# COMPLETELY RIGHT PURE MONOIDS

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ABSTRACT

A monoid $S$ is *completely right pure* if all its right $S$-systems are absolutely pure; we show that this is equivalent to all right $S$-systems satisfying a weaker notion of purity that we call *almost pure*. This approach produces a new characterisation of completely right pure monoids in terms of right ideals and right congruences that is analogous to a characterisation of completely right injective monoids, given by several authors. Using this result we prove that a monoid $S$ is a completely right pure union of groups if and only if it has local left zeros and is such that for each finitely generated right ideal $I$ and each finite subset $A$ of $S\setminus I$, there is an idempotent generator of $I$ that commutes with all elements of $A$.

## 1. Introduction

Throughout this paper $S$ will denote a given monoid, that is, a semigroup with identity. By the term *$S$-system* we shall mean a right unitary $S$-system over $S$. So an $S$-system is a set $A$ together with a function $\varphi : A \times S \rightarrow A$ such that $\varphi(a, 1) = a$ and $\varphi(a, st) = \varphi(\varphi(a, s),t)$ for any $a \in A$ and $s, t \in S$. For $\varphi(a, s)$ we write simply $as$.

A monoid is *completely right injective* if every $S$-system is injective. Such monoids have been studied by several authors. In [3] Feller and Gantos give a characterisation of completely right injective monoids that are unions of groups as those monoids with 0 for which every right ideal $I$ is generated by an idempotent that commutes with all elements of $S\setminus I$. The papers [4], [8] and [10] each give characterisations of completely right injective monoids, generalising the result of [3].

If all $S$-systems are absolutely pure then we shall say that $S$ is *completely right pure*. A characterisation of completely right pure monoids was given in [5], but is clearly not satisfactory. This paper attempts to remedy this by giving a characterisation of completely right pure monoids in terms of right ideals and right congruences that is a closer analogue of proposition 2.1 of [4].

The main tool used is the notion of an almost pure $S$-system, defined by weakening the concept of absolute purity. Monoids for which all $S$-systems are almost pure can be described in an analogous, but much simpler, fashion to proposition 4.5 of [5], which characterises completely right pure monoids. If $S$ is completely right pure then clearly all $S$-systems are almost pure. However, it is not difficult to show that the converse of this is also true. Thus we have a new characterisation of completely right pure monoids. This result is then used to describe completely right pure monoids that are unions of groups as those monoids that have local left zeros and are such that for each finitely generated right ideal $I$ and each finite subset $A$ of $S\backslash I$, $I$ has an idempotent generator that commutes with all the elements of $A$.

## 2. Preliminaries

This section contains definitions and background results needed for the rest of the paper. As far as possible we follow the notation of [7]. In particular, $E(S)$ will always denote the set of idempotents of $S$. We assume a basic knowledge of semigroup theory such as may be found in [2] or [7].

We shall be considering equations with constants from an $S$-system $A$. Such equations are said to be *equations over* $A$. An equation over $A$ has one of the three forms

$$xs = a \quad xs = xt \quad xs = yt$$

where $s,\ t \in S$, $a \in A$ and $x,\ y$ are variables.

If $A$ is an $S$-subsystem of an $S$-system $B$ then $A$ is (*almost*) *pure in* $B$ if every finite system of equations (in one variable) over $A$, having a solution in $B$, must also have a solution in $A$. If $A$ is (almost) pure in all $S$-systems containing it then $A$ is *absolutely pure* (*almost pure*).

A system of equations $\Sigma$ over an $S$-system $A$ is *consistent* if $\Sigma$ has a solution in some $S$-system containing $A$. Thus $A$ is absolutely pure (almost pure) if and only if all finite consistent systems of equations (in one variable) over $A$ have a solution in $A$. It is known that $A$ is injective if and only if all consistent systems of equations over $A$ have a solution in $A$ [5].

Our first result gives some justification for the introduction of almost pure $S$-systems.

**Proposition 2.1.** *A monoid $S$ is completely right pure if and only if all $S$-systems are almost pure.*

PROOF. We need only show that if all $S$-systems are almost pure then all $S$-systems are absolutely pure.

Suppose then that all $S$-systems are almost pure. Let $n$ be a natural number and assume that for any $S$-system $A$, all finite consistent systems of equations over $A$, in no more than $n$ variables, have a solution in $A$. Clearly the assumption is true for $n = 1$.

