CONTRIBUTIONS TO GENERAL ALGEBRA 3
Proceedings of the Vienna Conference, June 21–24, 1984
Verlag Hölder–Pichler–Tempsky, Wien 1985 – Verlag B. G. Teubner, Stuttgart

John Fountain and Victoria Gould

COMPLETELY O-SIMPLE SEMIGROUPS OF QUOTIENTS II

A new concept of completely 0-simple semigroup of quotients has been introduced by Mario Petrich and the first author. The main purpose of this paper is to give several new characterisations of those semigroups which have such a semigroup of quotients.

1. Introduction.

Many definitions of semigroups of quotients have been proposed and studied. In a recent paper [2] Petrich and the first author obtain a new concept of completely 0-simple semigroup of quotients by examining the relationship between simple Artinian rings and completely 0-simple semigroups. This definition has since been extended to the class of all semigroups, giving a definition of semigroup of quotients which may be regarded as an analogue of the classical ring of quotients.

A famous theorem of Goldie [3] classifies those rings which have a simple Artinian ring of quotients. Correspondingly, the main result of [2], quoted as Theorem 1 below, characterises those semigroups which have a completely 0-simple semigroup of quotients. The conditions of Theorem 1, however, are more akin to those of Ore's theorem than to the ones occurring in Goldie's theorem. In this note we give several alternative characterisations of semigroups having a completely 0-simple semigroup of quotients which are, perhaps, closer in spirit to Goldie's result for rings.

We start in Section 2 by giving the basic definitions and quoting the main theorem of [2]. We then use another result from [2] to give a minor variation of the main theorem. We introduce and demonstrate the significance of uniform one-sided ideals in Section 3. Annihilator conditions are the concern of Section 4 and we conclude the paper in Section 5 by considering the case of completely 0-simple semigroups which satisfy the descending chain conditions for right and for left ideals.

2. The basic ideas and a minor variation.

We begin by summarising the relevant definitions and giving the main result of [2]. We refer the reader to [2] for further details.

Let S be a semigroup. An element a of S is square-cancellable if for all x,y in S^1,

$$a^2x = a^2y \Rightarrow ax = ay$$

and $$xa^2 = ya^2 \Rightarrow xa = ya.$$

This is equivalent to saying that $a\mathcal{H}^*a^2$ where $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$ and two elements of S are $\mathcal{L}^*$-related [$\mathcal{R}^*$-related] if and only if they are $\mathcal{L}$-related [$\mathcal{R}$-related] in some oversemigroup of S. If S is a subsemigroup of a completely 0-simple semigroup, then it is easy to see that an element of S is square-cancellable if and only if a = 0 or $a^2 \neq 0$.

We make the convention that if a is an element of a semigroup S, then whenever we write a^{-1} we implicitly assert that a is in a group $\mathcal{H}$-class of some oversemigroup of S and that a^{-1} is the inverse of a in this subgroup. Observe that a necessary condition for a to be in a subgroup of an oversemigroup is that a be square-cancellable.

We can now give the basic definition. A semigroup Q is a semigroup of quotients of its subsemigroup S if

(i) every square-cancellable element of S lies in a subgroup of Q,

(ii) every element q of Q may be written as $q = a^{-1}b = cd^{-1}$ for some elements a,b,c,d of S.

Before we can state the main theorem of [2] we must give some more definitions. Let S be a semigroup with zero. We recall from [1] that 0 is a prime ideal of S if $aSb \neq 0$ for any non-zero elements a,b of S. Also, S is categorical at 0 if for elements a,b,c of S, abc is non-zero whenever both ab and bc are non-zero.

Following [5] we define the relations ρ,λ on S thus:-

$$a\rho b \Leftrightarrow a = b \text{ or } aS \cap bS \neq 0,$$

$$a\lambda b \Leftrightarrow a = b \text{ or } Sa \cap Sb \neq 0.$$

We can now quote from [2],

THEOREM 1. A semigroup S has a completely 0-simple semigroup of quotients if and only if it satisfies conditions (A),(B),(C),(D) listed below and the duals (A'),(B') of (A) and (B).

For elements a,b,c,u,v of S,

(A) *if $a\rho b$, $au \neq 0$ and $bv \neq 0$, then $au\rho bv$,*

(B) *if $a\rho b$ and $ca = cb \neq 0$, then $a = b$,*

(C) *S is categorical at 0,*

(D) *0 is a prime ideal of S.*

In order to get our first alternative characterisations we need the following result from [2] which describes the restrictions of Green's relations $\mathcal{L}$ and $\mathcal{R}$ on a completely 0-simple semigroup Q to a subsemigroup S which has Q as a semigroup of quotients.

