorder compatible with multiplication, with associated equivalence relation $\mathcal{H}^*$. If S is a (weak) left order in Q then Green's relations and their preorders will always refer to relations on Q; the starred versions will refer to relations on S.

We end this section by gathering together some elementary remarks concerning the relations $\mathcal{H}^*$ and $\mathcal{H}$ on a commutative semigroup.

Lemma 2.2 *Let T be a commutative semigroup. Then*
(i) $\mathcal{H}^*$ is a congruence on T and $\mathcal{S}(T)$ is a subsemigroup of T;
(ii) $\mathcal{H}$ is a congruence on T and $\mathcal{H}(T)$ is a subsemigroup, in fact $\mathcal{H}(T)$ is a semilattice of the group $\mathcal{H}$-classes of T;
(iii) for all $a, b \in \mathcal{H}(T)$, $(ab)^* = b^*a^* = a^*b^*$.
Further, if S is an order in a commutative semigroup Q, then $\mathcal{S}(S)$ is an order in $\mathcal{H}(Q)$.

3 Left orders satisfying a permutation identity

In proving our first result it is convenient to make a slight adjustment in notation. If a is an element of a semigroup Q and a lies in a subgroup of Q then for any positive integer n we write a^{-n} for $(a^*)^n$.

Theorem 3.1 *Let S be a weak left order in Q and suppose that S satisfies a permutation identity*

$$x_1 ... x_n = x_{1\pi} ... x_{n\pi} \ (\dagger)$$

for some permutation π for which $1 \neq 1\pi$ and $n \neq n\pi$. Then Q is commutative.

Proof Put $k = 1\pi^{-1} > 1$ and $\ell = n\pi^{-1} < n$. Observe that, for all $a, b \in S$ two applications of $(\dagger)$ give

$$a^{2n-2}b = (a^2)^{n-1}b = (a^2)^{\ell-1}b(a^2)^{n-\ell}$$

$$= (a^2)^{\ell-1}(ba)a(a^2)^{n-\ell-1} = a^{2n-3}ba.$$

Thus for any positive integer K and any integer $M \geq 2n-3$,

$$a^{M+K}b = a^M ba^K.$$

Suppose now that $a, b \in S$, that a lies in a subgroup of Q and $b \leq_{\mathcal{R}} a$. Then, since $a^{-1}ab = b$, the above yields

$$ab = a^{-2n+3}a^{2n-2}b = a^{-2n+3}a^{2n-3}ba = ba,$$

and so also

$$a^{-1}b = a^{-2}ab = a^{-2}ba = a^{-2}ba^2a^{-1} = a^{-2}a^2ba^{-1} = ba^{-1}.$$

Now suppose that $s, t \in S$ and s lies in a subgroup of Q. Then, since $s^k t \leq_{\mathcal{R}} s^{k-1}$, from another application of $(\dagger)$,

$$st = s^{-k+1}(s^k t) = (s^k t)s^{-k+1} = s^{k-2}s^2 ts^{n-k}s^{-n+1} = ts^n s^{-n+1} = ts,$$

and so also, as $st \leq_{\mathcal{R}} s^2$,

$$s^{-1}t = s^{-2}(st) = (st)s^{-2} = (ts)s^{-2} = ts^{-1}.$$

If further t lies in a subgroup of Q then

$$s^{-1}t^{-1} = s^{-1}tt^{-2} = ts^{-1}t^{-2} = t^{-2}t^3s^{-1}t^{-2}$$

$$= t^{-2}s^{-1}t^3t^{-2} = t^{-2}s^{-1}t = t^{-2}ts^{-1} = t^{-1}s^{-1}.$$

Finally, consider $h, k \in S$. As commented in Section 2, $h \leq_{\mathcal{R}} a$ for some element a of S lying in a subgroup of Q. Then $hk = a^{-(n-2)}a^{\ell-1}ha^{n-\ell-1}k$, since $a^{-1}ah = aa^{-1}h = h$, and since all powers of a commute with all elements of S. (If $\ell - 1 = 0$ or $n - \ell - 1 = 0$ we consider the corresponding power of a to be the empty word.) Using (†) we have that, for some integer p

$$hk = a^p k a^{-p} h = k a^p a^{-p} h = kh.$$

It follows that S and Q are commutative.

Corollary 3.2 is now immediate.

Corollary 3.2 *Let S be a commutative weak left order in Q. Then Q is commutative.*

If Q is a regular semigroup then clearly Q is a weak left order in itself. The following corollary appears as Theorem 6 in [10].

Corollary 3.3 *[10] Let Q be a regular semigroup satisfying a permutation identity*

$$x_1 \ldots x_n = x_{1\pi} \ldots x_{n\pi}$$

for some permutation π for which $1 \neq 1\pi$ and $n \neq n\pi$. Then Q is commutative.

In fact, if Q is a regular semigroup then it is easy to see that $\mathcal{R}^* = \mathcal{R}$ and $\mathcal{L}^* = \mathcal{L}$. Hence $\mathcal{H}^* = \mathcal{H}$ and $\mathcal{H}(Q) = \mathcal{S}(Q)$. Thus Q is an order in itself.

4 Characterisation of commutative orders

In this section we give necessary and sufficient conditions for a commutative semigroup S to be an order. Necessary conditions are easy to obtain; as is usual with these problems, proof of sufficiency is more involved. The most general description of left orders to date is given in [5], where *straight* left orders are described. Commutative orders, although in many ways less complex than arbitrary orders, need not be straight, so we cannot here make use of the results of [5] or previous work on orders. Examples of non-straight commutative orders are given in Section 7.

The proof of our first lemma is straightforward.

