ORDERS AND STRAIGHT LEFT ORDERS IN COMPLETELY REGULAR SEMIGROUPS

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ABSTRACT. A subsemigroup S of a completely regular semigroup Q is an *order* in Q if every element of Q can be written as $a^\sharp b$ and as $cd^\sharp$ where $a, b, c, d \in S$ and $x^\sharp$ is the inverse of $x \in Q$ in a subgroup of Q. If only the first condition holds and one insists also that $a \,\mathcal{R}\, b$ in Q, then S is said to be a *straight left order* in Q. This paper characterizes those semigroups that are straight left orders in completely regular semigroups. A consequence of this result, together with some technicalities concerning lifting of morphisms, is a description of orders in completely regular semigroups.

1. INTRODUCTION

A subsemigroup S of a semigroup Q is a *left order* in Q and Q is a *semigroup of left quotients of* S if every element of Q can be written as $a^\sharp b$ for some $a, b \in S$ and if, in addition, every element of S that satisfies a weak cancellation property called *square cancellation*, lies in a subgroup of Q. Here $a^\sharp$ denotes the inverse of a in a subgroup of Q. This subgroup does not have to be the group of units, indeed this concept of a semigroup of left quotients, introduced by Fountain and Petrich in [5], is designed to reflect the equal importance of *all* subgroups of a semigroup. *Right orders* and *semigroups of right quotients* are defined dually; if S is both a left order and a right order in Q then S is an *order* in Q and Q is a *semigroup of quotients* of S.

In the case where Q is a group, our concept of a semigroup of left quotients becomes the classical one of a group of left quotients. A theorem of Ore and Dubreil [2] says that a semigroup S has a group of left quotients if and only if it is right reversible, that is, for all $a, b \in S$, $Sa \cap Sb \neq \emptyset$ and cancellative. The question then arises of characterising left orders in semigroups from larger classes than the class of groups. Of course, one chooses classes whose semigroups have a structure that is well understood, the point being that the left orders inherit some of this structure.

Up to isomorphism, a semigroup has at most one group of left quotients. This is not the case for semigroups of left quotients in general. An example is given in [5] of a semigroup having two non-isomorphic semigroups of left quotients. Under what circumstances then are two semigroups of left quotients isomorphic? More generally, if S is a left order in Q and ϕ is a morphism from

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S to a semigroup P, when is it possible for ϕ to be extended to a morphism with domain Q?

This paper makes contributions to the questions arising in each of the two preceding paragraphs. After a section on the technical results that we need to tackle these problems, we answer in Section 3 the morphism extension problem.

The remainder of the paper is concerned with characterising left orders and orders in completely regular semigroups. These questions were the central topic of the D. Phil. thesis of the second author; our results are variations on those of [13].

A left order S in a semigroup Q is *straight* if every element of Q can be written as $a^\sharp b$ for some $a, b \in S$ with $a \,\mathcal{R}\, b$ in Q. Many left orders are straight; it is shown in [9] that if Green's relation $\mathcal{H}$ is a congruence on Q then every left order in Q is straight. If Q is completely regular, then $\mathcal{H}$ is a congruence on Q if and only if Q is a band of groups. Orders in bands of groups are the topic of [6]. In Section 4 we show that every order in a completely regular semigroup is straight as a left order (and dually as a right order).

Section 5 contains a characterisation of straight left orders in completely regular semigroups. Our result is a reformulation of that in [13]. As in [13] we use the general description from [8] of straight left orders. This involves the use of $*$-pairs, the details of which are given in Section 2.

Our final Section 6 puts together the description of straight one-sided orders, the fact that every order in a completely regular semigroup is straight, and the morphism lifting results of Section 3, to give a characterisation of orders in completely regular semigroups.

2. Embeddable $*$-pairs

This section gives background on preorders on a semigroup S that are essential in determining the existence and uniqueness of semigroups of (left) quotients of S.

We assume a familiarity with the basic notions of semigroup theory, as in [2] and [11], in particular, with Green's relations. We bear in mind the fact that Green's relations $\mathcal{L}$ and $\mathcal{R}$ are the equivalence relations associated with the preorders $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ respectively, where for elements p and q of a semigroup Q,

$$p \leq_{\mathcal{L}} q \text{ if and only if } Q^1 p \subseteq Q^1 q$$

and dually,

$$p \leq_{\mathcal{R}} q \text{ if and only if } pQ^1 \subseteq qQ^1.$$

Notice that the relations $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ are respectively right and left compatible with multiplication. To avoid ambiguity, where there is more than one semigroup under consideration, we may use a superscript on a relation to denote the semigroup concerned.

For any element q of a semigroup Q, lying in a subgroup of Q, $q^\sharp$ denotes the inverse of q in this subgroup. Use of the notation $q^\sharp$ implies that q lies in a subgroup of Q; $q^\sharp$ is then uniquely determined as the (group) inverse of q in the subgroup H_q of Q. The union of subgroups of Q is denoted by $\mathcal{H}(Q)$.

For convenience we list some properties of $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ that will be used repeatedly.

Lemma 2.1. *Let p, q, r be elements of a semigroup Q. The following statements and their left-right duals hold:*
(i) if $q \leq_{\mathcal{L}} p$ and $p \in \mathcal{H}(Q)$, then $qp \,\mathcal{R}\, q$;
(ii) if $q, r \leq_{\mathcal{L}} p, p \in \mathcal{H}(Q)$ and $qp = rp$, then $q = r$;
(iii) if $q, r \leq_{\mathcal{L}} p, p \in \mathcal{H}(Q)$ and $qp \,\mathcal{L}\, rp$, then $q \,\mathcal{L}\, r$;
(iv) if $q \leq_{\mathcal{R}} p$ and $q \in \mathcal{H}(Q)$, then $qp \,\mathcal{R}\, q$.

Proof. For (i), (ii) and (iii) use the fact that $\leq_{\mathcal{L}}$ ($\leq_{\mathcal{R}}$) is right (left) compatible with multiplication and if $q \leq_{\mathcal{L}} p$ ($q \leq_{\mathcal{R}} p$) where $p \in \mathcal{H}(Q)$, then $q = qpp^\sharp$ ($q = p^\sharp pq$). For (iv) $q \leq_{\mathcal{R}} p$ gives that

$$q \,\mathcal{H}\, q^2 \leq_{\mathcal{R}} qp \leq_{\mathcal{R}} q$$

so that $q \,\mathcal{R}\, qp$. ☐

We make repeated use of two futher preorders, namely $\leq_{\mathcal{L}^*}$ and $\leq_{\mathcal{R}^*}$. For elements a and b of a semigroup S, $a \leq_{\mathcal{L}^*} b$ if and only if for all $x, y \in S^1$,

$$bx = by \text{ implies that } ax = ay.$$

Clearly $\leq_{\mathcal{L}^*}$ is a preorder that is right compatible with multiplication so that the associated equivalence relation, denoted $\mathcal{L}^*$, is a right congruence. The relations $\leq_{\mathcal{R}^*}$ and $\mathcal{R}^*$ are defined dually.

Our next lemma is well known and can easily be proved by embedding S in the (dual of the) full transformation semigroup on the set S^1.

Lemma 2.2. [4],[12] *Let S be a semigroup and let $a, b \in S$. Then $a \leq_{\mathcal{L}^*} b$ if and only if $a \leq_{\mathcal{L}} b$ in an oversemigroup of S. The corresponding statements are true for the relations $\leq_{\mathcal{R}^*}, \mathcal{L}^*$ and $\mathcal{R}^*$.*

By Green's Theorem, q lies in a subgroup (so that $q^\sharp$ exists) if and only if $q \,\mathcal{H}\, q^2$. Denoting by $\mathcal{H}^*$ the intersection of $\mathcal{L}^*$ and $\mathcal{R}^*$, it is then clear that a necessary condition for an element a of a semigroup S to lie in a subgroup of an oversemigroup is that $a \,\mathcal{H}^*\, a^2$.

We say that an element a of a semigroup S is *square cancellable* if $a \,\mathcal{H}^*\, a^2$ and we denote by $\mathcal{S}(S)$ the set of square cancellable elements of S. A subsemigroup S of a semigroup Q is a *left order* in Q and Q is a *semigroup of left quotients* of S if every element of Q can be written in the form $a^\sharp b$ for some $a, b \in S$ and if, in addition, every square cancellable element of S lies in a subgroup of Q. The concepts of *right order* and *semigroup of right quotients* are defined dually. If S is both a left order and a right order in Q then S is an *order* in Q and Q is a *semigroup of quotients* of S. For much of this paper we are concerned only with (left) orders in completely regular semigroups; in this case the condition that every square cancellable element lies in a subgroup is redundant as completely regular semigroups are unions of groups.