Let $A$ be an $S$-system and let $\Sigma$ be a finite consistent system of equations over $A$ in variables $x_1, \ldots, x_{n+1}$. Since $\Sigma$ is consistent, $\Sigma$ has a solution $(b_1, \ldots, b_{n+1})$ in some $S$-system $B$ containing $A$. Note that $C = A \cup b_1S \cup \ldots \cup b_nS$ is an almost pure $S$-system and $A \subseteq C \subseteq B$.

From $\Sigma$ we construct a new system of equations by considering only those equations in which $x_{n+1}$ appears and replacing in these equations each $x_i$, $i \neq n+1$, by $b_i$. Denote by $\Sigma'$ this new system of equations. Then $\Sigma'$ is a finite system of equations over $C$ in one variable. Since $\Sigma'$ has solution $b_{n+1}$ in $B$, $\Sigma'$ is consistent and as $C$ is almost pure, $\Sigma'$ has a solution $c_{n+1}$ in $C$. Thus $(b_1, \ldots, b_n, c_{n+1})$ is a solution of $\Sigma$ in $C$.

If $c_{n+1} \in A$ then let $\Sigma''$ be the system of equations obtained from $\Sigma$ by replacing $x_{n+1}$ with $c_{n+1}$. Then $\Sigma''$ has $n$ variables, constants from $A$ and solution $(b_1, \ldots, b_n)$. By assumption, $\Sigma''$ has a solution $(c_1, \ldots, c_n)$ in $A$. Then $(c_1, \ldots, c_n, c_{n+1})$ is a solution of $\Sigma$ in $A$.

Otherwise, $c_{n+1} \in b_1S \cup \ldots \cup b_nS$. Without loss of generality we may suppose that $c_{n+1} = b_ns$, $s \in S$. In this case let $\Sigma''$ be the system of equations obtained from $\Sigma$ by replacing $x_{n+1}$ with $x_ns$. Then $\Sigma''$ has $n$ variables, constants in $A$ and solution $(b_1, \ldots, b_n)$ in $C$. By assumption, $\Sigma''$ has a solution $(c_1, \ldots, c_n)$ in $A$ and then $(c_1, \ldots, c_n, c_ns)$ is a solution of $\Sigma$ in $A$.

The result follows by induction.

For an $S$-system $A$ and a subset $H$ of $A \times A$ we denote by $\rho(H)$ the congruence generated by $H$, that is, the smallest congruence $\nu$ over $A$ such that $H \subseteq \nu$.

**Lemma 2.2** [9]. *The ordered pair $(a, b)$ is in $\rho(H)$ if and only if $a = b$ or there exists a natural number $n$ and a sequence*

$$a = c_1t_1,\ d_1t_1 = c_2t_2, \ldots, d_{n-1}t_{n-1} = c_nt_n,\ d_nt_n = b,$$

*where $t_1, \ldots, t_n$ are elements of $S$ and for each $i \in \{1, \ldots, n\}$ either $(c_i, d_i)$ or $(d_i, c_i)$ is in $H$.*

A sequence as in Lemma 2.2 will be referred to as a $\rho(H)$*-sequence of length $n$*. For any congruence $\rho$ on $A$, the set of congruence classes of $\rho$ can be made into an $S$-system, with the obvious action of $S$. We write $A/\rho$ to denote this $S$-system and $[a]_\rho$, or simply $[a]$ where $\rho$ is understood, for the $\rho$-class of an element $a$ of $A$.

It is convenient to state as a lemma an easy result, which we shall use without further comment.

**Lemma 2.3.** *The following are equivalent for a monoid $S$:*

(i) *$S$ is regular and its principal right ideals are linearly ordered with respect to inclusion,*

(ii) *every finitely generated right ideal of $S$ is generated by an idempotent.*

### 3. Almost pure $S$-systems

An $S$-system $A$ is *finitely presented* if $A$ is isomorphic to $F/\rho$ where $F$ is a finitely generated free $S$-system and $\rho$ is a finitely generated congruence on $F$. A discussion of finitely presented $S$-systems is contained in section 4 of [6]; a particular consequence of this is that an $S$-system is finitely presented and cyclic if and only if it is isomorphic to $S/\rho$ for some finitely generated right congruence $\rho$ on $S$.