Lemma 4.1 *Let S be a commutative order in a semigroup Q and denote $\leq_{\mathcal{H}} \cap (S \times S)$ by $\leq$, where $\mathcal{H}$ is Green's relation on Q. Then $\leq$ is a preorder compatible with multiplication such that:*
(A) for all $b, c \in S, bc \leq b$;
(B) for all $b, c \in S$ and $a \in \mathcal{S}(S)$,

$$b \leq a, c \leq a, ab = ac$$

implies that $b = c \leq ba$;
(C′) for all $b, c \in S, b \leq c$ implies that $bx = cy$ for some $x \in \mathcal{S}(S), y \in S$ with $b \leq x$.

For our construction proof we make use of a weaker version of (C′), namely:
(C) for all $b \in S$ there exists $a \in \mathcal{S}(S)$ with $b \leq a$.

Theorem 4.2 *Let S be a commutative semigroup. Then S is an order in some semigroup Q if and only if there exists a preorder $\leq$ on S which is compatible with multiplication such that conditions (A), (B) and (C) hold.*

Further, Q may be chosen such that

$$\leq_{\mathcal{H}} \cap (S \times S) \subseteq \leq$$

and for all $b \in S$ and $a \in \mathcal{S}(S)$

$$b \leq a \text{ if and only if } b \leq_{\mathcal{H}} a.$$

Proof The necessity of the conditions is immediate from Lemma 4.1 and the comment which follows it.

Suppose conversely that S satisfies conditions (A), (B) and (C). We aim to construct a semigroup Q in which S is an order.

Let $\equiv$ be the equivalence relation on S associated with the preorder $\leq$; since $\leq$ is compatible with multiplication, $\equiv$ is a congruence on S. By conditions (A) and (B) (with $b = c = a$) we have that if $a \in \mathcal{S}(S)$ then $a \equiv a^2$. Put

$$\sum = \{(a,b) \in \mathcal{S}(S) \times S : b \leq a\}$$

so that $\sum \neq \emptyset$ by (C). Define a relation $\sim$ on $\sum$ by the rule that for $(a,b),(c,d) \in \sum$,

$$(a,b) \sim (c,d) \text{ if and only if } ad = bc \text{ and } b \equiv d.$$

Clearly $\sim$ is reflexive and symmetric. Suppose that $(a,b),(c,d),(e,f) \in \sum$ and

$$(a,b) \sim (c,d) \sim (e,f).$$

Then $ad = bc, cf = de$ and $b \equiv d \equiv f$; since $\equiv$ is a congruence, $b \equiv f$. Then

$$(af)c = (cf)a = (de)a = (ad)e = (bc)e = (be)c.$$

By (A), $af \leq f \equiv d \leq c$ and $be \leq b \equiv d \leq c$ so that (B) gives $af = be$. Thus $\sim$ is an equivalence relation on $\sum$.

Put $Q = \sum / \sim$ and denote the $\sim$-equivalence class of (a,b) by $[a,b]$. Define a multiplication on Q by

$$[a,b][c,d] = [ac, bd],$$

which makes sense since by Lemma 2.2 $\mathcal{S}(S)$ is a subsemigroup of S. This multiplication on Q is well defined, for if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then $b \equiv b', d \equiv d', ab' = ba'$ and $cd' = dc'$. Now

$$(ac)(b'd') = (ab')(cd') = (ba')(dc') = (a'c')(bd)$$

and further, $bd \equiv b'd'$ so that $(ac, bd) \sim (a'c', b'd')$. Clearly the multiplication in Q is associative and commutative.

We now show that S is embedded in Q. For $b \in S$ there exists $a \in \mathcal{S}(S)$ with $b \leq a$, by (C); (A) and (B) together give that

$$ab \equiv b \ (\ddagger),$$

so that $ab \leq a$ and $(a, ab) \in \sum$. Further, if $a' \in \mathcal{S}(S)$ and $b \leq a'$, then $(a, ab) \sim (a', a'b)$. Thus $\theta : S \to Q$ is well defined where $b\theta = [a, ab]$ and

$a \in \mathcal{S}(S)$ is chosen with $b \leq a$. It is easy to see that θ is a homomorphism. Suppose now that $b, d \in S$ and $b\theta = d\theta$. Thus $(a, ab) \sim (c, cd)$ for some $a, c \in \mathcal{S}(S)$ with $b \leq a$ and $d \leq c$. From the definition of $\sim$ and from (‡) it follows that $b \equiv ab \equiv cd \equiv d$. Also $acd = acb$ and $cb \leq b \leq a$ so that $cd \leq a$ and (B) gives that $cd = cb$. Another application of (B) gives that $d = b$, hence $\theta : S \to Q$ is an embedding.

We now verify that $S\theta$ is an order in Q. If $a \in \mathcal{S}(S)$ then $a\theta = [a, a^2]$. Note also that $(a^2, a) \in \sum$ and $(a, a) \in \sum$. It is easy to check that $a\theta$ lies in a subgroup of Q with identity $[a, a]$ and $(a\theta)^* = [a^2, a]$. Finally, suppose that $[a, b] \in Q$. Then

$$(a\theta)^* b\theta = [a^2, a][a, ab] = [a^3, a^2 b] = [a, b],$$

which completes the proof that $S\theta$ is an order in Q.

To prove the last assertions of the theorem, let $b, d \in S$ and suppose that $b\theta \leq_{\mathcal{H}} d\theta$ in Q. Then there exists $[x, y] \in Q$ with

$$[a, ab] = [c, cd][x, y] = [cx, cdy]$$

where $a, c \in \mathcal{S}(S), b \leq a$ and $d \leq c$. This gives that $b \equiv ab \equiv cdy \leq d$ so that $b \leq d$. Further, given $m \in S$ and $n \in \mathcal{S}(S)$ with $m \leq n$, it is easy to check that

$$[n, n^2][n, m] = [n, nm]$$

giving $m\theta \leq_{\mathcal{H}} n\theta$.