It is easy to see that if S is a left order in Q, then any element of Q may be written as $a^\sharp b$ where $a, b \in S$ and $b \leq_{\mathcal{R}} a$ in Q. In this case the dual of Lemma 2.1 (i) gives that $a^\sharp b \,\mathcal{L}\, b$ in Q. Thus S intersects every $\mathcal{L}$-class of Q. If moreover S intersects every $\mathcal{H}$-class of Q, then S inherits more of the structure of Q. This condition is ensured by the concept of *straightness*.

Let S be a left order in Q. Then S is *straight* in Q if every element of Q can be written as $a^\sharp b$ where $a, b \in S$ and $a \,\mathcal{R}\, b$ in Q. For such elements a and b it is easy to see that $a^\sharp b \,\mathcal{H}\, b$ in Q, so that S intersects every $\mathcal{H}$-class of Q. The notion of *straight right order* is dual; if S is a straight left order and a straight right order in Q then S is a *straight order* in Q.

An ordered pair $(\leq_l, \leq_r)$ of preorders on a semigroup S is a $*$*-pair* if $\leq_l$ is right compatible with multiplication, $\leq_r$ is left compatible, $\leq_l \subseteq\ \leq_{\mathcal{L}^*}$ and $\leq_r \subseteq\ \leq_{\mathcal{R}^*}$. Clearly $(\leq_{\mathcal{L}^*}, \leq_{\mathcal{R}^*})$ is a $*$-pair for any semigroup S. From Lemma 2.2 we have the following.

Corollary 2.3. *Let S be a subsemigroup of Q. Then*

$$(\leq_{\mathcal{L}^Q} \cap (S \times S), \leq_{\mathcal{R}^Q} \cap (S \times S))$$

is a $*$-pair for S, the $*$-pair for S induced by Q.

In particular the above applies when S is a (left) order in Q. Our immediate aim is to describe a number of properties held by certain induced $*$-pairs. This will lead us to the notion of an *embeddable $*$-pair*. On the other hand, we know from [8] that an embeddable $*$-pair satisfying certain conditions is *always* induced by a semigroup of left quotients.

For a $*$-pair $(\leq_l, \leq_r)$, we denote by $\mathcal{L}'$ and $\mathcal{R}'$ the equivalence relations associated with $\leq_l$ and $\leq_r$ respectively. Notice that $\mathcal{L}'$ is a right congruence contained in $\mathcal{L}^*$ and $\mathcal{R}'$ is a left congruence contained in $\mathcal{R}^*$; $\mathcal{H}'$ denotes the intersection of $\mathcal{L}'$ and $\mathcal{R}'$.

In the following lemma, the slightly eccentric notation is that inherited from [8].

Lemma 2.4. [8] *Let S be a subsemigroup of a regular semigroup Q such that S intersects every $\mathcal{H}$-class of Q. Then the $*$-pair for S induced by Q satisfies the following conditions and the duals (Eii)(r), (Ev)(r), (Evi)(r) and (Evii)(r) of (Eii)(l), (Ev)(l), (Evi)(l) and (Evii)(l) respectively, where $\mathcal{G}(S) = \{a \in S : a\mathcal{H}' a^2\}$.*

(Ei) $\mathcal{L}' \circ \mathcal{R}' = \mathcal{R}' \circ \mathcal{L}'$.

(Eii)(l) For all $b, c \in S$, $b \leq_l c$ if and only if $b \,\mathcal{L}'\, dc$ for some $d \in S$.

(Eiii) Every $\mathcal{L}'$-class and every $\mathcal{R}'$-class contains an element from $\mathcal{G}(S)$.

(Ev)(l) For all $a \in \mathcal{G}(S)$ and $b \in S$, if $b \leq_l a$, then $ba \,\mathcal{R}'\, b$.

(Evi)(l) For all $a \in \mathcal{G}(S)$ and $b, c \in S$, if $b, c \leq_l a$ and $ba = ca$, then $b = c$.

(Evii)(l) For all $a \in \mathcal{G}(S)$ and $b, c \in S$, if $b, c \leq_l a$ and $ba \,\mathcal{L}'\, ca$, then $b \,\mathcal{L}'\, c$.

Proof. The semigroup S inherits these properties from the properties of Green's relations in Q; use Lemma 2.1. $\square$

As in [8] we say that a $*$-pair satisfying conditions (Ei), (Eii)(l), (Eiii), (Ev)(l), (Evi)(l), (Evii)(l) and their duals where appropriate is an *embeddable $*$-pair.*

Lemma 2.5. *Let S be a straight left order in Q. Then the $*$-pair for S induced by Q is an embeddable $*$-pair.*

Proof. As in [8], Q must be regular. By remarks above, S intersects every $\mathcal{H}$-class of Q, so the result holds by Lemma 2.4. □

3. Extension of morphisms

In the founding paper [5] Fountain and Petrich showed that a semigroup can have non-isomorphic semigroups of left quotients. This led to the question, answered in [7], of under what circumstances are semigroups of left quotients of a given semigroup isomorphic. More generally, if S is a left order in a semigroup Q and ϕ is a morphism from S to a semigroup P, when can ϕ can be extended to a morphism from Q to P? We answer the latter question in this section. One consequence of our results is that in Section 6 we deduce the characterisation of orders in completely regular semigroups from the characterisation of one-sided orders.

The first lemma of this section is elementary.

Lemma 3.1. *Let ϕ be a morphism from a semigroup Q to a semigroup P. Then*
(i) if $p, q \in Q$ and $p \leq_{\mathcal{L}} q$ in Q, then $p\phi \leq_{\mathcal{L}} q\phi$ in P;
(ii) $\mathcal{H}(Q)\phi \subseteq \mathcal{H}(P)$;
(iii) if $p \in \mathcal{H}(Q)$ then $(p\phi)^\sharp = p^\sharp\phi$.

Let S be a subsemigroup of Q. If every element of Q can be written as $a^\sharp b$ where $a, b \in S$, then S is a *weak left order* in Q. Straight weak left orders are defined in the obvious manner.

Lemma 3.2. *Let S be a weak left order in Q and let ϕ be a morphism from S to a semigroup P. Then ϕ can be extended to at most one morphism $\overline{\phi} : Q \to P$.*

Proof. By Lemma 3.1, if $\overline{\phi}$ exists, it must be given by

$$(a^\sharp b)\overline{\phi} = (a\phi)^\sharp b\phi$$

where $a, b \in S$. □

Thus if S is a weak left order in Q and $\phi : S \to P$ is a morphism, any extension of ϕ to Q is unique. To answer the question of the existence of such an extension, we consider a ternary relation on S, introduced in [7].

Let S be a subsemigroup of R. The ternary relation $\mathcal{T}(R, S)$ on S is defined by the rule that for $a, b, c \in S$,

$$(a, b, c) \in \mathcal{T}(R, S) \text{ if and only if } a, c \in \mathcal{H}(R), b \leq_{\mathcal{R}} a \text{ and } a^\sharp b \leq_{\mathcal{R}} c \text{ in } R.$$

Theorem 3.3. *Let S be a weak left order in Q and let $\phi : S \to P$ be a morphism. Then ϕ can be extended to a morphism from Q to P if and only if*

$$(a,b,c) \in \mathcal{T}(Q,S) \text{ implies that } (a\phi, b\phi, c\phi) \in \mathcal{T}(P, S\phi) \ (*).$$

Proof. The proof may be lifted from part of the proof of Theorem 3.1 of [7], which considers the case where ϕ is an embedding. □

The following corollary may be deduced from Lemma 3.1 and Theorem 3.3.

Corollary 3.4. *Let S be a weak left order in Q. If $\psi : Q \to R$ is a morphism then $S\psi$ is a weak left order in $Q\psi$. Thus if $\phi : S \to P$ is a morphism such that $(*)$ holds, then $S\phi$ is a weak left order in $Q\overline{\phi}$.*

We now consider when a morphism ϕ from a weak left order S in Q to a semigroup P can be lifted to an onto morphism, so that P is the image of Q.

Lemma 3.5. *Let S be a weak left order in Q and let $\phi : S \to P$ be a morphism. Then ϕ can be extended to an onto morphism if and only if $S\phi$ is a weak left order in P and $(*)$ and $(**)$ hold where $(**)$ is the following condition:*

for any $a \in S$ such that $a\phi \in \mathcal{H}(P)$, there are elements $u, v \in S$ with $u \in \mathcal{H}(Q)$ and $a\phi \leq_{\mathcal{R}} u\phi$ such that

$$u\phi a\phi = v\phi(a\phi)^2 \quad (**).$$

Proof. Suppose that ϕ can be extended to an onto morphism $\overline{\phi}$. By Theorem 3.3, (*) holds and by Corollary 3.4, $S\phi$ is a weak left order in P. Let $a \in S$ be such that $a\phi \in \mathcal{H}(P)$. Then $(a\phi)^\sharp$ exists so that as $\bar{\phi}$ is onto and S is a weak left order in Q, there are elements $u, v \in S$ with $u \in \mathcal{H}(Q)$ and $v \leq_{\mathcal{R}} u$ such that $(a\phi)^\sharp = (u\phi)^\sharp v\phi$. Thus

$$a\phi \,\mathcal{H}\, (a\phi)^\sharp = (u\phi)^\sharp v\phi \leq_{\mathcal{R}} u\phi$$

and (**) holds.