**Proposition 3.1.** *An $S$-subsystem $A$ of an $S$-system $B$ is almost pure in $B$ if and only if for every cyclic finitely presented $S$-system $M$, for every $S$-homomorphism $\psi: M \to B$ and for every finite set $L$ of $M$ such that $\psi(L) \subseteq A$, there exists an $S$-homomorphism*

*$\bar{\psi} : M \rightarrow A$ with $\bar{\psi}(a) = \psi(a)$ for each $a \in L$.*

Proof. Suppose that $A$ is almost pure in $B$ and $M$, $\psi$ are as given. We may assume that $M = S/\rho$ where $\rho$ is a finitely generated right congruence on $S$. If $\rho \neq I_S$, the identity congruence on $S$, let $\rho$ be generated by $\{(s_i, t_i) : 1 \leqq i \leqq n\}$. If $L \neq \emptyset$ put $L = \{[m_1], \ldots, [m_r]\}$ and $a_j = \psi([m_j])$ for $j \in \{1, \ldots, r\}$.

If $\rho \neq I_S$ and $i \in \{1, \ldots, n\}$, then

$$\psi([1])s_i = \psi([s_i]) = \psi([t_i]) = \psi([1])t_i$$

and if $L \neq \emptyset$ and $j \in \{1, \ldots, r\}$,

$$\psi([1])m_j = \psi([m_j]) = a_j.$$

Thus if $\rho \neq I_S$ or $L \neq \emptyset$,

$$\sum = \{xs_i = xt_i : 1 \leqq i \leqq n\} \cup \{xm_j = a_j : 1 \leqq j \leqq r\}$$

is a system of equations over $A$ in one variable with solution $\psi([1])$ in $B$. By assumption, $\Sigma$ has a solution $a$ in $A$ and the required $S$-homomorphism is given by $\bar{\psi}([s]) = as$. For the case where $\rho = I_S$ and $L = \emptyset$, choose any $a \in A$ and define $\bar{\psi}$ as above.

Conversely, suppose that the given condition holds. Let $\Sigma$ be a finite system of equations over $A$ in one variable with solution $b$ in $B$. Consider the $S$-system $S/\rho$, where $\rho$ is generated by $\{(s, t) : xs = xt \in \Sigma\}$. Define $\psi : S/\rho \rightarrow B$ by $\psi([s]) = bs$. Then for each equation $xp = a \in \Sigma$, $\psi([p]) = bp = a \in A$. By assumption, there is an $S$-homomorphism $\bar{\psi} : S/\rho \rightarrow A$ with $\bar{\psi}([p]) = \psi([p])$ for each equation $xp = a \in \Sigma$. It is not difficult to see that $\bar{\psi}([1])$ is a solution of $\Sigma$ in $A$.

It is convenient in the next proposition to use the fact that any $S$-system $A$ can be embedded in a minimal injective extension $I(A)$ of $A$ that is unique up to isomorphism [1]. The $S$-system $I(A)$ is the *injective envelope* of $A$.

**Proposition 3.2.** *The following conditions are equivalent for an S-system $A$:*

(i) *$A$ is almost pure;*

(ii) *given any diagram of S-systems and S-homomorphisms*

$$\begin{tikzcd} & A \\ M & N \arrow[l, "\iota"'] \arrow[u, "\theta"'] \end{tikzcd}$$

*where $M$ is cyclic finitely presented, $N$ is finitely generated and $\iota : N \rightarrow M$ is an injection, there exists an S-homomorphism $\psi : M \rightarrow A$ such that*

$$\begin{tikzcd} & A \\ M \arrow[ur, "\psi"] & N \arrow[l, "\iota"'] \arrow[u, "\theta"'] \end{tikzcd}$$

*is commutative; further, for any* $s_1, \ldots, s_n$ *in* $S$ *there is an element* $a$ *in* $A$ *with* $a = as_1 = \ldots = as_n$.

PROOF. $(i) \Rightarrow (ii)$. The first part is just as in [5, proposition 3.8].

It is easy to see that any injective $S$-system contains a one-element $S$-subsystem. Hence given any elements $s_1, \ldots, s_n$ of $S$,

$$\sum = \{xs_i = x : 1 \leqq i \leqq n\}$$

has a solution in $I(A)$. As $A$ is almost pure in $I(A)$, $\Sigma$ has a solution $a$ in $A$, so that $a = as_1 = \ldots = as_n$.

