RESTRICTION SEMIGROUPS AND INDUCTIVE CONSTELLATIONS

VICTORIA GOULD AND CHRISTOPHER HOLLINGS

Abstract. The Ehresmann-Schein-Nambooripad (ESN) Theorem, stating that the category of inverse semigroups and morphisms is isomorphic to the category of inductive groupoids and inductive functors, is a powerful tool in the study of inverse semigroups. Armstrong and Lawson have successively extended the ESN Theorem to the classes of ample, weakly ample and weakly E-ample semigroups. A semigroup in any of these classes must contain a semilattice of idempotents, but need not be regular. It is significant here that these classes are each defined by a set of conditions *and their left-right duals*.

Recently, a class of semigroups has come to the fore that is a one-sided version of the class of weakly E-ample semigroups. These semigroups appear in the literature under a number of names: in category theory they are known as *restriction semigroups*, the terminology we use here. We show that the category of restriction semigroups, together with appropriate morphisms, is isomorphic to a category of partial semigroups we dub *inductive constellations*, together with the appropriate notion of ordered map, which we call *inductive radiant*. We note that such objects have appeared outside of semigroup theory in the work of Exel. In a subsequent article we develop a theory of partial action and expansion for inductive constellations, along the lines of that of Gilbert for inductive groupoids.

Introduction

We introduce in this article the notions of *constellation* and *inductive constellation*, together with appropriate structure-preserving maps we name *radiant* and *ordered radiant*. An inductive constellation is a set with a partial binary operation and a partial order, satisfying a number of axioms reminiscent of those for inductive categories. However, we stress that an inductive constellation is not, in general, an inductive category, neither is the converse true. Our reason for studying these structures is to elucidate the class of restriction semigroups, inspired by the celebrated result below, named to reflect its diverse authorship as the *Ehresmann-Schein-Nambooripad (ESN) Theorem*.

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Theorem 0.1. (Lawson, 1998) *The category of inverse semigroups and morphisms is isomorphic to the category of inductive groupoids and inductive functors.*

We give further details of Theorem 0.1, and its generalisations relevant to our purposes, in the next section.

For a set X we denote by $\mathcal{PT}_X$ the semigroup of partial maps on X, under composition of partial functions *from left to right.* A semigroup S is a *restriction semigroup* (properly a *left restriction semigroup*) if it is (isomorphic to) a subsemigroup of $\mathcal{PT}_X$ that is closed under the (unary) operation $\alpha \mapsto \alpha^+ = I_{\operatorname{dom}\alpha}$, where $I_{\operatorname{dom}\alpha}$ is the identity mapping on $\operatorname{dom}\alpha$. We refer to idempotents of the form I_Y, $Y \subseteq X$, as *local identities* and note that $E_X = \{I_Y : Y \subseteq X\}$ forms a semilattice. Clearly, $\mathcal{PT}_X$ itself is a restriction semigroup. By the Wagner-Preston Representation Theorem, inverse semigroups are restriction semigroups, but the latter class is much wider. It includes, for example, all right cancellative monoids, indeed all unipotent monoids, that is, monoids having a single idempotent, and all semigroups that are semidirect products of semilattices by unipotent monoids. It is worth remarking that from the very nature of functions, restriction semigroups are defined in a manner that is *not* left-right dual.

Restriction semigroups appear in the literature under a plethora of names. They are first seen in the work of Schweizer and Sklar (1960, 1961, 1965, 1967) on *function systems.* The latter are one instance of algebras that arise from attempts to find axiomatisations of semigroups embedded in $\mathcal{PT}_X$ for some set X, and *enriched with additional operations.* Function systems were revisited by Schein (1970a), correcting a misconception of Schweizer and Sklar (1967). A survey of this material, in the setting of relation algebras, was given by Schein in the first ever Semigroup Forum article (Schein, 1970b) and revisited in Jackson and Stokes (to appear). A more recent (and not readily available) survey appears in Chapter 2 of the PhD thesis of the second author (Hollings, 2007a). Restriction semigroups (under another name) appear for the first time as a class in their own right in the work of Trokhimenko (1973). They arose in the early 1980s as the *type SL2 γ-semigroups* of Batbedat (1981) (see also Batbedat and Fountain (1981)). More recently, they appear in the work of Jackson and Stokes (2001) in the guise of *(left) twisted C-semigroups* and in that of Manes (2006) as *guarded semigroups*, motivated by consideration of closure operators and categories, respectively. The current authors formerly referred to restriction semigroups as *weakly left E-ample semigroups* (Gould and Hollings, to appear).