Conversely, if $S\phi$ is a weak left order in P and (*) and (**) hold, then we know by Theorem 3.3 that ϕ can be extended to $\bar{\phi} : Q \to P$. As $S\phi$ is a weak left order in P, for an arbitrary $p \in P$ we can write p as $p = (a\phi)^\sharp b\phi$ for some $a, b \in S$ with $b\phi \leq_{\mathcal{R}} a\phi$ in P. Now $a\phi \in \mathcal{H}(P)$ so that $u, v \in S$ exist as in (**), giving us

$$a\phi = (u\phi)^\sharp v\phi(a\phi)^2.$$

Now

$$p = (a\phi)^\sharp b\phi = a\phi(a\phi)^\sharp(a\phi)^\sharp b\phi = (u\phi)^\sharp v\phi(a\phi)^2(a\phi)^\sharp(a\phi)^\sharp b\phi = (u\phi)^\sharp v\phi b\phi$$

as $b\phi \leq_{\mathcal{R}} a\phi$. Thus $p = (u\phi)^\sharp(vb)\phi = (u^\sharp vb)\overline{\phi}$ so that $\overline{\phi}$ is onto as required. □

In a large number of cases we can avoid condition (**).

Corollary 3.6. *Let S be a weak left order in Q and let $\phi : S \to P$ be a morphism such that for all $a \in S$,*

$$a\phi \in \mathcal{H}(P) \text{ implies that } a \in \mathcal{H}(Q).$$

Then ϕ can be extended to an onto morphism if and only if (*) holds and $S\phi$ is a weak left order in P.

Proof. In view of Lemma 3.5 it is enough to show that (**) always holds.

Let $a \in S$ be such that $a\phi \in \mathcal{H}(P)$. By the given condition, $a \in \mathcal{H}(Q)$. Put $u = a^2$ and $v = a$. Then

$$a\phi \, \mathcal{H} \, (a\phi)^2 = a^2\phi = u\phi.$$

□

To avoid the ternary relation in (*) we introduce the notion of $\mathcal{R}$-boundedness as in [7].

Let S be a subsemigroup of Q. We say that S is *$\mathcal{R}$-bounded in Q* if for any $q \in Q$ and $a, c \in \mathcal{H}(Q) \cap S$ with $q \leq_{\mathcal{R}} a, c$ in Q, there exists $w \in S$ with $q \leq_{\mathcal{R}} w \leq_{\mathcal{R}} a, c$ in Q.

Remark If S intersects every $\mathcal{R}$-class of Q, certainly S is $\mathcal{R}$-bounded in Q. As in Section 2, if S is a straight weak left order in Q, or a weak right order in Q, then S intersects every $\mathcal{R}$-class of Q.

Proposition 3.7. *Let S be a weak left order in Q and let $\phi : S \to P$ be a morphism. Suppose that S is $\mathcal{R}$-bounded in Q. Then ϕ can be extended to a morphism from Q to P if and only if*

for all $b, d \in S$,

$$b \leq_{\mathcal{R}} d \text{ in } Q \text{ implies that } b\phi \leq_{\mathcal{R}} d\phi \text{ in } P$$

and

for all $a \in S$,

$$a \in \mathcal{H}(Q) \text{ implies that } a\phi \in \mathcal{H}(P).$$

Proof. If ϕ is extendable, the result follows by Lemma 3.1.

Conversely, suppose that the given conditions hold. Let $(a, b, c) \in \mathcal{T}(Q, S)$. Then $a, c \in \mathcal{H}(Q)$, $b \leq_{\mathcal{R}} a$ and $a^\sharp b \leq_{\mathcal{R}} c$ in Q. By hypothesis, $a\phi, c\phi \in \mathcal{H}(P)$ and as S is $\mathcal{R}$-bounded in Q, there exists $w \in S$ with

$$a^\sharp b \leq_{\mathcal{R}} w \leq_{\mathcal{R}} a, c.$$

This gives that

$$b \leq_{\mathcal{R}} aw$$

and so as we are given that ϕ preserves $\leq_{\mathcal{R}}$,

$$b\phi \leq_{\mathcal{R}} (aw)\phi = a\phi w\phi, w\phi \leq_{\mathcal{R}} a\phi, c\phi.$$

Thus

$$(a\phi)^\sharp b\phi \leq_{\mathcal{R}} (a\phi)^\sharp a\phi w\phi = w\phi \leq_{\mathcal{R}} c\phi$$

and so $(a\phi, b\phi, c\phi) \in \mathcal{T}(P, S\phi)$. Thus $(*)$ holds so that ϕ is extendable by Theorem 3.3. □

As in the **Remark**, if S intersects every $\mathcal{R}$-class of Q, certainly S is $\mathcal{R}$-bounded in Q. In this case we can further tighten the conditions of Proposition 3.7.

Proposition 3.8. *Let S be a weak left order in Q that intersects every $\mathcal{R}$-class of Q and let $\phi : S \to P$ be a morphism. Then ϕ is extendable to a morphism from Q to P if and only if*
for all $b, d \in S$,

$$b \,\mathcal{R}\, d \text{ in } Q \text{ implies that } b\phi \,\mathcal{R}\, d\phi \text{ in } P$$

and for all $a \in S$,

$$a \in \mathcal{H}(Q) \text{ implies that } a\phi \in \mathcal{H}(P).$$

Proof. In view of Proposition 3.7 we need only show that if ϕ preserves $\mathcal{R}$, then it preserves $\leq_{\mathcal{R}}$.

Let $b, d \in S$ with $b \leq_{\mathcal{R}} d$ in Q. Then $b = d$ and so $b\phi \leq_{\mathcal{R}} d\phi$ in P, or

$$b = dq \text{ for some } q \in Q.$$

By assumption, $q \,\mathcal{R}\, w$ for some $w \in S$, so that

$$b \,\mathcal{R}\, dw.$$

Since ϕ preserves $\mathcal{R}$ we have $b\phi \,\mathcal{R}\, d\phi w\phi$ in P so that in this case also, $b\phi \leq_{\mathcal{R}} d\phi$ in P. □

Notice that the hypotheses of the above proposition are certainly fulfilled if S is a straight weak left order in Q and $\phi : S \to P$ is a morphism such that for all $b, d \in S$

$$b \,\mathcal{R}\, d \text{ in } Q \text{ implies that } b\phi \,\mathcal{R}\, d\phi \text{ in } P$$

and

$$b \,\mathcal{L}\, d \text{ in } Q \text{ implies that } b\phi \,\mathcal{L}\, d\phi \text{ in } P.$$

This has a translation into the terminology of induced $*$-pairs.

We use Proposition 3.8 to obtain Theorem 3.2 of [1].

Corollary 3.9. *Let S be a left order in a completely 0-simple semigroup Q and let $\phi : S \to T$ be a 0-restricted morphism from S to a completely 0-simple semigroup T such that for all $a, b \in S$*

$$a \,\mathcal{R}\, b \text{ in } Q \text{ implies that } a\phi \,\mathcal{R}\, b\phi \text{ in } T.$$

Then ϕ has a unique extension to a morphism $\overline{\phi} : Q \to T$. Further, if $S\phi$ is a left order in T, then $\overline{\phi}$ is onto.

Proof. By Proposition 2 of [9], S is a straight left order in Q, so that S intersects every $\mathcal{R}$-class of Q. Let $a \in S \cap \mathcal{H}(Q)$. Then $a \,\mathcal{H}\, a^2$ so that by hypothesis, $a\phi \,\mathcal{R}\, (a\phi)^2$ in T. Since T is completely 0-simple, it follows that $a\phi \,\mathcal{H}\, (a\phi)^2$ in T. By Proposition 3.8, ϕ is extendable to a morphism $\overline{\phi}$ from Q to T, which by Lemma 3.2 is unique.

Suppose now that $S\phi$ is a left order in T. Let $a \in S$ and suppose that $a\phi \in \mathcal{H}(T)$. If $a\phi = 0$, then as ϕ is 0-restricted, $a = 0$ and certainly $a \in \mathcal{H}(Q)$. If $a\phi \neq 0$, then $a^2\phi = (a\phi)^2 \neq 0$ and $a^2 \neq 0$. But Q is completely 0-simple, so that $a \in \mathcal{H}(Q)$. Since if ϕ is extendable, $(*)$ holds by Theorem 3.3. Corollary 3.6 now gives that $\overline{\phi}$ is onto. □

We now consider the case where ϕ is an embedding.