$(ii) \Rightarrow (i)$. Let $A$ be an $S$-subsystem of an $S$-system $B$. Let $M = S/\rho$ be a cyclic finitely presented $S$-system, $\psi : M \to B$ an $S$-homomorphism and $L$ a finite subset of $M$ such that $\psi(L) \subseteq A$. We consider first the case where $L = \emptyset$. If $\rho \neq I_S$, let $\{(s_i, \ t_i) : 1 \leqq i \leqq n\}$ be a generating set for $\rho$ and choose $a \in A$ with $a = as_1 = at_1 = as_2 = \ldots = at_n$. If $\rho = I_S$, pick any $a \in A$. Then one obtains a well-defined $S$-homomorphism $\bar{\psi} : M \to A$ by putting $\bar{\psi}([s]) = as$.

Suppose now that $L = \{[m_1], \ldots, [m_r]\}$ where $r \in \mathbb{N}$.

Let $N = [m_1]S \cup \ldots \cup [m_r]S$ and let $\theta$ be the $S$-homomorphism $\psi|N : N \to A$. By assumption there is an $S$-homomorphism $\bar{\psi} : M \to A$ such $\bar{\psi}\iota = \theta$, where $\iota : N \to M$ is the inclusion mapping. For any $j \in \{1, \ldots, r\}$, $\bar{\psi}([m_j]) = \bar{\psi}\iota([m_j]) = \theta([m_j]) = \psi([m_j])$. It follows from Proposition 3.1 that $A$ is almost pure in $B$.

At this point we make a new definition. For a monoid $S$ we say that $S$ has *local left zeros* if, given any finite subset $\{s_1, \ldots, s_n\}$ of $S$, there is an elememt $s$ in $S$ with $s = ss_1 = \ldots = ss_n$.

**Proposition 3.3.** *The following conditions are equivalent for a monoid* $S$:

(i) *all* $S$*-systems are almost pure*;

(ii) $S$ *is completely right pure*;

(iii) $S$ *has local left zeros and satisfies* (*):
(*) *given any finitely generated right congruence* $\rho$ *on* $S$ *and any finitely generated right ideal* $I$ *of* $S$, *there is an element* $s$ *of* $I$ *such that for any* $u$, $v$ *in* $S$, *if* $u\rho v$ *then* $su\rho sv$ *and for any* $w \in I$, $w\rho sw$.

PROOF $(i) \Leftrightarrow (ii)$. This is Proposition 2.1.

$(ii) \Rightarrow (iii)$. It is immediate from proposition 4.5 of [5] that $S$ satisfies (*).

The existence of local left zeros follows from (ii) of Proposition 3.2 and the fact that $S$ is an almost pure $S$-system.

$(iii) \Rightarrow (i)$. Let $A$ be an $S$-system, $M$ a cyclic finitely presented $S$-system, $N$ a finitely generated $S$-subsystem of $M$ and $\theta : N \to A$ an $S$-homomorphism. We may assume that $M = S/\rho$ for some finitely generated right congruence $\rho$ on $S$ and $N = [1]I$ for some finitely generated right ideal $I$ of $S$. Let $s \in I$ be chosen to satisfy (*). Defining $\psi : M \to A$ by $\psi([t]) = \theta([st])$, one sees that $\psi$ is an $S$-homomorphism extending $\theta$.

If $s_1, \ldots, s_n$ are elements of $S$ then choose $u \in S$ with $u = us_1 = \ldots = us_n$ and let $b \in A$. Putting $a = bu$ we have $a = as_1 = \ldots = as_n$. From Proposition 3.2 we have that $A$ is almost pure.

**Corollary 3.4.** *Let $I$ be a finitely generated right ideal of a completely right pure monoid $S$. Then $I = eS$ for some $e \in E(S)$.*

PROOF. Putting $\rho = I_S$ in (*), there is an element $e \in I$ such that $w = ew$ for any $w \in I$. In particular, $e = e^2$ so that $e \in E(S)$ and the result follows.

## 4. Completely right pure monoids that are unions of groups

We begin this section by quoting two results of [3], where they are given for monoids in which every right ideal is generated by an idempotent. However, the proofs only require that every finitely generated right ideal be generated by an idempotent, a condition that is satisfied by completely right pure monoids.

An ideal $I$ of a semigroup $S$ is *prime* if $ab \in I$ implies that $a \in I$ or $b \in I$.