THEOREM 2. *Let Q be a completely 0-simple semigroup of quotients of a semigroup S. Then*

(i) $\rho = \mathcal{R} \cap (S \times S)$,

(ii) $\lambda = \mathcal{L} \cap (S \times S)$.

We can now give our first result.

THEOREM 3. *Let S be a semigroup in which 0 is a prime ideal and suppose that S satisfies condition (B) of Theorem 1 and its dual. Then the following are equivalent:*

(i) *S has a completely 0-simple semigroup of quotients,*

(ii) *λ is a right congruence and ρ is a left congruence,*

(iii) *λ is a right congruence and ρ is an equivalence.*

Proof. Since it is well-known that $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence, it is clear from Theorem 2 that (ii) follows from (i). Obviously (ii) implies (iii). That (i) is a consequence of (iii) follows from results in [2] and [5] but we give a short direct proof. To see that condition (A) holds, let a,b,u,v be elements of S with $a\rho b$, $au \neq 0$ and $bv \neq 0$. Since 0 is a prime ideal of S, there are elements x,y in S such that $aux \neq 0$ and $bvy \neq 0$. Hence $aS \cap auS \neq 0$ and $bS \cap bvS \neq 0$ so that $a\rho au$ and $b\rho bv$. Since ρ is an equivalence we get $au\rho bv$. In a similar manner we see that (A') holds.

Now suppose that a,b,c are elements of S such that ab and bc are non-zero. Since 0 is a prime ideal, $b\lambda ab$ and as λ is a right congruence, $bc\lambda abc$. As bc is non-zero, it follows from the definition of λ that abc is non-zero. Thus S is categorical at 0.

We can now apply Theorem 1 to get that (i) holds.

3. Uniform right ideals.

In this section we examine the way in which the concept of uniform one-sided ideal fits into our results and make some comparisons with ring theory.

Let S be a semigroup with zero. A right [left] ideal $\mathcal{U}$ of S is uniform if $P \cap Q \neq 0$ for any non-zero right [left] ideals, P,Q of S contained in $\mathcal{U}$.

We remark that one has the same definition in ring theory.

Clearly, if $\mathcal{U}$, V are right [left] ideals of S with $\mathcal{U} \subseteq V$ and V uniform, then $\mathcal{U}$ is also uniform.

We extend the concept of 0-direct union to one-sided ideals so that we will write

S is a 0-direct union of right [left] ideals if S is a union of right [left] ideals and any two of these right [left] ideals intersect in 0.

A right [left] ideal of S is indecomposable if it cannot be written as a 0-direct union of two non-zero right [left] ideals of S.

It is clear that uniform one-sided ideals are indecomposable but the converse is not always true.

A subsemigroup R of S is right 0-cancellative if whenever $ba = ca \neq 0$ for elements a,b,c of R, then $b = c$. We define left 0-cancellative dually.

Our next result shows that in Theorem 1, conditions (A),(B) and their duals may be replaced by conditions on the ideal structure of S.

THEOREM 4. For a semigroup S which is categorical at 0 and in which 0 is a prime ideal, the following conditions are equivalent:

(i) S has a completely 0-simple semigroup of quotients,

(ii) S is both a 0-direct union of uniform right ideals and of uniform left ideals where the right [left] ideals involved are left [right] 0-cancellative,

(iii) indecomposable one-sided ideals of S are uniform and uniform right [left] ideals are left [right] 0-cancellative.

Proof. (i) => (ii). Let Q be a completely 0-simple semigroup of quotients of S. We may assume that Q is a Rees matrix semigroup $\mathcal{M}^0(G;I,\Lambda;P)$ over a group G with index sets I,Λ and sandwich matrix P. It is well-known that for elements (i,g,λ), (j,g,ν) of Q

$$(i,g,\lambda) \,\mathcal{R}\, (j,h,\nu) \iff i = j.$$

We denote by R_i the $\mathcal{R}$-class $\{(i,g,\lambda); g \in G, \lambda \in \Lambda\}$ of Q and put $U_i = \{0\} \cup (R_i \cap S)$. It is easy to see that each U_i is a right ideal of S and that S is the 0-direct union of the U_i. If a,b are non-zero elements of U_i, then $a \mathcal{R} b$ in Q so that by Theorem 2, $a\rho b$ in S, that is, $aS \cap bS \neq 0$. It follows that U_i is uniform. From condition (B) of Theorem 1 we have that U_i is left 0-cancellative.