Corollary 3.10. *Let S be a left order in Q and let $\phi : S \to P$ be an embedding such that $S\phi$ is a weak left order in P. The following conditions are equivalent:*
(i) ϕ can be extended to Q;
(ii) $(*)$ holds;
(iii) ϕ can be extended to an onto morphism from Q to P.

Proof. The equivalence of the first two conditions comes from Theorem 3.3. It remains to prove that these conditions imply (iii). Suppose then that ϕ is extendable and $a\phi \in \mathcal{H}(P)$. Then $a\phi$ is square cancellable in $S\phi$ so that as ϕ is an embedding, a is square cancellable in S. But S is a left order in Q, so that $a \in \mathcal{H}(Q)$. The result follows from Corollary 3.6. □

Notice that if any of the conditions in the previous corollary hold, then $S\phi$ is a left order in P. The specialisation of Propositions 3.7, 3.8 and Corollary 3.10 to the case where S is commutative gives Theorem 5.1 of [3].

Corollary 3.11. *Let S be a weak left order in Q that intersects every $\mathcal{R}$-class of Q. Suppose that $\phi : S \to P$ is an embedding and $\mathcal{S}(S\phi) = \mathcal{H}(P) \cap S\phi$. Then ϕ can be extended to a morphism from Q to P if and only if for all $b, d \in S$,*

$$b \,\mathcal{R}\, d \text{ in } Q \text{ implies that } b\phi \,\mathcal{R}\, d\phi \text{ in } P.$$

Proof. Let $a \in \mathcal{H}(Q)$. Then $a \in \mathcal{S}(S)$ so that $a\phi \in \mathcal{S}(S\phi) = \mathcal{H}(P) \cap S\phi$ by assumption. The result follows from Proposition 3.8. □

The proof of the next lemma follows immediately from Lemma 3.1.

Lemma 3.12. *Let S be a weak left or right order in Q and let $\phi : S \to P$ be an embedding such that $S\phi$ is a weak left or right order in P. If both ϕ and ϕ^{-1} are extendable to Q and P respectively, then they are mutually inverse isomorphisms.*

We can use Lemma 3.1 to specialise Theorem 3.3 and Corollary 3.11 to answer the question of when two semigroups of (weak) left quotients of a given semigroup are isomorphic. These results appear in [7]. For the purposes of this paper we need a different application of Corollary 3.11.

Theorem 3.13. *Let S be a left order in Q that intersects every $\mathcal{R}$-class of Q and let $\phi : S \to P$ be an embedding such that $S\phi$ is a* right *order in P that intersects every $\mathcal{L}$-class of P. Suppose in addition that for all $b, d \in S$,*

$$b \,\mathcal{R}\, d \text{ in } Q \text{ implies that } b\phi \,\mathcal{R}\, d\phi \text{ in } P$$

and

$$b\phi \,\mathcal{L}\, d\phi \text{ in } P \text{ implies that } b \,\mathcal{L}\, d \text{ in } Q.$$

Then ϕ can be extended to an isomorphism from Q to P. In particular, S is also a right order in Q and Q is a semigroup of quotients of S.

Proof. Corollary 3.11 and its dual give that, ϕ and ϕ^{-1} are extendable to morphisms which by Lemma 3.12 are mutually inverse isomorphisms.

Let $q \in Q$: as $\overline{\overline{\phi}}$ is onto, $q = p\overline{\overline{\phi}}$ for some $p \in P$. But $S\phi$ is a right order in P, so that $p = (d\phi)(c\phi)^\sharp$ for some $c, d \in S$. But then

$$q = p\overline{\overline{\phi}} = (d\phi(c\phi)^\sharp)\overline{\overline{\phi}} = d\phi\phi^{-1}(c\phi\phi^{-1})^\sharp = dc^\sharp$$

so that S is a right order, hence an order, in Q. □

Using the terminology of induced $*$-pairs we have the following.

Corollary 3.14. *Let S be a left order in Q that intersects every $\mathcal{R}$-class of Q and let S be a* right *order in P that intersects every $\mathcal{L}$-class of P. Suppose in addition that Q and P induce the same $*$-pair for S. Then Q is isomorphic to P under an isomorphism that restricts to the identity on S. In particular, Q is a semigroup of quotients of S.*

4. Every order in a completely regular semigroup is straight

The main results of the second author's D. Phil. thesis [13] describe straight left orders and straight orders in completely regular semigroups. These two results were proved using different techniques. In this paper we give a version of Smith's result characterising straight left orders in completely regular semigroups and use this, together with its dual and Theorem 3.13, to characterise straight orders in completely regular semigroups. In fact, this yields a description of all orders in completely regular semigroups, since we show in this section that every order in a completely regular semigroup is straight.

We will use repeatedly the fact that every completely regular semigroup Q has a unique decomposition as a semilattice Y of completely simple semigroups $Q_\alpha, \alpha \in Y$ and each of the Q_α's has a unique decomposition as a rectangular band $I_\alpha \times \Lambda_\alpha$ of groups. The Q_α's are the $\mathcal{D}$-classes of Q and the union of the I_α's (Λ_α's) indexes the $\mathcal{R}$-classes ($\mathcal{L}$-classes) of Q.

Lemma 4.1. *Let S be an order in a completely regular semigroup Q. Then S intersects every $\mathcal{H}$-class of Q.*

Proof. As remarked in Section 2, S intersects every $\mathcal{L}$-class of Q and dually, every $\mathcal{R}$-class of Q. Let $q \in Q$. Then there exist $a, b \in S$ with

$$a \,\mathcal{R}\, q \,\mathcal{L}\, b.$$

Since Q is completely regular, $ab \in H_q \cap S$, by the remarks immediately preceding this lemma. □

Lemma 4.2. *Let S be a left order in a completely regular semigroup Q. Let H be an $\mathcal{H}$-class of Q. If $S \cap H \neq \emptyset$, then $S \cap H$ is a left order in H.*

Proof. Consider Q as a semilattice Y of completely simple semigroups $Q_\alpha, \alpha \in Y$.

Let $s \in S \cap H$ and suppose that $H \subseteq Q_\alpha$. Let $q \in H$. Then $sq \in H$ and as S is a left order in Q, $sq = a^\sharp b$ for some $a, b \in S$ with $b \leq_{\mathcal{R}} a$. We know that $b \,\mathcal{L}\, q$ and as $b = ap$ for some $p \in Q$, we have $a \in Q_\beta$ for some β with $\alpha \leq \beta$. From $sq = a^\sharp b$ we have

$$b = asq \,\mathcal{R}\, as.$$

Now

$$q = s^\sharp a^\sharp b = s^\sharp a^\sharp (as)^\sharp asb =$$

$$= s^\sharp a^\sharp (as)^\sharp as^\sharp s^2 b = xv$$

where $x = s^\sharp a^\sharp (as)^\sharp as^\sharp$ and $v = s^2 b$. Notice that as $(as)^\sharp \in Q_\alpha$ and $\alpha \leq \beta$, certainly $a^\sharp (as)^\sharp a \in Q_\alpha$ so that $x \in H$. Since $b \,\mathcal{L}\, s$, we have $v = s^2 b \in H$ also.

Let $y = s^2 as$. Since $sa \in Q_\alpha$ we have $y = s(sa)s \in S \cap H$ and

$$\begin{aligned} xy &= s^\sharp a^\sharp (as)^\sharp as^\sharp s^2 as \\ &= s^\sharp a^\sharp (as)^\sharp asas \\ &= s^\sharp a^\sharp as \\ &= s^\sharp s \end{aligned}$$

since

$$s \,\mathcal{H}\, sq = a^\sharp b \leq_{\mathcal{R}} a.$$

Thus x and y are mutually inverse elements of the group H, so that

$$q = xv = y^\sharp v$$

where $y, v \in S \cap H$ as required. □

Theorem 4.3. *Let S be an order in a completely regular semigroup Q. Then S is straight in Q.*

Proof. Let H be an $\mathcal{H}$-class of Q. By Lemma 4.1, $S \cap H \neq \emptyset$. By Lemma 4.2 and its dual, $S \cap H$ is a left order and a right order in H. Thus any $q \in Q$ can be written as

$$q = a^\sharp b = dc^\sharp$$

for some $a, b, c, d \in S \cap H$. Since $a \,\mathcal{H}\, b$ and $c \,\mathcal{H}\, d$, certainly S is straight in Q. □

We end this section with an example of a left order in a completely regular semigroup that is not straight.

Example 4.4 Let $G = \{x^i : i \in \mathbb{Z}\}$ be the free group on single generator x and let L be the left zero semigroup on the set $\mathbb{Z}$. Extend the multiplication in G and L to $Q = G \cup L$ by defining

$$x^i z = i + z$$

and

$$zx^i = z$$

for all $i, z \in \mathbb{Z}$. It is tedious but routine to check that Q is a semigroup, which is clearly completely regular.