From their definition, we see that restriction semigroups are semigroups of mappings that may be equipped with an additional unary operation modelling the domain of the map. Such a modelling also appears in recent work on *domain semirings* (see for example Desharnais and Struth (2008)), in the area of theoretical computer science. The work of Manes has a forerunner in the *restriction categories* of Cockett and Lack (2002), from whose work we take the terminology 'restriction semigroup'. Cockett and Lack were also influenced by considerations of theoretical computer science. If one takes a restriction category $\mathbf{C}$, puts $S = \mathrm{Mor}\ \mathbf{C} \cup \{0\}$, where 0 is an element not appearing in Mor $\mathbf{C}$, then extends the partial binary operation in $\mathbf{C}$ to S by putting all undefined products to be 0, then one obtains a restriction semigroup. Conversely, any restriction monoid may be thought of as a restriction category with one object. This is an elementary indication of the close connection between restriction semigroups and restriction categories. Cockett has suggested another connection, reminiscent of our construction of constellations (Cockett, 2008). Cockett's construction produces a category from a restriction semigroup, but uses rather more machinery than we require. Another means of producing a category from a restriction semigroup is given in Jackson and Stokes (2003); the correlation with a restriction semigroup is, however, not a bijection on the underlying sets, due to the fact that the codomain of an element in a restriction semigroup cannot be well defined in general. We comment further on the construction in Jackson and Stokes (2003) in Section 4.

The terminology 'weakly left E-ample' was first used in Fountain et. al. (1999), and was arrived at from the starting point of the left ample semigroups of Fountain (1977, 1979) via the route of replacing considerations of the relation $\mathcal{R}^*$ on a semigroup S by those of $\widetilde{\mathcal{R}}$ (hence the 'weakly') and by making reference to a specific set of idempotents E (which may not be the whole of $E(S)$).

A semigroup that is both a restriction semigroup, and which satisfies the left-right dual conditions together with a compatibility condition, explained in Section 1, is called a *two-sided restriction semigroup* or, in alternative terminology, a *weakly E-ample semigroup.* As we explain below, Lawson (1991) extended Theorem 0.1 to the class of two-sided restriction semigroups.

As may be seen from their representation as semigroups of partial maps, a restriction semigroup is defined by a set of conditions that are not left-right dual. Herein lies the difficulty in developing an 'ESN' type theorem — categories are defined via a symmetric set of axioms, and as such will not do for us here. It is these considerations that lead us to develop the notions of inductive constellation and ordered radiant. Our main result, Theorem 4.13, shows that the category of restriction semigroups and appropriate morphisms is isomorphic to the category of inductive constellations and ordered radiants.

The structure of the paper is as follows. In Section 1 we give further details of restriction semigroups and related classes, and comment further on Theorem 0.1 and its generalisations to certain classes of non-regular semigroups. In Section 2 we introduce constellations and radiants, giving a number of examples including one arising from partial mappings of a set; we show later that this constellation is inductive, and indeed the canonical such. Section 3 provides the specialisation to inductive constellations and inductive radiants, proving a number of technical results. The next section presents Theorem 4.13, which is the analogue of

Theorem 0.1 for restriction semigroups and inductive constellations. Finally in Section 5 we briefly specialise our construction to weakly left ample, left ample, and inverse semigroups.

Gilbert (2005) studies partial actions of inductive groupoids to inform that of partial actions of inverse semigroups. He proves in particular that the partial action of an inductive groupoid I lifts to an action of an inductive groupoid which we call $\mathrm{Sz}_G(I)$. This result involving the 'expansion' of I to $\mathrm{Sz}_G(I)$ is analogous to those previously obtained for groups (Kellendonk and Lawson, 2004) and inverse semigroups (Lawson et. al., 2006), and subsequently proved for monoids (Hollings, 2007b) and restriction semigroups (Gould and Hollings, to appear). Gilbert connects his approach using inductive groupoids with that for inverse semigroups in a very natural way. In a subsequent paper, we detail a similar expansion result for actions and partial actions of inductive constellations, making the connection with the approach for restriction semigroups as described in Gould and Hollings (to appear).