We now show that by strengthening condition (C) in Theorem 4.2 Q may be chosen so that $\leq_{\mathcal{H}} \cap (S \times S) = \leq$, where $\leq$ is the given preorder on S. The significance of this becomes apparent in the next section where we consider conditions under which one semigroup of quotients of S is a homomorphic image of another.

Theorem 4.3 *Let S be a commutative semigroup. Then S is an order in a semigroup Q such that $\leq_{\mathcal{H}} \cap (S \times S) = \leq$ if and only if $\leq$ is a preorder on S compatible with multiplication satisfying conditions (A),(B) and (C′).*

Proof The necessity of the conditions is immediate from Lemma 4.1.

Conversely, suppose that $\leq$ is a preorder on S compatible with multiplication such that conditions (A), (B) and (C$'$) hold. By Theorem 4.1, S is an order in a semigroup Q such that

$$\leq_{\mathcal{H}} \cap (S \times S) \subseteq \leq$$

and for all $b \in S$ and $a \in \mathcal{S}(S)$

$$b \leq a \text{ if and only if } b \leq_{\mathcal{H}} a.$$

Suppose that $b, c \in S$ and $b \leq c$. By condition (C$'$) there exists $x \in \mathcal{S}(S), y \in S$ with $b \leq x$ and $bx = cy$. Since $x \in \mathcal{S}(S)$ we have $b \leq_{\mathcal{H}} x$ and $b = bxx^* = cyx^*$ so that $b \leq_{\mathcal{H}} c$. Thus $\leq_{\mathcal{H}} \cap (S \times S) = \leq$.

If S is a commutative order in Q, then $\leq_{\mathcal{H}} \cap (S \times S) \subseteq \leq_{\mathcal{H}^*}$; in the case where $S = \mathcal{S}(S)$ we show that S has some semigroup of quotients in which $\leq_{\mathcal{H}} \cap (S \times S) = \leq_{\mathcal{H}^*}$.

Corollary 4.4 *Let S be a commutative semigroup with $S = \mathcal{S}(S)$. Then S is an order if and only if the $\mathcal{H}^*$-classes of S are cancellative. In this case, S has a semigroup of quotients in which $\leq_{\mathcal{H}} \cap (S \times S) = \leq_{\mathcal{H}^*}$.*

Proof Note that for a commutative semigroup S with $S = \mathcal{S}(S)$, conditions (A) and (C$'$) always hold for $\leq_{\mathcal{H}^*}$. If S is an order in Q and a, b, c are $\mathcal{H}^*$-related elements of S with $ab = ac$, then $b^2 = bc = c^2$ so that $b \mathcal{H} c$ in Q and $b = c$ since H_b is a group. Thus the $\mathcal{H}^*$-classes of S are cancellative.

Conversely, suppose that the $\mathcal{H}^*$-classes of S are cancellative and $a, b, c \in S$ with

$$b \leq_{\mathcal{H}^*} a, c \leq_{\mathcal{H}^*} a \text{ and } ab = ac.$$

Since $S = \mathcal{S}(S)$ we have $b \mathcal{H}^* b^2 \leq_{\mathcal{H}^*} ba \leq_{\mathcal{H}^*} b$ so that $b \mathcal{H}^* ba$. Similarly, $c \mathcal{H}^* ca$ so that $b \mathcal{H}^* c$ and from the fact that the $\mathcal{H}^*$-class of b is cancellative, $(ba)b = (ba)c$ yields $b = c$. Thus condition (B) holds. Theorem 4.3 says that S is an order in some semigroup where $\leq_{\mathcal{H}} \cap (S \times S) = \leq_{\mathcal{H}^*}$.

If a commutative semigroup S with $\mathcal{S}(S) = S$ is an order in Q then it is not necessarily the case that $\leq_{\mathcal{H}} \cap (S \times S) = \leq_{\mathcal{H}^*}$. For example, a commutative cancellative semigroup (here $\mathcal{H}^*$ is universal) can be an order in a non-trivial semilattice of groups (see [4]). In the final section we give examples of commutative orders for which the relation $\mathcal{H}^*$ does not satisfy conditions (A),(B) and (C$'$). By Theorem 4.3, such an order S cannot have a semigroup of quotients in which $\leq_{\mathcal{H}} \cap (S \times S) = \leq_{\mathcal{H}^*}$.

5 Maximum and minimum semigroups of quotients

As remarked in the introduction, a commutative semigroup S may have non-isomorphic semigroups of quotients. In this section we determine when two semigroups of quotients of S *are* isomorphic; this is a corollary of our first result which gives necessary and sufficient conditions for one semigroup of quotients of S to be a homomorphic image of another, under a homomorphism which restricts to the identity on S. Such a homomorphism is called an S-homomorphism. More precisely, if S is an order in Q_1 and Q_2 where $\iota : S \to Q_1$ and $\tau : S \to Q_2$ are the embeddings of S in Q_1 and Q_2 respectively, then a homomorphism $\theta : Q_1 \to Q_2$ is an *S-homomorphism* if $\iota\theta = \tau$. If θ is a bijective S-homomorphism then we say that Q_1 and Q_2 are *isomorphic over* S.