**Lemma 4.1** [3]. *Let $S$ be a monoid in which every finitely generated right ideal is generated by an idempotent.*

(i) *For any $e, f \in E(S)$, if $Se = Sf$, then $e = f$.*

(ii) *For any $a$, $b \in S$, if $a \mathscr{L} b$ then $a' \mathscr{R} b'$, for any inverses $a'$, $b'$ of $a$, $b$ respectively.*

(iii) *Let $I$ be an ideal of $S$ and $a$, $b$, $c \in S$:*

  (a) *if $a \in I$ then every inverse $a'$ of $a$ is in $I$;*
  (b) *if $a \notin I$* and *$c \in I$, then $Sac = Sc$;*
  (c) *$I$ is a prime ideal of $S$.*

**Proposition 4.2** [3]. *Let $S$ be a monoid in which every finitely generated right ideal is generated by an idempotent. The following conditions are equivalent:*

(i) *$S$ is a union of groups;*

(ii) *every $\mathscr{L}$*-class of S is a group;

(iii) *every finitely generated right ideal of $S$ is two-sided.*

We now show that for monoids that are unions of groups the statement of Proposition 3.3 can be weakened. In this case we can also characterise the completely right pure monoids in terms of elements and ideals.

**Proposition 4.3.** *Let $S$ be a monoid that is a union of groups. The following conditions* *are equivalent*:

(i) *S is completely right pure*;

(ii) *S has local left zeros and satisfies*(**):
(**) *given any finitely generated right congruence $\rho$ on S and any finitely generated right ideal I of S that is a union of $\rho$-classes, there is an $s \in I$ such that for any u, v in S, if $u\rho v$ then $su\rho sv$, and for any $w \in I$ we have $w\rho sw$*;

(iii) *S has local left zeros and satisfies* (†):
(†) *given any finitely generated right ideal I of S and any elements $a_1, \ldots, a_n \in S \backslash I$, where $n \in \mathbb{N}$, there is an $f \in E(S)$ with $I = fS$ and for any $i, j \in \{1, \ldots, n\}$*

$$a_i'fa_i = a_j'fa_j$$

*for any inverses $a_i', a_j'$ of $a_i, a_j$ respectively.*

PROOF. *(i)⇒(ii)*. This is immediate from Proposition 3.3.

*(ii)⇒(iii)*. Let $I$ be a finitely generated right ideal of $S$. As in the proof of Corollary 3.4 one can show that $I = eS$ for some $e \in E(S)$.

Now consider a finite number of elements $a_1, \ldots, a_n$ of $S\backslash I$. Define $\rho$ to be the right congruence on $S$ generated by $\{(a_i,a_j) : 1 \leqq i, j \leqq n\}$. We show that $I$ is a union of $\rho$-classes. From Lemma 4.1 and Proposition 4.2, $I$ is a prime ideal of $S$. Let $h \in I$ and suppose that $h\rho k$. If $h \neq k$ then there is a $\rho$-sequence

$$h = c_1t_1, d_1t_1 = c_2t_2, \ldots, d_lt_l = k$$

connecting $h$ and $k$. As $I$ is prime and $c_1 \notin I$ we have that $t_1 \in I$ and so $d_1t_1 \in I$. This gives that $t_2 \in I$ and so $d_2t_2 \in I$. Continuing in this way one sees that $k \in I$ and it follows that $I$ is a union of $\rho$-classes.

We apply (**) to obtain an $x \in I$ such that if $u, v \in S$ and $u\rho v$, then $xu\rho xv$ and $xw\rho w$ for all $w \in I$.

Let $h \in I$ and let $h\rho k$ as above. If $h = k$ then clearly $Sh = Sk$, otherwise we may use Lemma 4.1 to obtain

$$Sh = Sc_1t_1 = St_1 = Sd_1t_1 = Sc_2t_2 = \ \ldots\ = Sd_lt_l = Sk$$

and so $h \mathscr{L} k$.

From $e\rho xe$ we have $e\mathscr{L}xe$ and so $e\mathscr{L}(xe)'xe$ for any inverse $(xe)'$ of $xe$. Again using Lemma 4.1 we have that $e = (xe)'xe$. Every finitely generated right ideal of $S$ is generated by an idempotent; from this it is easy to see that $S$ is orthodox, that is, $E(S)$ is a subsemigroup. Theorem IV 1.1 of [7] gives that $ex'$ is an inverse of $xe$ for any inverse $x'$ of $x$ and so also $e = ex'xe$. Put $f = x'x$, where $x'$ is an inverse of $x$. Note that $f = x''x$ for any inverse $x''$ of $x$. Now $f \in E(S) \cap I$, which gives that $fS \subseteq eS$ and $ef = f$. Then from $e = efe$ we have $eS = feS \subseteq fS$ and so $fS = eS = I$.