Let

$$S = \{x^i : i \geq 0\} \cup \{z \in \mathbb{Z} : z \geq 0\}.$$

Clearly, S is a subsemigroup of Q. We claim that S is a left order in Q.

For any $g \in G$, if $g = x^i$ where $i \geq 0$, then $g = (x^0)^{-1}x^i$, where $x^0, x^i \in S$. On the other hand, if $i < 0$, then $g = (x^{-i})^{-1}x^0$ and $x^{-i}, x^0 \in S$. If $z \in \mathbb{Z}$ and $z \geq 0$, certainly $z = z^{-1}z$ and $z \in S$. If $z < 0$ then

$$z = x^z 0 = (x^{-z})^{-1}0$$

and $x^{-z}, 0 \in S$. Thus S is a left order in Q.

Notice that

$$S \cap H_{-1} = S \cap \{-1\} = \emptyset$$

so that by Proposition 4 of [9], S is not straight in Q.

5. Straight left orders in completely regular semigroups

The aim of this section is to give a characterisation of straight left orders in completely regular semigroups. Our result is a version of the description of such semigroups given by the second author in her thesis [13].

Let S be a semilattice Y of semigroups $S_\alpha, \alpha \in Y$. Suppose that each S_α is a $I_\alpha \times \Lambda_\alpha$ matrix of semigroups $S_{i\lambda}, (i, \lambda) \in I_\alpha \times \Lambda_\alpha$. We define relations $\mathcal{L}'$ and $\leq_l$ on S as follows.

Definition 5.1. *For $a, b \in S$,*

$a \,\mathcal{L}'\, b \iff a \in S_{i\lambda}, b \in S_{j\lambda}$ *for some* $(i, \lambda), (j, \lambda) \in I_\alpha \times \Lambda_\alpha$ *and some* $\alpha \in Y$
and
$a \leq_l b \iff a \,\mathcal{L}'\, cb$ *for some* $c \in S$.

The relations $\mathcal{R}'$ and $\leq_r$ are defined dually. As usual we put $\mathcal{H}' = \mathcal{L}' \cap \mathcal{R}'$.

Lemma 5.2. *Let $a, b \in S$. Then $a \leq_l b$ if and only if $a \in S_\alpha, b \in S_\beta$ for some $\alpha \leq \beta$ and $a \,\mathcal{H}'\, cb$ for some $c \in S_\alpha$.*

Proof. Suppose that $a \leq_l b$. By definition, there exists $c' \in S$ with $a \in S_{i\lambda}$ and $c'b \in S_{j\lambda}$ for some $(i, \lambda), (j, \lambda) \in I_\alpha \times \Lambda_\alpha$ where $a \in S_\alpha$. It follows that $c' \in S_\gamma$ and $b \in S_\beta$ for some $\beta, \gamma \in Y$ with $\alpha \leq \beta, \gamma$. Now $c = ac' \in S_\alpha$ and $cb = ac'b \in S_{i\lambda}$ so that $a \,\mathcal{H}'\, cb$.

The converse is clear. □

Since each of the S_α's is a subsemigroup, it is clear that $\leq_l$ and $\leq_r$ are reflexive. Since $\mathcal{L}'$ and $\mathcal{R}'$ are obtained from partitions of S, they are equivalences. It is also worth pointing out that for any $a, b \in S$,

$$ab \leq_l b \text{ and } ab \leq_r a.$$

Further,

$$\mathcal{G}(S) = \{a \in S : a\,\mathcal{H}'\,a^2\} = S.$$

Lemma 5.3. *For any $a, b \in S$,*

$$a \leq_l b \text{ and } b \leq_l a$$

if and only if

$$a\,\mathcal{L}'\,b.$$

Proof. Suppose that $a \leq_l b$ and $b \leq_l a$. By Lemma 5.2, $a, b \in S_\alpha$ for some $\alpha \in Y$ and there exist $c, d \in S_\alpha$ with

$$a\,\mathcal{H}'\,cb \text{ and } b\,\mathcal{H}'\,da.$$

It follows that $a \in S_{i\lambda}$ and $b \in S_{j\lambda}$ for some $(i, \lambda), (j, \lambda) \in I_\alpha \times \Lambda_\alpha$ and so $a\,\mathcal{L}'\,b$.

Conversely, suppose that $a\,\mathcal{L}'\,b$. Then $a\,\mathcal{L}'\,b^2$ and $b\,\mathcal{L}'\,a^2$ so that by definition, $a \leq_l b$ and $b \leq_l a$. □

Lemma 5.4. *Let S be a straight left order in a completely regular semigroup Q, where Q is a semilattice Y of completely simple semigroups $Q_\alpha, \alpha \in Y$. Suppose that each Q_α is a $I_\alpha \times \Lambda_\alpha$ matrix of group $\mathcal{H}$-classes $H_{i\lambda}, (i, \lambda) \in I_\alpha \times \Lambda_\alpha$. For $\alpha \in Y$ put $S_\alpha = S \cap Q_\alpha$ and for $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$ put $S_{i\lambda} = S \cap H_{i\lambda}$. Then S is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$ and each S_α is a $I_\alpha \times \Lambda_\alpha$ matrix of right reversible cancellative semigroups $S_{i\lambda}, (i, \lambda) \in I_\alpha \times \Lambda_\alpha$.*

Further, $(\leq_l, \leq_r)$ is an embeddable $*$-pair for S, induced by Q.

Proof. By Proposition 4 of [9], for any $\mathcal{H}$-class H of Q, $S \cap H \neq \emptyset$ and $S \cap H$ is a left order in H. Thus each of the $S_{i\lambda}$'s is a left order in the group $\mathcal{H}$-class $H_{i\lambda}$ and so is right reversible and cancellative by the theorem of Ore and Dubreil. Thus S has the given decomposition.

To prove the final statement, it is enough to show that Q induces $(\leq_l, \leq_r)$. It then follows from Corollary 2.3 and Lemma 2.5 that $(\leq_l, \leq_r)$ is an embeddable $*$-pair.

Let $a, b \in S$. If $a \leq_l b$, then by definition, $a\,\mathcal{L}'\,cb$ in S so that $a \in S_{i\lambda}$ and $cb \in S_{j\lambda}$ for some $(i, \lambda), (j, \lambda) \in I_\alpha \times \Lambda_\alpha$ and $\alpha \in Y$. But $S_{i\lambda} \subseteq H_{i\lambda}$ and $S_{j\lambda} \subseteq H_{j\lambda}$ so that $a\,\mathcal{L}\,cb$ in Q and so $a \leq_{\mathcal{L}} b$ in Q.

Conversely, suppose that $a, b \in S$ and $a \leq_{\mathcal{L}} b$ in Q. Since Q is regular, $a = qb$ for some $q \in Q$ and as there exists $d \in S \cap H_q$,

$$a = qb\,\mathcal{L}\,db$$

and as Q is completely regular,

$$a\,\mathcal{H}\,adb.$$

Hence $a, adb \in S_{i\lambda}$ for some $(i,\lambda) \in I_\alpha \times \Lambda_\alpha, \alpha \in Y$ so that $a\,\mathcal{L}'\,adb$ and $a \leq_l b$ in S.

We have shown that

$$\leq_{\mathcal{L}^Q} \cap (S \times S) = \leq_l;$$

the dual argument completes the proof of the lemma. □

The main result of this section is a converse to Lemma 5.4.

Theorem 5.5. *Let S be a semilattice Y of semigroups $S_\alpha, \alpha \in Y$, such that each S_α is a $I_\alpha \times \Lambda_\alpha$ matrix of right reversible cancellative semigroups $S_{i\lambda}, (i,\lambda) \in I_\alpha \times \Lambda_\alpha$. Let $\mathcal{L}', \mathcal{R}', \leq_l$ and $\leq_r$ be as in Definition 5.1.*

The following conditions are equivalent:

(I) $(\leq_l, \leq_r)$ is an embeddable $*$-pair;

(II) S satisfies conditions (A),(B),(C) and (D) where

(A) $\leq_l$ and $\leq_r$ are transitive;

(B) $\mathcal{R}'$ is a left congruence;

(C) $\mathcal{L}' \subseteq \mathcal{L}^*$ and $\mathcal{R}' \subseteq \mathcal{R}^*$;

(D) for all $a, b, c \in S$,

$$a^2b\,\mathcal{R}'\,a^2c \text{ implies that } ab\,\mathcal{R}'\,ac;$$

(III) S is a straight left order in a semigroup Q inducing $(\leq_l, \leq_r)$.

If any of these conditions hold, then Q is completely regular and is a semilattice Y of completely simple semigroups $Q_\alpha, \alpha \in Y$. Each S_α is a left order in Q_α and Q_α is a $I_\alpha \times \Lambda_\alpha$ matrix of groups $H_{i\lambda}, (i,\lambda) \in I_\alpha \times \Lambda_\alpha$ such that $S_{i\lambda} = S \cap H_{i\lambda}$.