Theorem 5.1 *Let S be a commutative semigroup and an order in semigroups Q_1 and Q_2. The following are equivalent:*
(i) there is an onto S-homomorphism $\theta : Q_1 \to Q_2$;
(ii) for all $a, b \in S$

$$a \leq_{\mathcal{H}} b \text{ in } Q_1 \text{ implies } a \leq_{\mathcal{H}} b \text{ in } Q_2;$$

(iii) for all $a, b \in S$

$$a\mathcal{H}b \text{ in } Q_1 \text{ implies } a\mathcal{H}b \text{ in } Q_2.$$

Moreover, if the above conditions hold then θ is uniquely defined.

Proof The first two implications are immediate; we prove (iii) implies (i). To clarify the notation we denote the inverse of $a \in S$ in a subgroup of Q_2 by $a^\sharp$, using the usual notation a^* for the inverse of a in a subgroup of Q_1. Since S is an order in both Q_1 and Q_2, a^* exists if and only if $a \in \mathcal{S}(S)$ if and only if $a^\sharp$ exists.

Suppose that (iii) holds. If $b \in S$ and $a \in \mathcal{S}(S)$, then $b \leq_{\mathcal{H}} a$ in Q_1 implies $b\mathcal{H}ba$ in Q_1 so that by hypothesis $b\mathcal{H}ba$ and $b \leq_{\mathcal{H}} a$ in Q_2. Define $\theta : Q_1 \to Q_2$ by $(a^*b)\theta = a^\sharp b$ where $b \leq_{\mathcal{H}} a$ in Q_1. It is straightforward to show that θ is an S-homomorphism from Q_1 onto Q_2 so that (i) holds.

Now if ϕ is any S-homomorphism from Q_1 to Q_2 then for all $a^*b \in Q_1$, $(a^*b)\phi = (a\phi)^\sharp b\phi = a^\sharp b = (a^*b)\theta$.

We can now determine immediately when two semigroups of quotients of S are isomorphic over S.

Corollary 5.2 *Let S be a commutative semigroup and an order in Q_1 and Q_2. Then Q_1 is isomorphic to Q_2 over S if and only if $\mathcal{H}^{Q_1} \cap (S \times S) = \mathcal{H}^{Q_2} \cap (S \times S)$.*

Let S be a commutative order. A semigroup of quotients Q of S is *maximum* if, given any semigroup of quotients P of S, there is an S-homomorphism from Q onto P. Dually, Q is *minimum* if, given any semigroup of quotients P of S, there is an S-homomorphism from P onto Q. Clearly, a maximum (minimum) semigroup of quotients of S is unique up to isomorphism over S. In view of Corollary 4.4 and the definition of $\mathcal{H}^*$, if $S = \mathcal{S}(S)$ then a minimum semigroup of quotients of S exists. Using Theorem 3.1 of [4] we can improve this result considerably.

At this point we note that if S is a commutative order, then since each semigroup of quotients of S is a homomorphic image of the free semigroup on $T \cup S$ where $T \cap S = \emptyset$ and T is in one-one correspondence with $\mathcal{S}(S)$, the isomorphism classes (over S) of the semigroups of quotients of S form a *set*.

Proposition 5.3 *Let S be a commutative order for which $\mathcal{S}(S) = S$ and let $\mathcal{Q}$ be the set of semigroups of quotients of S. Define a relation $\preceq$ on $\mathcal{Q}$ by the rule that $Q \preceq P$ if there is an S-homomorphism from P onto Q. Then $\mathcal{Q}$ is a complete lattice under $\preceq$. In particular, S has maximum and minumum semigroups of quotients.*

Proof It is implicit in the proof of Theorem 3.1 of [4] that a commutative semigroup T is an order in a semilattice Y of (commutative) groups $G_\alpha, \alpha \in Y$, if and only if T is a semilattice Y of (commutative) cancellative semigroups T_α where T_α is an order in $G_\alpha, \alpha \in Y$. This result is also obtained as a corollary of Theorem 4.3 in the next section of this paper.

If S is an order in Q then, using the facts that $S = \mathcal{S}(S)$ and S has nonempty intersection with every $\mathcal{H}$-class of Q, it follows from Theorem 3.1 that

Q is a commutative regular semigroup, hence a semilattice of commutative groups.

By Corollary 4.4, $\mathcal{H}^*$ is a semilattice congruence on S, all of whose classes are cancellative. Now if S is an order in Q then as Q is a semilattice of groups, $\mathcal{H}^Q \cap (S \times S)$ is a semilattice congruence contained in $\mathcal{H}^*$. Moreover, if ρ is any semilattice congruence contained in $\mathcal{H}^*$, then each ρ-class is a cancellative semigroup and S is a semilattice of its ρ-classes. Thus S is an order in Q_ρ, where $\mathcal{H}^{Q_\rho} \cap (S \times S) = \rho$. It follows from Corollary 5.2 that there is a bijection between $\mathcal{C}$ and $\mathcal{Q}$, where $\mathcal{C}$ is the set of semilattice congruences on S contained in $\mathcal{H}^*$. Further, if $\rho, \mu \in \mathcal{C}$ then by Theorem 5.1, $\rho \subseteq \mu$ if and only if there is an S-homomorphism from Q_ρ onto Q_μ, that is, $Q_\mu \preceq Q_\rho$. Since $\mathcal{C}$ is a complete lattice, so also is $\mathcal{Q}$.

If we relax the condition on a commutative order S that $S = \mathcal{S}(S)$ then it is not always the case that S has a maximum or minimum semigroup of quotients as we show in the final section. We also give examples of commutative orders S where $S \neq \mathcal{S}(S)$ such that the semigroups of quotients of S form a complete lattice.

We finish this section with a necessary and sufficient condition for a commutative order to have a maximum semigroup of quotients. The idea for the construction was suggested to us by P.N. Anh.