Let $i, j \in \{1, \ldots, n\}$. We have $a_i\rho a_j$ and so $xa_i\rho xa_j$. But $xa_i, xa_j \in I$ so that $xa_i\mathscr{L}xa_j$

and $(xa_i)'xa_i = (xa_j)'xa_j$ for any inverses $(xa_i)'$, $(xa_j)'$ of $xa_i$, $xa_j$ respectively. It follows that $a_i'f\,a_i = a_i'x'xa_i = a_j'x'xa_j = a_j'\,f\,a_j$ for any inverses $a_i'$, $a_j'$ of $a_i$, $a_j$ respectively.

$(iii) \Rightarrow (i)$. Let $I$ be a finitely generated right ideal of $S$. If $I \neq S = 1S$, there is an element $a = a_1 \in S \backslash I$. An application of (†) gives that $I$ is generated by an idempotent.

Now suppose that $A$ is an $S$-system and that $A$ is an $S$-subsystem of an $S$-system $B$. It follows from the fact that $S$ has local left zeros that every finite system of equations without constants has a solution in $A$.

Let $\Sigma$ be a finite system of equations over $A$ in one variable, where a constant appears in at least one equation of $\Sigma$, and suppose that $\Sigma$ has a solution $b$ in $B$. Write $\Sigma$ as

$$\sum = \{xs_i = xt_i : 1 \leqq i \leqq n\} \cup \{xm_j = a_j : 1 \leqq j \leqq r\} \tag{1}$$

or

$$\sum = \{xm_j = a_j : 1 \leqq j \leqq r\} \tag{2}$$

as appropriate.

If $\Sigma$ has form (1), define a subset $K$ of $\{1, \ldots, m\}$ by putting $K = \{i : bs_i = bt_i \in A\}$ and let

$$I = \bigcup_{j \in \{1, \ldots, r\}} m_jS \cup \bigcup_{i \in K} (s_iS \cup t_iS).$$

If $\Sigma$ has form (2) put

$$I = \bigcup_{j \in \{1, \ldots, r\}} m_jS.$$

In either case, if $s \in I$ then $bs \in A$. In particular, this shows that for any $i \in \{1, \ldots, n\}$, $s_i \in I$ if and only if $t_i \in I$. Using (†) one can choose $f \in E(S)$ with $I = fS$ and such that if $i \in \{1, \ldots, n\} \backslash K$, then $s_i'fs_i = t_i'ft_i$ for any inverses $s_i', t_i'$ of $s_i$, $t_i$ respectively.

As $f \in I$ we have $bf \in A$. If $j \in \{1, \ldots, r\}$, then $bfm_j = bm_j = a_j$, since $f$ is a left identity for $I$. If $i \in K$ then $bfs_i = bs_i = bt_i = bft_i$. If $i \in \{1, \ldots, n\} \backslash K$ then $s_i, t_i \notin I$ and as the principal right ideals are linearly ordered we have that $fS \subset s_iS$ and $fS \subset t_iS$. Since $S$ is regular we can choose inverses $s_i'$, $t_i'$ of $s_i$, $t_i$ respectively. From $fS \subset s_iS = s_is_i'S$ we have $s_is_i'f = f$ and, similarly, $t_it_i'f = f$. Then since $b$ is a solution of $\Sigma$, we have

$$bfs_i = bs_is_i'fs_i = bt_it_i'ft_i = bft_i.$$

Thus $bf$ is a solution of $\Sigma$ in $A$. If follows that all $S$-systems are almost pure and so from Proposition 2.1, $S$ is completely right pure.

We now give an analogue of theorem 3.2 of [3].