Proof. (I) implies (II) Conditions (A), (B) and (C) are immediate from the definition of $*$-pair. Let $a, b, c \in S$ and suppose that $a^2b\,\mathcal{R}'\,a^2c$. We know that $ab, ac \leq_{\mathcal{R}} a$ and $\mathcal{G}(S) = S$, so by (Evii)(r), $ab\,\mathcal{R}'\,ac$. Thus (D) holds.

(II) implies (I) We must show that the remaining conditions hold for $(\leq_l, \leq_r)$ to be an embeddable $*$-pair. We proceed via a series of lemmas.

Lemma 5.6. *The relation $\mathcal{L}'$ is a right congruence.*

Proof. Let $a, b, c \in S$ and suppose that $a\,\mathcal{L}'\,b$. Then $a \in S_{i\lambda}$ and $b \in S_{j\lambda}$ for some $(i,\lambda), (j,\lambda) \in I_\alpha \times \Lambda_\alpha$ for some $\alpha \in Y$. Now $ab \in S_{i\lambda}$ and as $S_{i\lambda}$ is right reversible, there are elements $x, y \in S_{i\lambda}$ with

$$xa = yab.$$

Suppose that $c \in S_\beta$. We have

$$(cx)(ac) = (cya)(bc)$$

and $cx, ac, cya, bc \in S_{\alpha\beta}$ so that as $S_{\alpha\beta}$ is a $I_{\alpha\beta} \times \Lambda_{\alpha\beta}$ matrix of semigroups,

$$ac\,\mathcal{L}'\,(cx)(ac) = (cya)(bc)\,\mathcal{L}'\,bc$$

as required. □

Lemma 5.7. *The relations $\leq_l$ and $\leq_r$ are respectively right and left compatible preorders.*

Proof. We know that $\leq_l$ is reflexive, by (A) $\leq_l$ is transitive, hence a preorder. Let $a, b, c \in S$ and suppose that $a \leq_l b$. Then $a \, \mathcal{L}' \, db$ for some $d \in S$. From Lemma 5.6, $\mathcal{L}'$ is a right congruence, so that $ac \, \mathcal{L}' \, dbc$ and by definition of $\leq_l$, $ac \leq_l bc$. The argument for $\leq_r$ is dual, using (B) in place of Lemma 5.6. □

By Lemma 5.3, $\mathcal{L}'$ and $\mathcal{R}'$ are the equivalence relations associated with the preorders $\leq_l$ and $\leq_r$ respectively.

Lemma 5.8. *The pair $(\leq_l, \leq_r)$ is a $*$-pair.*

Proof. We need only show that $\leq_l \subseteq \leq_{\mathcal{L}^*}$ and $\leq_r \subseteq \leq_{\mathcal{R}^*}$.

Let $a, b \in S$ with $a \leq_l b$ and suppose that $x, y \in S^1$ are such that $bx = by$. Now $a \, \mathcal{L}' \, cb$ for some $c \in S$ and as $\mathcal{L}' \subseteq \mathcal{L}^*$ by (C), $a \, \mathcal{L}^* \, cb$. Thus $cbx = cby$ gives $ax = ay$ so that $a \leq_{\mathcal{L}^*} b$.

The proof for $\leq_r$ and $\leq_{\mathcal{R}^*}$ is dual. □

Since each S_α is a union of $\mathcal{L}'$-classes and a union of $\mathcal{R}'$-classes and is a $I_\alpha \times \Lambda_\alpha$ matrix of semigroups, it is easy to see that

$$\mathcal{L}' \circ \mathcal{R}' = \mathcal{R}' \circ \mathcal{L}',$$

that is, condition (Ei) from Lemma 2.4 holds. Clearly (Eii)(l) and (Eii)(r) hold by defintion of $\leq_l$ and $\leq_r$. Since $S = \mathcal{G}(S)$, (Eiii) is immediate.

Lemma 5.9. *For $b \in S_\alpha$ and $a \in S_\beta$ with $\alpha \leq \beta$, we have*

$$ab \, \mathcal{L}' \, b \, \mathcal{R}' \, ba.$$

Proof. Certainly $ab \in S_\alpha$ and

$$b \, \mathcal{R}' \, bab \leq_r ba \leq_r b.$$

By Lemma 5.3, $b \, \mathcal{R}' \, ba$. Dually for ab. □

Notice that if $b \in S_\alpha$ and $a \in S_\beta$ and $b \leq_l a$ or $b \leq_r a$, then $\alpha \leq \beta$. Thus (Ev)(l) and (Ev)(r) follow from Lemma 5.9.

We prove that (Evii)(l) and (Evii)(r) hold in order to prove that (Evi)(l) and (Evi)(r) hold. Condition (Evii)(r) is straightforward: if $a, b, c \in S, b, c \leq_r a$ and $ab \, \mathcal{R}' \, ac$, then there exist $x, y \in S$ with $b \, \mathcal{R}' \, ax$ and $c \, \mathcal{R}' \, ay$. By condition (B),

$$a^2x \, \mathcal{R}' \, ab \, \mathcal{R}' \, ac \, \mathcal{R}' \, a^2y$$

so that by (D),

$$b \, \mathcal{R}' \, ax \, \mathcal{R}' \, ay \, \mathcal{R}' \, c.$$

Lemma 5.10. *Condition (Evii)(l) holds.*

Proof. Let $a, b, c \in S$ with $b, c \leq_l a$ and suppose that $ba\, \mathcal{L}'\, ca$. Then $b, c \in S_\alpha$ and $a \in S_\beta$ for some $\alpha \leq \beta$ and there exist $x, y \in S$ with $b\, \mathcal{L}'\, xa$ and $c\, \mathcal{L}'\, ya$. By Lemma 5.2 we may assume $x, y \in S_\alpha$. Since $\mathcal{L}'$ is a right congruence,

$$xa^2\, \mathcal{L}'\, ba\, \mathcal{L}'\, ca\, \mathcal{L}'\, ya^2$$

so that $xa^2 \in S_{i\lambda}, ya^2 \in S_{j\lambda}$ for some $(i, \lambda), (j, \lambda) \in I_\alpha \times \Lambda_\alpha$. Then $xa^2ya^2 \in S_{i\lambda}$ and as $S_{i\lambda}$ is right reversible

$$hxa^2ya^2 = kxa^2$$

for some $h, k \in S_{i\lambda}$. But $\mathcal{R}' \subseteq \mathcal{R}^*$ so that $a\, \mathcal{R}^*\, a^2$ and we have

$$hxa^2ya = kxa.$$

Since $hxa^2, ya, k, xa \in S_\alpha$,

$$b\, \mathcal{L}'\, xa\, \mathcal{L}'\, kxa = hxa^2ya\, \mathcal{L}'\, ya\, \mathcal{L}'\, c$$

as required. □

Lemma 5.11. *Condition (Evi)(l) holds.*

Proof. Let $a, b, c \in S$ with $b, c \leq_l a$ and $ba = ca$. Then $b, c \in S_\alpha$ and $a \in S_\beta$ for some $\alpha \leq \beta$. Certainly $ba\, \mathcal{L}'\, ca$ so that by (Evii)(l), $b\, \mathcal{L}'\, c$. By Lemma 5.9,

$$b\, \mathcal{R}'\, ba = ca\, \mathcal{R}'\, c$$

so that $b, c \in S_{i\lambda}$ for some $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$. Since $S_{i\lambda}$ is right reversible, $xb = yc$ for some $x, y \in S_{i\lambda}$. Now

$$xbab = ycab = ybab$$

and $bab \in S_{i\lambda}$, so that as $S_{i\lambda}$ is cancellative, $x = y$. Now $xb = yc$ gives $xb = xc$ and cancelling again in $S_{i\lambda}$, $b = c$ as required. □

Lemma 5.12. *Condition (Evi)(r) holds.*

Proof. Let $a, b, c \in S$ with $b, c \leq_r a$ and $ab = ac$. Then $b, c \in S_\alpha$ and $a \in S_\beta$ for some $\alpha \leq \beta$. As in Lemma 5.11, $b\, \mathcal{H}\, c\, \mathcal{H}'\, bab$. Thus there exist $u, v \in S_{i\lambda}$ with

$$ubab = vb.$$

Now $b\, \mathcal{R}'\, c$ and $\mathcal{R}' \subseteq \mathcal{R}^*$ so that

$$vb = ubab = ubac = vc.$$

Since $S_{i\lambda}$ is cancellative, $b = c$. □

This completes the proof that $(\leq_l, \leq_r)$ is an embeddable $*$-pair.

The proof of (III) implies (I) is immediate in view of Lemma 2.5. To prove the final implication of Theorem 5.5 we call upon Theorem 4.1 of [8], which for convenience we state here.

Result 5.13. *Let S be a semigroup having an embeddable $*$-pair $(\leq_l, \leq_r)$. Then S is a straight left order in a semigroup Q inducing $(\leq_l, \leq_r)$ if and only if S satisfies the following conditions.*

(Gi) $\mathcal{S}(S) = \mathcal{G}(S)$.