Proposition 5.4 *Let S be a commutative order. Then S has a maximum semigroup of quotients if and only if for every $s \in S$ there exists $a \in \mathcal{S}(S)$ with $s \leq_{\mathcal{H}} a$ in* every *semigroup of quotients of S.*

Proof Suppose that S has a maximum semigroup of quotients P. If $s \in S$ then $s \leq_{\mathcal{H}} a$ in P for some $a \in \mathcal{S}(S)$; now if S is an order in Q then there is an S-homomorphism $\theta : P \to Q$ so that $s\theta \leq_{\mathcal{H}} a\theta$ in Q, that is, $s \leq_{\mathcal{H}} a$ in Q.

Conversely, suppose that for each $s \in S$ there exists $a \in \mathcal{S}(S)$ such that $s \leq_{\mathcal{H}} a$ in every semigroup of quotients of S. Let $\{Q_i : i \in I\}$ be the set of distinct semigroups of quotients of S and let $a^{*(i)}$ denote the inverse of $a \in \mathcal{S}(S)$ in a subgroup of Q_i.

Put $P = \Pi\{Q_i : i \in I\}$ and define $\phi : S \to P$ by $s\phi = (s)$, so that ϕ is an embedding of S into P. If $a \in \mathcal{S}(S)$ then $a^{*(i)}$ exists for all $i \in I$. Clearly (a) lies in a subgroup of P with inverse $(a^{*(i)})$. Let

$$Q = \langle (a)^*, (b) : a \in \mathcal{S}(S), b \in S \rangle,$$

so that ϕ is an embedding of S into Q. It is easy to see that

$$Q = \{(a)^*(b) : a \in \mathcal{S}(S), b \in S\}$$

and it follows that S is an order in Q. Now if $\pi_i : P \to Q_i$ is the ith projection, $\pi_i : Q \to Q_i$ is an onto homomorphism and for any $s \in S$, $s\phi\pi_i = (s)\pi_i = s$, so that π_i is an S-homomorphism. Thus Q is the maximum semigroup of quotients of S.

Corollary 5.5 *Let S be a commutative order. If S is a monoid, or if $S = \mathcal{S}(S)$, then S has a maximum semigroup of quotients.*

6 Semilattice decompositions of commutative orders

A useful tool in studying semigroups is to break them down into smaller, hopefully simpler, constituent parts. One way of doing this is to consider *semilattice decompositions* of a semigroup. This philosophy has proved particularly useful in the study of left orders. For example, the question of whether a semigroup S is a left order in a regular semigroup with central idempotents (a *Clifford semigroup*) can be reduced to the study of semilattice congruences on S and the question of when a semigroup T is a left order in a group. This latter question is answered in [CP, Theorem 1.24]; T is a left order in a group if and only if T is right reversible and cancellative. Now if S is a left order in a Clifford semigroup Q, then as is well known, Q is a semilattice Y of groups $G_\alpha, \alpha \in Y$, and it is not difficult to see that $S_\alpha = S \cap G_\alpha$ is an order in G_α for each $\alpha \in Y$. Thus each S_α is a right reversible cancellative semigroup and S is a semilattice Y of the semigroups $S_\alpha, \alpha \in Y$. On the other hand, given a semigroup T that is a semilattice Z of right reversible cancellative semigroups $T_\gamma, \gamma \in Z$, then T is an order in a Clifford semigroup P, where P is a semilattice Z of the groups of left quotients of the semigroups $T_\gamma, \gamma \in Z$. This is implicit in Theorem 3.1 of [4]. A similar approach is used in [9] to study straight left orders in completely regular semigroups. With this in mind we ask the following question, answered in the proposition below: if a commutative semigroup S is a semilattice Y of (commutative) semigroups $S_\alpha, \alpha \in Y$, and each S_α is an order in Q_α, when is S an order in a semigroup that is the union of the Q_αs?

Proposition 6.1 *Let S be a commutative semigroup. Suppose that S is a semilattice Y of (commutative) semigroups $S_\alpha, \alpha \in Y$, where each S_α is an order in Q_α. Put*

$$Q = \bigcup\{Q_\alpha : \alpha \in Y\}$$

$$\mathcal{H}'_\alpha = \mathcal{H}^{Q_\alpha} \cap (S_\alpha \times S_\alpha),\ (\alpha \in Y)$$

and

$$\mathcal{H}' = \bigcup\{\mathcal{H}'_\alpha : \alpha \in Y\}$$

so that $\mathcal{H}'$ is an equivalence relation on S. Then Q is a semigroup of left quotients of S (under a multiplication extending that of each Q_α) if and only if $\mathcal{H}'$ is a congruence on S and $\mathcal{S}(S) = \bigcup\{\mathcal{S}(S_\alpha) : \alpha \in Y\}$. If these conditions hold then in addition Q is a semilattice Y of the semigroups $Q_\alpha, \alpha \in Y$.

Proof Suppose first that S is an order in Q. It is clear from Lemma 2.2 that Q is a semilattice Y of the semigroups $Q_\alpha, \alpha \in Y$. Now from the fact that for any $\alpha \in Y$ and any $q \in Q_\alpha$ there is an idempotent $e \in Q_\alpha$ with $q = eq$, it is easy to see that if $p, q \in Q$, then $p\mathcal{H}q$ in Q if and only if $p, q \in Q_\alpha$ and $p\mathcal{H}q$ in Q_α, for some $\alpha \in Y$. It then follows that $\mathcal{H}'$ is a congruence on S. Clearly $\mathcal{S}(S) \subseteq \bigcup\{\mathcal{S}(S_\alpha) : \alpha \in Y\}$ and the opposite inclusion also holds, since Q is a union of the quotient semigroups $Q_\alpha, \alpha \in Y$.