We assume in this paper only a very elementary knowledge of algebraic semigroup theory, most technical terms being explained when used. The exception is the notion of an inverse semigroup, but this is essentially used only for illustration: for further details we recommend Lawson (1998). We remark that in this article we consider both general categories and small categories: where we are aiming to associate the morphisms of a category with the elements of a semigroup we will always assume, without explicit mention, that the category is small.

1. Preliminaries

We have introduced a restriction semigroup S via its representation as a subsemigroup of $\mathcal{PT}_X$ (for some set X), closed under $\alpha \mapsto \alpha^+ = I_{\operatorname{dom}\alpha}$. Here we present without proof abstract descriptions of restriction semigroups, and those of semigroups in the more specialised classes mentioned in this paper. Until the final section these latter classes are used only by way of illustration, so the unfamiliar reader may safely bypass their definitions until that point. Further details of the material and claims made in the rest of this section may be found in the collection of notes Gould (2007).

Formally, a *left restriction semigroup* is an algebra of type $(2,1)$, that is, possessing one binary and one unary operation, where the binary operation is denoted by juxtaposition and the unary operation by $a \mapsto a^+$, satisfying the identities (first appearing in Jackson and Stokes (2001)):

$$(xy)z = x(yz),$$

$$x^+x = x,\ x^+y^+ = y^+x^+,\ (x^+y)^+ = x^+y^+ \text{ and } xy^+ = (xy)^+x.$$

Thus left restriction semigroups are a variety of algebras; we require that morphisms between restriction semigroups preserve both basic operations. To stress this, we may refer to them as $(2,1)$-*morphisms*.

Let S be a left restriction semigroup; it is clear from the first identity that S is indeed a semigroup! We let

$$E = \{x^+ : x \in S\}.$$

Notice that for any $x^+ \in E$,

$$x^+x^+ = (x^+x)^+ = x^+,$$

so that as elements of E commute, $E \subseteq E(S)$ is a semilattice. We say that E is the *distinguished semilattice* of the left restriction semigroup S.

Right restriction semigroups are defined dually; we denote the unary operation in this case by $a \mapsto a^*$. An algebra of type $(2, 1, 1)$ with unary operations $a \mapsto a^+$ and $a \mapsto a^*$ is a *two-sided restriction semigroup* if it is a left restriction semigroup with respect to $^+$ and a right restriction semigroup with respect to * *and the distinguished semilattices coincide*. The reader may see that this latter condition is equivalent to the identities

$$(x^+)^* = x^+ \text{ and } (x^*)^+ = x^*.$$

Morphisms between two-sided restriction semigroups must preserve the three basic operations, that is, they must be $(2, 1, 1)$-morphisms.

For ease of reference in this article we omit the 'left' in the term 'left restriction semigroup'.

Another approach to restriction semigroups and related classes, of which we implicitly make constant use, is via the generalisations $\mathcal{R}^*, \mathcal{L}^*, \widetilde{\mathcal{R}}_U$ and $\widetilde{\mathcal{L}}_U$ of Green's relations $\mathcal{R}$ and $\mathcal{L}$.

Let S be a semigroup with subset of idempotents U. The relation $\widetilde{\mathcal{R}}_U$ on S is defined by the rule that for any $a, b \in S$, $a\,\widetilde{\mathcal{R}}_U\,b$ if and only if for all $e \in U$,

$$ea = a \text{ if and only if } eb = b;$$

the relation $\mathcal{R}^*$ is defined by the rule that $a\,\mathcal{R}^*\,b$ if and only if for any $x, y \in S^1$,