(Gii) If $a \in \mathcal{S}(S)$ then H'_a is right reversible.

(Giii) If $b, c \in S$ then $b \leq_l c$ if and only if there exist $h \in \mathcal{S}(S)$ and $k \in S$ with $h \,\mathcal{R}'\, k, b \leq_r h$ and $hb = kc$.

(Giv) If $a, c \in \mathcal{S}(S)$ and $b \in S$ with $a \,\mathcal{R}'\, b$, then there exist $u, h \in \mathcal{S}(S), v, k \in S$ with $h \,\mathcal{R}'\, k \,\mathcal{R}'\, u \,\mathcal{R}'\, v, au \,\mathcal{R}'\, bc, v \leq_l c, u \leq_r a, k \leq_l a, hua = ka^2$ and $hvc^2 = kbc$.

We can now give the proof of (I) implies (III). Since $S = \mathcal{G}(S)$ and $\mathcal{G}(S) \subseteq \mathcal{S}(S)$ by the definition of $*$-pair, we have $S = \mathcal{S}(S)$ and (Gi) holds. The $\mathcal{H}'$-classes are the semigroups $S_{i\lambda}$ which we know are right reversible, so that (Gii) holds.

To see that (Giii) holds, suppose that $b, c \in S$ and $b \leq_l c$. By Lemma 5.3, $b \in S_\alpha, c \in S_\beta$ for some $\alpha \leq \beta$ and $b \,\mathcal{H}'\, uc$ for some $u \in S_\alpha$. Since H'_b is right reversible, there are elements $x, y \in H'_b$ such that $xb = yuc$. Now

$$b \,\mathcal{H}'\, x \,\mathcal{H}'\, y \,\mathcal{R}'\, yu;$$

put $h = x$ and $k = yu$.

Conversely, suppose that b, c, h and k are elements of S such that

$$h \,\mathcal{R}'\, k, b \leq_r h \text{ and } hb = kc.$$

By Lemmas 5.2 and 5.9

$$b \,\mathcal{L}'\, hb = kc$$

so that $b \leq_l c$. Thus (Giii) holds.

Before tackling (Giv), we need a preliminary result.

Lemma 5.14. *For any $b, c \in S$, there exist $p, q \in S$ with*

$$bc \,\mathcal{H}'\, p \,\mathcal{R}'\, q$$

and

$$pbc = qc^2.$$

Proof. Suppose that $bc \in S_\alpha$. We have

$$bc \leq_l c \,\mathcal{H}'\, c^2$$

so by Lemma 5.2, $bc \,\mathcal{H}'\, rc^2$ for some $r \in S_\alpha$. Since H'_{bc} is right reversible, there exist $s, t \in H'_{bc}$ with

$$sbc = trc^2.$$

Now $s, t \in S_\alpha$ so that

$$bc \,\mathcal{H}'\, s \,\mathcal{H}'\, t \,\mathcal{R}'\, tr$$

and the lemma holds with $p = s$ and $q = tr$. $\square$

Lemma 5.15. *Condition (Giv) holds.*

Proof. Let $a, b, c \in S$ with $a \,\mathcal{R}'\, b$. Then $a, b \in S_\alpha$ and $c \in S_\beta$ for some $\alpha, \beta \in Y$. From Lemma 5.14 there are elements $p, q \in S_{\alpha\beta}$ with

$$bc^2 \,\mathcal{H}'\, p \,\mathcal{R}'\, q$$

and $pbc^2 = qc^4$. But $c \,\mathcal{R}^*\, c^2$ so that

$$pbc = qc^3.$$

Now

$$bc \leq_r b \,\mathcal{R}'\, a \,\mathcal{H}'\, a^2$$

so by the dual of Lemma 5.2,

$$bc \,\mathcal{H}'\, a^2 t$$

for some $t \in S_{\alpha\beta}$. Put $u = atpa$ and $v = atqc$. Notice that $u, v \in S_{\alpha\beta}$,

$$v \leq_l c, u \leq_r a \text{ and } u \,\mathcal{R}'\, v \,\mathcal{R}'\, at.$$

Again by Lemma 5.14 there exist elements $h, k' \in S_{\alpha\beta}$ with

$$ua^2 \,\mathcal{H}'\, h \,\mathcal{R}'\, k'$$

and $hua^2 = k'a^4$, so that as $a^2 \,\mathcal{R}^*\, a$, $hua = k'a^3$. Put $k = k'a$. Now $k \leq_l a$ and by Lemma 5.9,

$$k \,\mathcal{R}'\, k' \,\mathcal{R}'\, h \,\mathcal{R}'\, u \,\mathcal{R}'\, v.$$

Further,

$$au = a^2tpa \,\mathcal{R}'\, a^2t \,\mathcal{R}' bc.$$

We know that $hua = ka^2$ and so $hatpa^2 = ka^2$, so that as $a^2 \,\mathcal{R}^*\, b$, $hatpb = kb$. Finally,

$$hvc^2 = hatqc^3 = hatpbc = kbc.$$

□

This completes the proof of the equivalence of conditions (I),(II) and (III).

We now suppose that any of conditions (I), (II) or (III) hold, so that S is a straight left order in Q where Q induces $(\leq_l, \leq_r)$.

In Section 2 we saw that if S is straight in Q, then S intersects every $\mathcal{H}$-class of Q. Thus if $q \in Q$, then $q \,\mathcal{H}\, s$ for some $s \in S$ and since Q induces $(\leq_l, \leq_r)$ and $s \,\mathcal{H}'\, s^2$, we have $s \,\mathcal{H}\, s^2$ in Q so that $H_q = H_s$ is a group $\mathcal{H}$-class. Hence Q is completely regular.

Let H be an $\mathcal{H}$-class of Q. Then $H \cap S = S_{i\lambda}$ for some $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$ and $\alpha \in Y$. Denote H by $H_{i\lambda}$. By Proposition 4 of [9], $S_{i\lambda}$ is a left order in the group $H_{i\lambda}$.

Let $p \in H_{i\lambda}$ and $q \in H_{j\mu}$, where $(i,\lambda) \in I_\alpha \times \Lambda_\alpha$ and $(j,\mu) \in I_\beta \times \Lambda_\beta$. Let $s \in S_{i\lambda}$ and $t \in S_{j\mu}$. Then

$$\begin{aligned} p\,\mathcal{R}\,q &\iff s\,\mathcal{R}\,t \\ &\iff s\,\mathcal{R}'\,t \\ &\iff \alpha = \beta \text{ and } i = j \end{aligned}$$

and dually,

$$p\,\mathcal{L}\,q \iff \alpha = \beta \text{ and } \lambda = \mu.$$

For $\alpha \in Y$ let

$$Q_\alpha = \bigcup\{H_{i\lambda} : (i,\lambda) \in I_\alpha \times \Lambda_\alpha\}.$$

It follows from the preceding paragraph that the Q_α's are the $\mathcal{D}$-classes of Q, so that the Q_α's are completely simple and Q is a semilattice Y of the semigroups $Q_\alpha, \alpha \in Y$.

Fix $\alpha \in Y$. Certainly $S_\alpha \subseteq Q_\alpha$ and as remarked above, for $(i,\lambda) \in I_\alpha \times \Lambda_\alpha$, $S_{i\lambda}$ is a left order in $H_{i\lambda}$. It follows that S_α is a straight left order in Q_α.

Further, we have seen that I_α indexes the $\mathcal{R}$-classes of Q_α and Λ_α the $\mathcal{L}$-classes, so that as Q_α is completely simple and therefore a matrix of group $\mathcal{H}$-classes, it follows that Q_α is a $I_\alpha \times \Lambda_\alpha$ matrix of groups $H_{i\lambda}, (i,\lambda) \in I_\alpha \times \Lambda_\alpha$. $\square$

Putting together Lemma 5.4 and Theorem 5.5 we have the following characterisation of straight left orders in completely regular semigroups.

Corollary 5.16. *Let S be a semigroup. Then S is a straight left order in a completely regular semigroup if and only if S is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$ such that each S_α is a $I_\alpha \times \Lambda_\alpha$ matrix of right reversible cancellative semigroups $S_{i\lambda}, (i,\lambda) \in I_\alpha \times \Lambda_\alpha$, such that $(\leq_l, \leq_r)$ is an embeddable $*$-pair.*

6. Orders in completely regular semigroups

In this final section, we specialise the characterisation of straight left orders given in Section 5 to the two-sided case. We use the fact that every order in a completely regular semigroup is straight, and the isomorphism result of Theorem 3.13.