Conversely, suppose that $\mathcal{H}'$ is a congruence on S and $\mathcal{S}(S) = \bigcup\{\mathcal{S}(S_\alpha) : \alpha \in Y\}$.

Define a relation $\leq$ on S by the rule that for all $b, c \in S$,

$$b \leq c \text{ if and only if } b\mathcal{H}'cd$$

for some $d \in S$. Note that if $b \in S_\alpha$ and $c \in S_\beta$ then $b \leq c$ implies that $\alpha \leq \beta$.

For any $\alpha \in Y$ we write $\leq_\alpha$ for the relation $\leq_{\mathcal{H}^{Q_\alpha}} \cap (S \times S)$. Then it is straightforward to show that for $b, c \in S_\alpha$,

$$b \leq c \text{ if and only if } b \leq_\alpha c.$$

It is now easy to check that $\leq$ is a preorder compatible with multiplication, with associated equivalence relation $\mathcal{H}'$. Conditions (A) and (C$'$) are immediately verified. To show that (B) holds, let $b, c \in S$ and $a \in \mathcal{S}(S)$ with $b \leq a, c \leq a$ and $ab = ac$. It follows that $b, c \in S_\alpha$ and $a \in S_\beta$ for some

$\alpha, \beta \in Y$ with $\alpha \leq \beta$. Choose $x \in \mathcal{S}(S_\alpha)$ with $b \leq_\alpha x$, so that $b \leq x$ and $x \in \mathcal{S}(S)$. Certainly every element of $\mathcal{S}(S)$ is $\mathcal{H}'$-related to its square, so if $b\mathcal{H}'au, u \in S$, then $ab\mathcal{H}'a^2u\mathcal{H}'au\mathcal{H}'b$ and $b \leq ab$. Further, $b\mathcal{H}'ab = ac\mathcal{H}'c$, and $b \leq x$ gives $xb\mathcal{H}'b$. Now $xa \in \mathcal{S}(S_\alpha)$ and $xab\mathcal{H}'xb\mathcal{H}'b$, so that $xab = xac$ and $c\mathcal{H}'_\alpha b \leq_\alpha xa$ yield $b = c$ as required.

By Theorem 4.3, S is an order in a semigroup P where $\leq_\mathcal{H} \cap (S \times S) = \leq$. It is routine to check that P is a semilattice of semigroups $P_\alpha = \{a^*b : a, b \in S_\alpha\}, \alpha \in Y$ and S_α is an order in $P_\alpha, \alpha \in Y$. As remarked in the first part of the proof, if $p, q \in P$ then $p\mathcal{H}q$ in P if and only if $p, q \in P_\alpha$ and $p\mathcal{H}q$ in P_α, for some $\alpha \in Y$. Now if $u, v \in S_\alpha$ then $u\mathcal{H}v$ in Q_α if and only if $u\mathcal{H}'v$; but this is equivalent to $u\mathcal{H}v$ in P. Thus $u\mathcal{H}v$ in Q_α if and only if $u\mathcal{H}v$ in P_α. By Corollary 5.2, P_α is isomorphic over S_α to Q_α, and the proposition follows.

Corollary 6.2 *[4] Let S be a commutative semigroup and suppose that S is a semilattice Y of cancellative semigroups $S_\alpha, \alpha \in Y$. Then S is an order in a commutative regular semigroup.*

Proof In fact we show the stronger statement that S is an order in Q, where Q is a semilattice Y of the groups of quotients G_α of $S_\alpha, \alpha \in Y$.

Let Q and $\mathcal{H}'$ be defined as in Proposition 6.1, where Q_α is replaced by $G_\alpha, \alpha \in Y$. Since $\mathcal{H}$ is the universal congruence on a group,

$$\mathcal{H}' = \bigcup\{S_\alpha \times S_\alpha : \alpha \in Y\},$$

so that $\mathcal{H}'$ is the *congruence* associated with the semilattice decomposition of S. Clearly $S = \bigcup\{\mathcal{S}(S_\alpha) : \alpha \in Y\}$. Let $a, x, y \in S$ where $a \in S_\alpha, x \in S_\beta$ and $y \in S_\gamma$ and suppose first that $xa^2 = ya^2$. Then $\alpha\beta = \alpha\gamma$ and $xa, ya \in S_{\alpha\beta}$. Now cancelling xa in $(xa)(ax) = (ya)(ax)$ we have $xa = ya$. On the other hand if $xa^2 = a^2$ then $xa \in S_\alpha$ and cancelling a gives $xa = a$. Thus $a \in \mathcal{S}(S)$ and so $S = \mathcal{S}(S) = \bigcup\{\mathcal{S}(S_\alpha) : \alpha \in Y\}$.

¿From Proposition 6.1, S is an order in Q and Q is a semilattice Y of the groups $G_\alpha, \alpha \in Y$.

We note that in the above corollary the hypotheses may be weakened slightly: if S is a semilattice of commutative cancellative semigroups then it is not difficult to see that S itself is commutative. The converse of the corollary is also true, as shown in [4].

We also comment that if a commutative semigroup S is an order in Q and Q is a semilattice Y of semigroups $Q_\alpha, \alpha \in Y$, then it is not always the case that $S_\alpha = S \cap Q_\alpha$ is an order in Q_α for each $\alpha \in Y$. This is illustrated by Example 7.1, where X^G is a chain of the group G and the null semigroup X. Now $X^T \cap X = X$, but as 0 is the only element of X lying in a subgroup, certainly X is not an order in itself.