$$xa = ya \text{ if and only if } xb = yb.$$

It is easy to see that

$$\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_U,$$

with equality if S is regular and $U = E(S)$. Another useful observation is that if $a \in S$ and $e \in U$, then $a\,\widetilde{\mathcal{R}}_U\,e$ if and only if $ea = a$ and for any $f \in U$, $fa = f$ implies that $fe = e$. The relations $\widetilde{\mathcal{L}}_U$ and $\mathcal{L}^*$ are defined dually. In case that $U = E(S)$ we may drop the subscript '$E(S)$' and write $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$ for $\widetilde{\mathcal{R}}_{E(S)}$ and $\widetilde{\mathcal{L}}_{E(S)}$. It is easy to see that (as is the case for $\mathcal{R}$ and $\mathcal{L}$) $\mathcal{R}^*$ and $\mathcal{L}^*$ are, respectively, left and right congruences. On the other hand, $\widetilde{\mathcal{R}}_U$ and $\widetilde{\mathcal{L}}_U$ need not be; if $\widetilde{\mathcal{R}}_U$ is a left congruence, then we say that *Condition (CL) holds (with respect to U)*, defining Condition (CR) dually.

Let S be a semigroup and let $E \subseteq E(S)$ be a semilattice; note that we are not assuming that $E = E(S)$. It is easy to see that if $e, f \in E$, then $e\,\widetilde{\mathcal{R}}_E\,f$ (or

$e\,\widetilde{\mathcal{L}}_E\,f$) if and only if $e = f$. Thus if S has the property that every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E, then we may define a unary operation on S by $a \mapsto a^+ \in E$, where $a\,\widetilde{\mathcal{R}}_E\,a^+$.

Proposition 1.1. *The following are equivalent for a semigroup S:*

(1) S may be equipped with a unary operation $^+$ such that $(S, \cdot, ^+)$ is a restriction semigroup;

(2) S is isomorphic to a subsemigroup of $\mathcal{PT}_X$, closed under $\alpha \mapsto \alpha^+$, for some set X;

(2) S may be equipped with a unary operation $^+$ such that $E = \{a^+ : a \in S\}$ is a semilattice, every $\widetilde{\mathcal{R}}_E$-class contains an idempotent of E, Condition (CL) holds, and S satisfies Condition (AL):

$$ae = (ae)^+a$$

for all $a \in S$ and $e \in E$.

Condition (AL) is known (for historical reasons) as the *(left) ample* condition. Roughly speaking, it gives some control over the position of idempotents in products.

For convenience we quote a result we use repeatedly, and which follows immediately from the facts that in a restriction semigroup S, $\widetilde{\mathcal{R}}_E$ is a left congruence, and for any $a \in S$, a^+ is the minimum left identity of a lying in E.

Lemma 1.2. *Let S be a restriction semigroup. Then for any $a, b \in S$,*

$$(ab)^+ = (ab^+)^+ \text{ and } (ab)^+ \leq a^+.$$

We record here that a restriction semigroup comes equipped with a natural partial order, defined by

$$a \leq b \text{ if } a = eb \text{ for some } e \in E.$$

That is, $\leq$ is a partial order, compatible with the semigroup multiplication, that restricts to the semilattice partial order on E. Notice that if $a, b \in S$ and $a \leq b$, that is, $a = eb$ for some $e \in E$, then

$$a^+ = (eb)^+ = eb^+ \leq b^+ \text{ and } a^+ \leq e;$$

consequently,

$$a^+b = a^+eb = a^+a = a.$$

In the spirit of Proposition 1.1 we define two further classes.

Definition 1.3. A semigroup S is *weakly left ample* if it is a restriction semigroup with $E = E(S)$ and *left ample* if in addition $\widetilde{\mathcal{R}} = \mathcal{R}^*$.

We remark that the classes of weakly left ample and left ample semigroups may also be defined via representations: a semigroup is weakly left ample if and only if it is isomorphic to a subsemigroup S of some $\mathcal{PT}_X$, closed under $^+$, and such that every idempotent of S is a local identity, i.e., S satisfies the quasi-identity

$x^2 = x \to x = x^+$ (see Proposition 1.4 below). Clearly, the only idempotents in a symmetric inverse semigroup $\mathcal{I}_X$ are local identities: a semigroup is left ample if and only if it is isomorphic to a subsemigroup of some $\mathcal{I}_X$ that is closed under $^+$.