A dual argument gives that S is also a 0-direct union of right 0-cancellative uniform left ideals.

(ii)=> (iii). Let S be the 0-direct union of uniform right ideals U_i, $i \in I$ and let J be an indecomposable right ideal of S. Then J is the 0-direct union of the right ideals $U_i \cap T$, $i \in I$ and hence $J \subseteq U_j$ for some $j \in I$. Thus J is uniform.

Since uniform right ideals are indecomposable, it follows that any uniform right ideal U of S is contained in a left 0-cancellative uniform right ideal. Thus U is also left 0-cancellative.

The dual condition follows similarly.

(iii) => (i). It is well-known that a semigroup can be written as a 0-direct union of indecomposable right ideals. Hence we have that S is the 0-direct union of left 0-cancellative uniform right ideals U_i, $i \in I$ for some index set I.

We show that $K_i = U_i \setminus \{0\}$ is a ρ-class. If a,b are elements of K_i, then the uniformity of U_i gives $aS \cap bS \neq 0$ so that $a\rho b$. If $a \in K_i$, $b \in S$ and $a\rho b$, then $aS \cap bS \neq 0$. Hence $bS^1 \cap U_i \neq 0$ and as principal right ideals are clearly indecomposable, it follows that $b \in U_i$. Hence $b \in K_i$. Since U_i is a right ideal, it is now clear that condition (A) of Theorem 1 holds.

Now suppose that a,b,c are elements of S such that $a\rho b$ and $ca = cb \neq 0$. We have that a,b are both in U_i for some i. Now 0 is a prime ideal of S and so there is an element d of S such that $adc \neq 0$. By categoricity at 0, $adca = adcb \neq 0$. Now adc is an element of U_i and U_i is left 0-cancellative, so $a = b$. Thus condition (B) of Theorem 1 holds.

Similarly, S satisfies conditions (A') and (B').

Thus by Theorem 1, S has a completely 0-simple semigroup of quotients.

At this point it might be worth comparing Theorem 4 with Goldie's Theorem which states that a ring R has a simple Artinian (classical) ring of quotients if and only if R is prime, satisfies the ascending chain condition on right and left annihilator ideals and has no infinite direct sum of one-sided ideals. In Theorem 4 we have that 0 is a prime ideal of the semigroup S which corresponds to the ring being prime. The condition that R has no infinite direct sum of one-sided ideals is equivalent to saying that R has an intersection large right [left] ideal which is a direct sum of a finite number of uniform right [left] ideals. The ring itself, however, need not be a direct sum of uniform right [left] ideals in contrast to the semigroup case where we have that the semigroup is both a 0-direct union of uniform right ideals and of uniform left ideals. Another apparent difference is that in the semigroup theorem we ask that uniform right [left] ideals be left [right] 0-cancellative. But as we now show, this is automatically true for uniform one-sided ideals in the ring case. We make use of Lemma 3.3 of [4] which tells us that if R is a ring with a simple Artinian ring of quotients and if U is a uniform right ideal of R, then for an element a of R, either $aU = 0$ or $au \neq 0$ for all elements u of U. Now, if $a,b,c \in U$ and $ab = ac \neq 0$, then $aU \neq 0$. But $b-c \in U$ and $a(b-c) = 0$ so that $b-c = 0$, that is, $b = c$.

4. Annihilator conditions.

In this section we consider two kinds of annihilators in semigroups with zero and consider conditions on them which lead to further characterisations of semigroups with completely 0-simple semigroups of quotients. Essentially, these conditions allow us to omit mention of one or both of categoricity at zero and left [right] 0-cancellation for uniform right [left] ideals.

Let S be a semigroup with zero. A right ideal I of S is a complement right ideal if there is a right ideal J such that $S = I \cup J$ and $I \cap J = 0$.

For subsets X,I of S we define the right annihilator of T in X to be

$$r_X(T) = \{x \in X : tx = 0 \text{ for all } t \in T\}.$$

Obviously, if X is a right ideal, then so is $r_X(T)$. A right annihilator ideal is a right ideal of the form $r_S(T)$ for some subset T of S.

Another kind of "annihilator" is the right equalizer of T in X defined by

$$\rho_X(T) = \{(x,y) \in X \times X : tx = ty \text{ for all } t \in T\} .$$

Clearly $\rho_X(T)$ is an equivalence relation and if X is a right ideal, then $\rho_X(T)$ is a right congruence.

When $T = \{a\}$, we write simply $r_X(a)$ and $\rho_X(a)$.