7 Examples

A left order S in Q is *straight* if any $q \in Q$ can be expressed as $q = a^*b$ where $a, b \in S$ and $a\mathcal{R}b$ in Q. As commented at the beginning of Section 4, a description of straight left orders is known [5]. It is easy to see that a commutative order S is straight (in any semigroup of quotients) if and only if $S = \mathcal{S}(S)$. With this in mind we comment that each of the orders presented in this section possesses elements that are not square-cancellable, and thus cannot be straight.

We show in Corollary 4.4 that if a commutative order S has the property that $\mathcal{S}(S) = S$, then S is an order if and only if the preorder $\leq_{\mathcal{H}^*}$ satisfies conditions (A), (B) and (C$'$). Our first example shows that if $S \neq \mathcal{S}(S)$ then S can be an order without $\leq_{\mathcal{H}^*}$ satisfying condition (C$'$). In spite of the fact that $S \neq \mathcal{S}(S)$, S still has the property that its semigroups of quotients form a complete lattice.

If S is any semigroup with zero then we denote by S^* the set of *non-zero* elements of S.

Example 7.1 .

Let R be any commutative semigroup and let X be a null semigroup disjoint from R. Then $R \cup X$ is a commutative semigroup under a multiplication extending that in R and X where

$$rx = x = xr$$

for all $r \in R$ and $x \in X$. We write this semigroup as X^R.

Let T be a commutative cancellative semigroup and let X be a null semigroup disjoint from T with $|X| \geq 3$. We consider the semigroup X^T.

It is easy to see that $\mathcal{S}(X^T) = \{0\} \cup T$ and if T is an order in Q then X^T is an order in X^Q. Conversely, suppose that X^T is an order in P. Using the notation of Lemma 2.2, $\mathcal{S}(X^T)$ is an order in $\mathcal{H}(P)$. Write $\mathcal{H}(P) = \{0\} \cup Q$, so that $\{0\} \cup T$ is an order in $\{0\} \cup Q$. Let $p, q \in Q$, so that $p\mathcal{H}a$ and $q\mathcal{H}b$ for some $a, b \in T$. Then $pq\mathcal{H}ab$ so that $pq \neq 0$ and Q is a subsemigroup, indeed a semigroup of quotients of T. If $a \in T$ and $x \in X$ then from $ax = x$ we have $x \leq_{\mathcal{H}} a$ in P, so that $x = a^*ax = a^*x$. It follows that $P = X^Q$.

Let $\mathcal{Q} = \{P : X^T \text{ is an order in } P\}$ so that from the above comments,

$$\mathcal{Q} = \{X^Q : T \text{ is an order in } Q\}.$$

Define a relation $\preceq$ on $\mathcal{Q}$ by $X^Q \preceq X^{Q'}$ if there is an X^T-homomorphism from Q' onto Q. At this point we note that if $x, y \in X^*$ and $x\mathcal{H}y$ in a semigroup of quotients X^Q of X^T, then from $x = ya^*b$ for some $a, b \in X^T$ it follows that $x = y$. With this in mind it is easy to see from Theorem 5.1 that $X^Q \preceq X^{Q'}$ if and only if there is a T-homomorphism from Q' onto Q. Proposition 5.3 now gives that $\mathcal{Q}$ is a complete lattice under $\preceq$. In particular, X^T has maximum and minimum semigroups of quotients.

T

X^*

0

Fig. 7.1

Fig. 7.1 is the Hasse diagram of the preorder $\leq_{\mathcal{H}^*}$ on X^T. Conditions (A), (B) and (C) hold for $\leq_{\mathcal{H}^*}$, but (C$'$) fails. For if b, c are distinct elements of X^* then $b \leq_{\mathcal{H}^*} c$, but $bx \neq cy$ for any $x \in \mathcal{S}(X^T), y \in X^T$ with $b \leq_{\mathcal{H}^*} x$.

It has been independently conjectured by P.N. Anh that a commutative semigroup is an order if and only if $\leq_{\mathcal{H}^*}$ satisfies conditions (A), (B) and (C). Our next example is a counterexample to this conjecture.

Example 7.2 .

Let R be a commutative semigroup and let X be a null semigroup disjoint from R. Then $R \cup X$ is a commutative semigroup under a multiplication extending that in R and X where

$$rx = 0 = xr$$

for all $r \in R$ and $x \in X$. We write this semigroup as R_X.

Let T be a commutative cancellative semigroup *with no idempotent* and let X be a null semigroup disjoint from T with $|X| \geq 3$. We consider the semigroup T^1_X.

It is straightforward to verify that $\mathcal{S}(T^1_X) = \{1\} \cup T \cup \{0\}$ and that if T is an order in Q then T^1_X is an order in Q^1_X. Conversely, suppose that T^1_X is an order in P. Write $\mathcal{H}(P)$ as $\mathcal{H}(P) = \{1\} \cup Q \cup \{0\}$; as in the previous example, $\{1\} \cup T$ is an order in $\{1\} \cup Q$. Suppose that $p \in \{1\} \cup Q$ and $p\mathcal{H}1$ in P. Then $p = a^*b$ where $a, b \in \{1\} \cup T$ and $b\mathcal{H}p$ in P. Let $x \in X^*$; then $bx\mathcal{H}1x = x$ so that $bx \neq 0$ and $b = 1$. Now as $p\mathcal{H}a$ we must also have that $ax \neq 0$ and so $a = 1$, giving also $p = 1$. In particular, if $p, q \in \mathcal{H}(P)$ and $pq = 1$, then $p = q = 1$. It follows that Q is a subsemigroup, T is an order in Q and further, $P = Q^1_X$. A similar argument to that in Example 7.1 shows that the set of semigroups of quotients of T^1_X is a complete lattice under $\preceq$, where $P \preceq P'$ if and only if there is a T^1_X-homomorphism from P' onto P.