The classes of *right restriction*, *weakly right ample* and *right ample* semigroups are defined in a dual manner to the left-handed versions, making use of the relations $\widetilde{\mathcal{L}}_E, \widetilde{\mathcal{L}}$ and $\mathcal{L}^*$, the unique idempotent of E in the $\widetilde{\mathcal{L}}_E$-class of $a \in S$, where S is a right restriction semigroup, being a^*. As commented above, a semigroup is a *two-sided restriction* semigroup if it is both a left and a right restriction semigroup (with respect to the same semilattice E); if in addition $E = E(S)$ we refer to it as being *weakly ample* and if further $\widetilde{\mathcal{R}} = \mathcal{R}^*$ and $\widetilde{\mathcal{L}} = \mathcal{L}^*$ then it is said to be *ample*.

The naturalness of the class of restriction semigroups over its close rivals is apparent from the following result, the first part of which we have already observed.

Proposition 1.4. *The class of restriction semigroups is a* variety *of algebras of type* $(2, 1)$*, and the class of two-sided restriction semigroups is a* variety *of type* $(2, 1, 1)$.

On the other hand, the classes of weakly left ample and left ample semigroups are quasi-varieties *of type* $(2, 1)$*; the classes of weakly ample and ample semigroups are* quasi-varieties *of type* $(2, 1, 1)$.

It is clear from our comments concerning representation that inverse semigroups are ample. Let S be an inverse semigroup, with binary operation denoted by juxtaposition. A favoured approach to studying S is to 'throw away' some of the products and consider only products of the form

$$a \cdot b = ab \text{ where } a^{-1}a = bb^{-1}.$$

In this way $(S, \cdot)$ becomes a category, in which the domain and range of elements are given by

$$\mathbf{d}(a) = aa^{-1} \text{ and } \mathbf{r}(a) = a^{-1}a.$$

Moreover, equipped with the natural partial order inherited from the semigroup S, the category $(S, \cdot)$ forms an *inductive groupoid* $\mathbf{G}(S) = (S, \cdot, \leq)$. Conversely, if $G = (G, \cdot, \leq)$ is an inductive groupoid, then we can build an inverse semigroup $\mathbf{I}(G)$ from G. Moreover, for an inverse semigroup S, we have that $S = \mathbf{I}(\mathbf{G}(S))$, and for an inductive groupoid G, we have that $G = \mathbf{G}(\mathbf{I}(G))$. This is comprised in the crucial Theorem 0.1, due variously to Ehresmann, Schein and Nambooripad. Further details may be found in Lawson (1998).

What Theorem 0.1 is saying to us is that *some* of the products in an inverse semigroup, together with the natural partial order, are together *enough* to determine all we need to know about our inverse semigroup. Naturally such a result begs extensions, indeed Nambooripad (1979) provides the generalisation to regular semigroups. Concentrating here on the case where a semigroup contains a semilattice of idempotents, Theorem 0.1 was essentially generalised by Armstrong (1984) to ample semigroups, although she made no explicit mention of category equivalence, and by Lawson (1985, 1991) to weakly ample semigroups and finally to two-sided restriction semigroups. In the first case the required inductive categories are cancellative, that is, multiplication where defined is cancellative on both left and right, and in the second they are unipotent, that is, the local submonoids are unipotent; no restriction is required in the third case. The generalisations essentially follow the pattern laid out in Theorem 0.1. For a semigroup S in any of these classes, we may construct a category $(S, \cdot)$ by restricting products to those of the form

$$a \cdot b = ab \text{ where } a^* = b^+,$$

so that $\mathbf{d}(a) = a^+$ and $\mathbf{r}(a) = a^*$. The category $(S, \cdot)$ inherits the natural partial order possessed by the semigroup S to become an inductive category $(S, \cdot, \leq)$.

It is now clear that to extend Theorem 0.1 to the class of restriction semigroups will not be straightforward. Essentially, in trying to construct a category from a restriction semigroup S, although we could define an operation $\mathbf{d}(\cdot)$ corresponding to the domain operation, and given by $\mathbf{d}(a) = a^+$, $a \in S$, we have no notion of the range $\mathbf{r}(a)$. Nevertheless, by throwing away rather fewer of the products in S, we obtain a partial algebra, equipped with a partial order, satisfying one-sided conditions reminiscent of the two-sided ones which define an inductive category. We name these structures *inductive constellations* and hope that they will be of interest in their own right.