A right ideal I of S is <u>intersection large</u> if I has non-zero intersection with every non-zero right ideal of S.

For $m = 1,2,3,4$ we say that S satisfies Ann_m if it satisfies both condition (m) and its dual where (1),(2),(3) and (4) are given below:

(1) for any element a of S,

$$\rho_S(a) \cap \rho = \iota \vee \bigcup_{\gamma \in \Gamma} (U_\gamma \times U_\gamma)$$

for some set $\{U_\gamma : \gamma \in \Gamma\}$ of complement uniform right ideals,

(2) for any uniform right ideal U of S and any element a of S,

$$\rho_U(a) = \iota \iff r_U(a) = 0,$$

(3) right annihilator ideals of S are complement right ideals,

(4) for a non-zero element a of S, $r_S(a)$ is not intersection large.

THEOREM 5. <u>Let S be a semigroup with zero in which 0 is a prime ideal. Suppose also that indecomposable one-sided ideals of S are uniform. Then the following are equivalent</u>:

(i) <u>S has a completely 0-simple semigroup of quotients</u>,

(ii) <u>S is categorical at 0 and satisfies Ann_1</u>,

(iii) <u>S is categorical at 0 and satisfies Ann_2</u>,

(iv) <u>S satisfies Ann_2 and Ann_3</u>,

(v) <u>S satisfies Ann_2 and Ann_4</u>,

<u>Proof</u>. <u>(i) => (ii)</u>. We may assume that the semigroup of quotients of S is a Rees matrix semigroup $\mathcal{M}^0(G;I,\Lambda;P)$ over a group G. Note that the complement uniform right ideals of S are precisely the right ideals

$$U_j = \{(j,g,\nu) : g \in G, \nu \in \Lambda\} \cap S$$

where $j \in I$. Furthermore, for elements b,c of S, $b\rho c$ if and only if b,c are both in U_j for some $j \in I$. Let $a = (i,g,\mu)$ be an element of S and let $\Gamma = \{j \in I : p_{\mu j} = 0\}$. Then we have

$$ab = ac \text{ and } b,c \in U_j \text{ for some } j \in I \text{ if and only if}$$

$$b = c \text{ or for some } j \in I,\ b,c \in U_j \text{ and } p_{\mu j} = 0$$

so that $(b,c) \in \rho_S(a) \cap \rho$ if and only if $b = c$ or $(b,c) \in \bigcup_{\gamma\in\Gamma}(U_\gamma \times U_\gamma)$. Together with its dual this shows that S satisfies Ann_1. By Theorem 4, S is categorical at 0, and so (ii) holds.

<u>(ii) => (iii)</u>. We must show that S satisfies Ann_2. Let U be a uniform right ideal of S and a be an element of S. Suppose that $r_U(a) = 0$ and that $ab = ac$ for some b,c in U. If $ab = 0$, then $r_U(a) = 0$ gives $b = c = 0$. If $ab \neq 0$, then using the fact that 0 is a prime ideal of S, we have that bS, cS are non-zero. As U is uniform, $bS \cap cS \neq 0$, that is, $b\rho c$. Thus (b,c) is in $\rho_S(a) \cap \rho$ so that by Ann_1, $b = c$ or $b,c \in U_\gamma$ for some complement uniform right ideal U_γ. Also $ab = au$ for all u in U so that, in particular, $ab = a.0 = 0$, a contradiction. Hence $\rho_U(a) = \iota$.

Conversely, if $\rho_U(a) = \iota$, and $ab = 0$ for b in U, we must have $b = 0$, that is, $r_U(a) = 0$.

Together with its dual this shows that (iii) holds.

<u>(iii) => (iv)</u>. Let $I = r_S(T)$ be a right annihilator ideal in S. Put $J = (S \setminus r_S(T)) \cup 0$. Let $a \in J$, $s \in S$. If $as \in r_S(T)$, then $tas = 0$ for all t in T. By categoricity at 0, we get either $as = 0$ or $ta = 0$ for all t in T. In the latter case, $a \in r_S(T)$ so that $a = 0$. Hence $as \in J$ and J is a right ideal.

It follows that (iv) holds.

<u>(iv) => (v)</u>. Let a be an element of S such that $r_S(a)$ is intersection large. By Ann_3, $r_S(a)$ is a complement right ideal and so $r_S(a) = S$. But 0 is a prime ideal and hence $aS = 0$ implies $a = 0$. Together with its dual, this shows that S satisfies Ann_4 and hence (v) holds.