Condition (B) fails for the relation $\leq_{\mathcal{H}^*}$ on T^1_X: if $a \in T$ and $b \in X^*$ then $b \leq_{\mathcal{H}^*} a$ but $ab = 0$ so $b \not\leq_{\mathcal{H}^*} ab$. Moreover, if b, c are distinct elements of X^* then $ab = ac = 0$.

We now present examples of orders which do not possess a lattice of semigroups of quotients. The first is an order which has a maximum semigroup of quotients but not a minumum, the second is an order which has a minimum semigroup of quotients but not a maximum. The following lemma is useful in verifying that the multiplication we define in our examples is associative.

Lemma 7.3 *Let X be a null semigroup and let T be a commutative semigroup disjoint from X which acts on X such that $0 \cdot t = 0$ for all $t \in T$. Then $S = X \cup T$ is a commutative semigroup under a multiplication extending that in X and T by*

$$tx = xt = x \cdot t$$

for all $t \in T$ and $x \in X$.

Example 7.4 .

Let $T = \{a_i : i \in \mathbf{Z}^+\}$ be the infinite monogenic semigroup generated by a and let

$$X = \{x, y, 0\} \cup \{z_i : i \in \mathbf{Z}^+\}$$

be a null semigroup disjoint from T. Define an action of T on X by

$$xa^i = ya^i = z_i, z_j a^i = z_{j+i}, 0a^i = 0$$

for all $i, j \in \mathbf{Z}^+$. Let $S = X \cup T$ be the semigroup with multiplication induced by this action; we consider the semigroup S^1. It is straightforward to check that $\mathcal{S}(S^1) = T \cup \{1, 0\}$.

Let $\leq$ be the preorder on S^1 given by the Hasse diagram

$$\{1\}$$

$$T \qquad\qquad\qquad\qquad \{y\}$$

$$\{x\} \cup \{z_i : i \in \mathbf{Z}^+\}$$

$$\{0\}$$

It is routine to show that $\leq$ is compatible with multiplication and satisfies conditions (A), (B) and (C$'$). By Theorem 4.3, S^1 is an order in a commutative semigroup Q where $\leq_{\mathcal{H}} \cap (S \times S) = \leq$. Thus in $Q, x \leq_{\mathcal{H}} a$.

Dually, S^1 is an order in a commutative semigroup Q' where in Q' we have $y \leq_{\mathcal{H}} a$. If S^1 had a minumum semigroup of quotients P, then Theorem 5.1 says that in P we have both $x \leq_{\mathcal{H}} a$ and $y \leq_{\mathcal{H}} a$. Calculating in P gives

$$x = a^* ax = a^* ay = y,$$

a contradiction. Thus no such P exists. However, from Corollary 5.5, S^1 has a maximum semigroup of quotients.

Example 7.5 .

Let $T = \{a^i b^j : i, j \in \mathbf{N}, i + j \geq 1\}$ be a subsemigroup of the free abelian group generated by a and b, and let

$$X = \{(i, j) : i, j \in \mathbf{N}\} \cup \{0\}$$

be a null semigroup disjoint from T. Define an action of T on X by

$$(i, j)a^k b^\ell = (i + k, j + \ell), 0a^k b^\ell = 0$$

for $(i, j) \in X$ and $a^k b^\ell \in T$. Let $S = X \cup T$ be the semigroup with multiplication induced by this action.

It is easy to see that $\mathcal{S}(S) = \{0\} \cup T$. Let $\leq$ be the preorder on S given by the Hasse diagram

$$\begin{array}{ccc} & \langle a \rangle & \\ T \setminus \langle a \rangle & & X_H \\ & X_L & \\ & \{0\} & \end{array}$$

where $\langle a \rangle = \{a^i : i \geq 1\}, X_H = \{(i, 0) : i \in \mathbf{N}\}$ and $X_L = \{(i, j) : i, j \in \mathbf{N}, j \geq 1\} = X^* \setminus X_H$. Then $\leq$ is compatible with multiplication and satisfies conditions (A), (B) and (C$'$). By Theorem 4.3, S is an order in a commutative semigroup Q such that $\leq_{\mathcal{H}} \cap (S \times S) = \leq$. If $u \in \mathcal{S}(S)$ then in Q,

$$(0, 0) \leq_{\mathcal{H}} u \text{ if and only if } u \in \langle a \rangle.$$

Dually, S is a left order in Q' where for $u \in \mathcal{S}(S)$,

$$(0, 0) \leq_{\mathcal{H}} u \text{ if and only if } u \in \langle b \rangle.$$

Suppose that S is an order in P where there exist S-homomorphisms from P onto Q and from P onto Q'. In P we must have $(0,0)\leq_{\mathcal{H}}u$ for some $u \in \mathcal{S}(S)$. By Theorem 5.1, $(0,0)\leq_{\mathcal{H}}u$ in Q and $(0,0)\leq_{\mathcal{H}}u$ in Q'. Thus $u \in \langle a\rangle \cap \langle b\rangle = \emptyset$, a contradiction. Thus no such P exists and S has no maximum semigroup of quotients.

The preorder $\leq_{\mathcal{H}^*}$ on S has Hasse diagram

$$T$$

$$X^*$$

$$\{0\}$$

Certainly $\leq_{\mathcal{H}^*}$ is compatible with multiplication, and it also satisfies conditions (A), (B) and (C$'$). Thus S is an order in a commutative semigroup R, where $\leq_{\mathcal{H}} \cap (S \times S) = \leq_{\mathcal{H}^*}$. By the nature of $\leq_{\mathcal{H}^*}$, R is the minimum semigroup of quotients of S.

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