2. Constellations and radiants

Let P be a set and $\cdot$ be a partial binary operation on P. If the product $x \cdot y$ is defined, for $x, y \in P$, then we will denote the fact by '$\exists x \cdot y$'. Whenever we write '$\exists (x \cdot y) \cdot z$', for example, it will be understood that we mean $\exists x \cdot y$ *and* $\exists (x \cdot y) \cdot z$. An element $e \in P$ is *idempotent* if $\exists e \cdot e$ and $e \cdot e = e$; the collection of all idempotents of P will be denoted by $E(P)$. An idempotent $e \in P$ is a *left identity* for $x \in P$ if $\exists e \cdot x$ and $e \cdot x = x$. A unary operation $a \mapsto a^+$ on P is said to be *image idempotent* if its image is contained in $E(P)$. In this case we say that

$$E = \{x^+ : x \in P\}$$

is *the distinguished subset* (of the unary operation).

Definition 2.1. Let P be a set, let $\cdot$ be a partial binary operation and let $^+$ be an image idempotent unary operation on P with distinguished subset E. We call $(P, \cdot, ^+)$ a *left constellation* if the following axioms hold:

(C1) $\exists x \cdot (y \cdot z) \Rightarrow \exists (x \cdot y) \cdot z$, in which case, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
(C2) $\exists x \cdot (y \cdot z) \Leftrightarrow \exists x \cdot y$ and $\exists y \cdot z$;
(C3) for each $x \in P$, x^+ is the unique left identity of x in E;
(C4) $a \in P$, $g \in E$, $\exists a \cdot g \Rightarrow a \cdot g = a$.

For ease of reference in this article we omit the 'left' and refer to a left constellation as a *constellation*; moreover we may refer to a constellation $(P, \cdot, ^+)$ more simply as 'P'. If $E = E(P)$, then we call P a *replete constellation*.

It is clear that if $e \in E$, then $e^+ = e$. We can therefore characterise E as the set of idempotents e for which $e^+ = e$; in a replete constellation, $e^+ = e$ for *all* idempotents e.

The following result shows that the $^+$ operation in a constellation mimics some of the behaviour of the domain operation $\mathbf{d}(\cdot)$ in a category:

Lemma 2.2. *In a constellation P, if $\exists a \cdot b$, then $(a \cdot b)^+ = a^+$.*

Proof. Since $\exists a^+ \cdot a$ and $\exists a \cdot b$, we have $\exists a^+ \cdot (a \cdot b)$, by (C2). Then $\exists (a^+ \cdot a) \cdot b$ and $(a^+ \cdot a) \cdot b = a^+ \cdot (a \cdot b)$, by (C1). Hence $a^+ \cdot (a \cdot b) = a \cdot b$, i.e., $(a \cdot b)^+ = a^+$. $\square$

Thus a constellation is a one-sided generalisation of a category in which we have an analogue of domain, namely $^+$, but no notion of range. This means that we can consider 'stars' in P, in the sense of Gilbert (2005, p. 177), but not the dual notion of 'costars'.[1]

Note also:

Lemma 2.3. *In a constellation P, $\exists a \cdot b \Longleftrightarrow \exists a \cdot b^+$.*

Proof. ($\Rightarrow$) Since $\exists a \cdot b$ and $b = b^+ \cdot b$, we have $\exists a \cdot (b^+ \cdot b)$. Then $\exists a \cdot b^+$, by (C2).

($\Leftarrow$) From $\exists a \cdot b^+$ and $\exists b^+ \cdot b$, we deduce that $\exists a \cdot (b^+ \cdot b)$, i.e., $\exists a \cdot b$. $\square$

We remark that any category C with set of identities C_o is certainly a constellation with distinguished subset C_o. From the results of Section 4, it will follow that we may extract a constellation from any restriction semigroup.

We end this section by offering further examples of a constellation, and a representation theorem. As remarked in the Introduction, for any set X, $\mathcal{PT}_X$ is a restriction semigroup, with distinguished semilattice E_X. We now construct a constellation $\mathcal{C}_X$ from $\mathcal{PT}_X$. *As a set*, $\mathcal{C}_X = \mathcal{PT}_X$; a restricted product is defined in $\mathcal{C}_X$ by

$$\alpha \cdot \beta = \begin{cases} \alpha\beta & \text{if } \alpha\beta^+ = \alpha \\ \text{undefined} & \text{otherwise} \end{cases}$$

or equivalently

$$\alpha \cdot \beta = \begin{cases} \alpha\beta & \text{if } \operatorname{im}\alpha \subseteq \operatorname{dom}\beta \\ \text{undefined} & \text{otherwise} \end{cases} \tag{2.1}$$

it is easy to see that $(\mathcal{C}_X, \cdot, ^+)$ is a constellation with distinguished subset of idempotents E_X, where $^+$ has the same meaning as for $\mathcal{PT}_X$. This observation was effectively made by Exel (2006).[2] We will call $\mathcal{C}_X$ the *function constellation* on X.