<u>(v) => (i)</u>. The first part of our argument is suggested by the proof of Lemma 3.3 of [4]. Let U be a uniform right ideal of S and suppose that $a,b,c \in U$ and $ab = ac$ but $b \neq c$. By Ann_2, $r_U(a) \neq 0$, that is, $r_S(a) \cap U \neq 0$. Now let $u \in U$ and let I be a non-zero right ideal of S.

If $auI = 0$, then $I \subseteq r_S(au)$ and so in particular, $I \cap r_S(au) \neq 0$. If $auI \neq 0$, then uI is a non-zero right ideal contained in U and since $r_S(a) \cap U$ is also non-zero, the uniformity of U gives $r_S(a) \cap U \cap uI \neq 0$. Let b be a non-zero element of $r_S(a) \cap U \cap uI$. Then $b = uv$ for some $v \in I$ and $auv = 0$. Since $v \neq 0$, we have $r_S(au) \cap I \neq 0$.

Thus $r_S(au)$ is intersection large and so by Ann_4, $au = 0$. Hence $aU = 0$ and so $ab = ac = 0$. It follows that U is left 0-cancellative.

A similar argument shows that uniform left ideals are right 0-cancellative.

Next we show that S is categorical at 0. Suppose that $a,b,c \in S$ and that ab, bc are non-zero. Since indecomposable right ideals are uniform, we have that b is a member of a uniform right ideal U. Now $aU \neq 0$ since $ab \neq 0$ and hence by the above argument, $r_U(a) = 0$. But $bc \in U$ and so we have $abc \neq 0$.

We can now apply Theorem 4 to get that S has a completely 0-simple semigroup of quotients, that is, (i) holds.

5. A special case.

In this final section we look briefly at 0-simple semigroups which satisfy the descending chain conditions for left and right ideals. These are precisely the completely 0-simple semigroups which have a finite number of $\mathcal{R}$-classes and $\mathcal{L}$-classes so that they may be represented as Rees matrix semigroups $\mathcal{M}^0(G;m,n;P)$ for some natural numbers m,n. It might be thought that these semigroups are more closely analogous to simple Artinian rings than arbitrary completely 0-simple semigroups are.

In looking for an analogue of Goldie's theorem, however, it appears that the extra finiteness conditions do not simplify our earlier results. The following theorem gives the additional conditions which have to be imposed on a semigroup in order that it have a semigroup of quotients of the form $\mathcal{M}^0(G;m,n;P)$.

THEOREM 6. For a semigroup S with 0, the following conditions are equivalent:

(i) S has a semigroup of quotients which is 0-simple and satisfies the descending chain conditions for left and right ideals,

(ii) S has a completely 0-simple semigroup of quotients and is both a finite 0-direct union of uniform right ideals and of uniform left ideals.

(iii) S has a completely 0-simple semigroup of quotients and satisfies the ascending chain conditions for complement right ideals and complement left ideals.

Proof. (i) $\Rightarrow$ (ii). The uniform one-sided ideals are found in the same way as in the proof of (i) $\Rightarrow$ (ii) of Theorem 4.

(ii) $\Rightarrow$ (iii). Since S is a finite 0-direct union of uniform right [left] ideals, there are only a finite number of complement right [left] ideals in S and so (iii) holds.

(iii) $\Rightarrow$ (i). We may assume that the semigroup of quotients of S is $\mathcal{M}^0(G;I,\Delta;P)$. For $i \in I$, we let

$$U_i = (\{(i,g,\lambda) : g \in G, \lambda \in \Lambda\} \cap S) \cup 0.$$

If I is infinite, we may suppose that $\mathbb{N} \subseteq I$ and then

$$U_1 \subset U_1 \cup U_2 \subset \ldots$$

is a strictly ascending chain of complement right ideals, contradicting our assumption.

Thus I is finite and similarly, Λ is finite.

REFERENCES

1. Clifford, A.H. and G.B. Preston, The algebraic theory of semigroups, Vol.I, Math. Surveys No.7, Amer. Math. Soc., Providence, R.I., 1961.
2. Fountain, J.B. and M. Petrich, Completely 0-simple semigroups of quotients, J. Algebra, to appear.
3. Goldie, A.W., The structure of prime rings under ascending chain conditions, Proc. London Math. Soc. 8 (1958), 589-608.
4. Goldie, A.W., Semi-prime rings with maximum condition, Proc. London Math. Soc. 10 (1960), 201-220.
5. Lallement, G. and M. Petrich, Décompositions I-matricielles d'un demi-groupe J. Math. Pures Appl. 45 (1966), 67-117.

Department of Mathematics
University of York
Heslington
YORK YO1 5DD
England.