**Theorem 4.4.** *The following conditions are equivalent for a monoid $S$:*

(i) *$S$ is completely right pure and is a union of groups;*

(ii) *S has local left zeros and, given any finitely generated right ideal I of S and any finite subset A of* $S\backslash I$, *then I has an idempotent generator which commutes with all elements in A.*

PROOF (i)⇒(ii). Let $I$ be a finitely generated right ideal of $S$. Corollary 3.4 gives that $I$ is generated by an idempotent, which takes care of the case where $A = \emptyset$. Now suppose that $A = \{a_1, \ldots, a_n\} \subseteq S\backslash I$. Since $I \neq S$, 1 is not in $I$ and we consider $B = A \cup \{1\} \subseteq S\backslash I$. From (†) there is an element $f \in E(S)$ with $I = fS$ and $f = 1f1 = a_j' f a_j$ for any inverse $a_j'$ of $a_j$ and for any $j \in \{1, \ldots, n\}$, giving that $a_j f = a_j a_j' f a_j$. As the principal right ideals of $S$ are linearly ordered, $a_j a_j' S \subseteq fS$ or $fS \subseteq a_j a_j' S$. The former is impossible since $a_j \mathscr{R} a_j a_j'$ and $a_j \notin I$. Thus $fS \subseteq a_j a_j' S$ and so $a_j f = f a_j$ as required.

*(ii)⇒(i).* Let $I$ be a finitely generated right ideal of $S$. Let $s \in S$, $b \in I$. If $s \in I$ then $sb \in I$. If $s \notin I$ then by assumption $I = fS$ for some $f \in E(S)$ with $fs = sf$. So $sb = sfb = fsb \in I$. Hence $I$ is a two-sided ideal and so Proposition 4.2 gives that $S$ is a union of groups.

Now take $I$ to be a finitely generated right ideal and let $a_1, \ldots, a_n \in S\backslash I$. Let $f$ be an idempotent generator for $I$ with $fa_i = a_i f$, $1 \leqq i \leqq n$. For any $i \in \{1, \ldots, n\}$ and for any inverse $a_i'$ of $a_i$, $a_i' a_i \in S\backslash I$ because $I$ is an ideal. Since the principal right ideals of $S$ are linearly ordered we have that $fS \subset a_i' a_i S$ which gives that $f = a_i' a_i f = a_i' f a_i$. It follows immediately that (†) holds.

**Corollary 4.5** [9]. *Let S be a monoid whose idempotents are central. Then S is completely right pure if and only if S has local left zeros and every finitely generated right ideal of S is generated by an idempotent.*

For our final result we recall from [2] that a semigroup $U$ is *right simple* if $U$ is the only right ideal of $U$. If in addition $U$ is left cancellative, then $U$ is said to be a *right group*. Theorem 1.27 of [2] states that a semigroup $U$ is a right group if and only if $U$ is right simple and contains an idempotent.

**Theorem 4.6.** *Let S be a completely right pure monoid that is a union of groups. Then each* $\mathscr{R}$*-class of S is a right group and S is a chain of its* $\mathscr{R}$*-classes.*

PROOF. From Corollary 3.4, the principal right ideals of $S$ are linearly ordered with respect to inclusion.

Let $R$ be an $\mathscr{R}$-class of $S$. Then $R = R_e$ for some $e \in E(S)$. Let $a \in R_e$ and let $a'$ be any inverse of $a$. It follows from the fact that $aS$ and $a'S$ are ideals of $S$ that $a \mathscr{R} a'$ and so $a' \in R_e$. If also $b \in R_e$ then clearly $abS \subseteq aS$. Since $abS$ is a prime ideal of $S$, either $a \in abS$ or $b \in abS$. So either $aS \subseteq abS \subseteq aS$ or $bS \subseteq abS \subseteq aS = bS$, which gives that $ab \in R_e$ and $R_e$ is a subsemigroup of $S$.

Fix $a \in R_e$. If $b \in R_e$ and $a'$ is an inverse of $a$ then $a'b \in R_e$ and $b = aa'b$, so that $R_e = aR_e$. Thus $R_e$ is a right simple semigroup containing an idempotent and so theorem 1.27 of [2] gives that $R_e$ is a right group.

For any $s \in S$ and inverse $s'$ of $S$ one may show as above that $s \mathscr{R} s' \mathscr{R} ss' \mathscr{R} s's$. Let $R_e$, $R_f$ be $\mathscr{R}$-classes of $S$ where $e, f \in E(S)$ and $eS \subset fS$. Let $a \in R_e$ and $b \in R_f$. Then $abS \subseteq aS = eS$ and since $abS$ is a prime ideal it follows that $abS = eS$ and $ab \in R_e$. Further, $aS \subset bS = b'bS$ so that $b'ba = a$. Now $baS$ is an ideal, so $a \in baS$

and as $aS$ is an ideal, $ba \in aS$. Consequently, $baS = aS$ and $ba \in R_e$. Thus $R_eR_f \subseteq R_e$ and $R_fR_e \subseteq R_e$, as required.

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