Theorem 6.1. *Let S be a semilattice Y of semigroups $S_\alpha, \alpha \in Y$ such that each S_α is a $I_\alpha \times \Lambda_\alpha$ matrix of reversible cancellative semigroups $S_{i\lambda}, (i,\lambda) \in I_\alpha \times \Lambda_\alpha$. Let $\mathcal{L}', \mathcal{R}', \leq_l$ and $\leq_r$ be as in Definition 5.1. The following conditions are equivalent:*

(I) $(\leq_l, \leq_r)$ is an embeddable $*$-pair;

(II) S satisfies conditions (A) and (C) where

(A) $\leq_l$ and $\leq_r$ are transitive;

(C) $\mathcal{L}' \subseteq \mathcal{L}^*$ and $\mathcal{R}' \subseteq \mathcal{R}^*$;

(III) S is an order in a completely regular semigroup Q and Q induces $(\leq_l, \leq_r)$.

If any of these conditions hold, then Q is a semilattice Y of completely simple semigroups $Q_\alpha, \alpha \in Y$ such that each S_α is an order in Q_α and Q_α is a $I_\alpha \times \Lambda_\alpha$ matrix of groups $H_{i\lambda}$, $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$, such that $S_{i\lambda} = S \cap H_{i\lambda}$.

Proof. <u>(I) implies (II)</u> This follows from the definition of embeddable $*$-pair.

<u>(II) implies (III)</u> As in the proof of Lemma 5.6 we can show that $\mathcal{L}'$ is a right congruence, and using the left reversibility of the $S_{i\lambda}$'s, that $\mathcal{R}'$ is a left congruence. Thus condition (B) of Theorem 5.5 and its dual hold.

Suppose now that $a, b, c \in S$ and $a^2b \,\mathcal{R}'\, a^2c$. Then for some $\alpha \in Y$, $a^2b \in S_{i\lambda}, a^2c \in S_{i\mu}$ for some $(i, \lambda), (i, \mu) \in I_\alpha \times \Lambda_\alpha$. Then $a^2ba^2c \in S_{i\mu}$ and as $S_{i\mu}$ is left reversible,

$$a^2ba^2cx = a^2cy$$

for some $x, y \in S_{i\mu}$. Since $a \,\mathcal{L}^*\, a^2$ we have

$$aba^2cx = acy.$$

Thus

$$ab \,\mathcal{R}'\, aba^2cx = acy \,\mathcal{R}'\, ac$$

and condition (D) holds. Clearly we can also obtain the dual of condition (D). By Theorem 5.5, S is a straight left order in a completely regular semigroup Q such that Q induces $(\leq_l, \leq_r)$. By the dual of Theorem 5.5, S is a straight right order in P, such that P induces $(\leq_l, \leq_r)$. By Corollary 3.14, S is an order in Q.

<u>(III) implies (I)</u> By Theorem 4.3, S is straight in Q so that (I) holds by Lemma 2.5.

The remaining statements of the theorem follow immediately from Theorem 5.5. □

Using the two-sided analogue of Lemma 5.4, we obtain the following corollary.

Corollary 6.2. *Let S be a semigroup. The S is an order in a completely regular semigroup if and only if S is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$ such that each S_α is a $I_\alpha \times \Lambda_\alpha$ matrix of reversible cancellative semigroups $S_{i\lambda}, (i, \lambda) \in I_\alpha \times \Lambda_\alpha$, such that $(\leq_l, \leq_r)$ is an embeddable $*$-pair.*

A completely regular semigroup is a *band of groups* if there is a congruence ρ on Q such that Q/ρ is a band and every ρ-class is a group. In view of the comments at the beginning of Section 4, if ρ exists, it must be the relation $\mathcal{H}$. Thus a completely regular semigroup Q is a band of groups if and only if $\mathcal{H}$ is a congruence on Q.

Corollary 6.3. [10] *A semigroup S is an order in a band B of groups if and only if S is a band B of reversible cancellative semigroups.*

Proof. If S is an order in Q, where Q is a band B of groups, then by Theorem 4.3, S is straight in Q. Using the two-sided version of Lemma 5.4, S has the desired structure.

Conversely, suppose that S is a band B of reversible cancellative semigroups. Then B is a semilattice Y of rectangular bands $I_\alpha \times \Lambda_\alpha$ and S is a semilattice Y of semigroups $S_\alpha, \alpha \in Y$ where each S_α is a $I_\alpha \times \Lambda_\alpha$ matrix of reversible cancellative semigroups $S_{i\lambda}, (i, \lambda) \in I_\alpha \times \Lambda_\alpha$. Further, the relation $\mathcal{H}'$ is a congruence on S.

Let $a, b, \in S$ and suppose that $a \leq_l b$ and $b \leq_l c$. By Lemma 5.2

$$a\,\mathcal{H}'\,xb \text{ and } b\,\mathcal{H}'\,yc$$

for some $x, y \in S$. But $\mathcal{H}'$ is a congruence so that

$$a\,\mathcal{H}'\,xyc$$

and $a \leq_l c$. Thus $\leq_l$ is transitive; dually, $\leq_r$ is transitive.

Suppose now that $a, b \in S$ and $a\,\mathcal{L}'\,b$. Let $x, y \in S^1$ be such that

$$ax = ay.$$

From $a\,\mathcal{L}'\,b$ we have $ba\,\mathcal{H}'\,b$ so that

$$bx\,\mathcal{H}'\,bax = bay\,\mathcal{H}'\,by.$$

Since H'_b is right reversible, there exist $u, v \in H'_b$ with

$$uba = vb.$$

Then

$$vbx = ubax = ubay = vby$$

and so

$$(bxv)(bxv) = (bxv)(byv).$$

Now $H'_{bxv} = H'_{byv}$ is cancellative, so that

$$bxv = byv.$$

Then

$$(bx)(vx) = (by)(vx)$$

and

$$bx\,\mathcal{H}'\,vx\,\mathcal{H}'\,by$$

so that as H'_{bx} is cancellative, $bx = by$. It follows that $a\,\mathcal{L}^*\,b$. Thus $\mathcal{L}' \subseteq \mathcal{L}^*$ and dually, $\mathcal{R}' \subseteq \mathcal{R}^*$.

By Theorem 6.1, S is an order in a completely regular semigroup Q such that Q induces $(\leq_l, \leq_r)$. Moreover, Q is a semilattice Y of completely simple semigroups $Q_\alpha, \alpha \in Y$. Each S_α is an order in Q_α and Q_α is a $I_\alpha \times \Lambda_\alpha$ matrix of groups $H_{i\lambda}, (i, \lambda) \in I_\alpha \times \Lambda_\alpha$, such that $S_{i\lambda} = S \cap H_{i\lambda}$. From Theorem 4.3, S is straight in Q. From Proposition 4 of [9], $S_{i\lambda}$ is an order in $H_{i\lambda}$. In view of the comments at the beginning of this section, it is enough to show that $\mathcal{H}$ is a congruence on Q.

Let $p \in H_{i\lambda}$ and $q \in H_{j\mu}$ where $(i,\lambda) \in I_\alpha \times \Lambda_\alpha$ and $(j,\mu) \in I_\beta \times \Lambda_\beta$. Then there are elements $a, b \in S_{i\lambda}$ and $c, d \in S_{j\mu}$ such that

$$p = a^\sharp b \text{ and } q = dc^\sharp.$$

Then

$$pq = a^\sharp bdc^\sharp \,\mathcal{R}\, a^\sharp bd.$$

Now $b \,\mathcal{H}'\, ab$ so that as $\mathcal{H}'$ is a congruence, $bd \,\mathcal{H}'\, abd$ so that

$$pq \,\mathcal{R}\, a^\sharp abd = bd.$$

Dually,

$$pq \,\mathcal{L}\, bd$$

so that $pq \,\mathcal{H}\, bd$.

We can now show that $\mathcal{H}$ is a congruence on Q. Let $p, q, r \in Q$ with $p \,\mathcal{H}\, r$. Then $p, r \in H_{i\lambda}$ and $q \in H_{j\mu}$ where $(i,\lambda) \in I_\alpha \times \Lambda_\alpha$ and $(j,\mu) \in I_\beta \times \Lambda_\beta$ for some $\alpha, \beta \in Y$. There are elements $a, b, x, y \in S_{i\lambda}$ and $c, d \in S_{j\mu}$ such that

$$p = a^\sharp b, r = x^\sharp y \text{ and } q = dc^\sharp.$$

Since $\mathcal{H}'$ is a congruence on S and

$$\mathcal{H}' = \mathcal{H} \cap (S \times S),$$

we have that

$$bd \,\mathcal{H}\, yd.$$

From the above,

$$pq \,\mathcal{H}\, bd \,\mathcal{H}\, yd \,\mathcal{H}\, rq$$

so that $\mathcal{H}$ is a right congruence on Q. Dually, it is a left congruence, hence a congruence as required. □

Finally we have Theorem 3.1 of [6].

Corollary 6.4. [6] *A semigroup S is an order in a normal band of groups if and only if it is a normal band of reversible cancellative semigroups.*

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