[1]An constellation is then, in some sense, a collection of 'stars', hence the name.

[2]The authors are grateful to M. V. Lawson for this observation.

We will see in the next section that $\mathcal{C}_X$ is in fact an inductive constellation, and in a sense made precise there, the canonical such. As regards constellations in general, we also present the following easily verifiable example, rather different in flavour.

Example 2.4. Let $(X, \leq)$ be a quasi-ordered set. We define a partial binary operation $*$ on X by the rule that

$$\exists x * y \text{ if and only if } x \leq y \text{ and then } x * y = x.$$

Setting $x^+ = x$ for all $x \in X$, $(X, *, ^+)$ is a constellation.

We observe that if X has the universal ordering, then the structure obtained as above is simply a left zero semigroup. Further, for an arbitrary quasi-order, the classes of the associated equivalence relation are left zero semigroups.

For any semigroup S we may define a quasi-order $\leq_{\mathcal{L}}$ by the rule that $a \leq_{\mathcal{L}} b$ if and only if $S^1a \subseteq S^1b$; clearly the equivalence relation associated with $\leq_{\mathcal{L}}$ is Green's relation $\mathcal{L}$. We may then construct the constellation $(S, \leq_{\mathcal{L}}, ^+)$ as in Example 2.4. It is well known that if B is a band, that is, a semigroup consisting entirely of idempotents, then the $\mathcal{L}$-classes of B are left zero semigroups; we further observe that $*$ is a restriction of the operation in B.

We now show that every constellation embeds into a direct product of a function constellation and a constellation constructed as in Example 2.4. We must first make the term 'embed' precise.

Definition 2.5. Let P and Q be constellations. A function $\rho : P \to Q$ is called a *radiant* if:

(R1) $\exists s \cdot t$ in $P \Rightarrow \exists (s\rho) \cdot (t\rho)$ in Q, in which case, $(s\rho) \cdot (t\rho) = (s \cdot t)\rho$;
(R2) $s^+\rho = (s\rho)^+$.

We note that a radiant $\rho : P \to Q$ is easily shown to map idempotents in P to idempotents in Q; condition (R2) is needed to ensure that E is mapped into F, where E is the distinguished subset of idempotents of P and F is that of Q. The radiant ρ is *strong* if, in addition, for all $s, t \in P$, if $\exists (s\rho) \cdot (t\rho)$ then $\exists s \cdot t$ and an *embedding* if it is both strong and injective.

Proposition 2.6. *Let P be a constellation. For each $s \in P$, let $\rho_s \in \mathcal{C}_P$ be defined by*

$$\operatorname{dom} \rho_s = \{x \in P : \exists x \cdot s\}$$

and for each $x \in \operatorname{dom} \rho_s$,

$$x\rho_s = x \cdot s.$$

Then $\rho : P \to \mathcal{C}_P$ given by $s\rho = \rho_s$ is a strong radiant.

Proof. Let $s, t \in P$ and suppose first that $\exists s \cdot t$. We wish to show that $\exists (s\rho) \cdot (t\rho)$, that is, $\operatorname{im} \rho_s \subseteq \operatorname{dom} \rho_t$. Let $x \cdot s \in \operatorname{im} \rho_s$. Then $\exists x \cdot s, \exists s \cdot t$ so by (C1) and (C2), $\exists (x \cdot s) \cdot t$, that is, $x \cdot s \in \operatorname{dom} \rho_t$.

We continue to assume $\exists s \cdot t$, so that also $\exists (s\rho) \cdot (t\rho)$. Let $y \in \text{dom } (s\rho) \cdot (t\rho)$, so that $\exists (y \cdot s) \cdot t$. Then by (C2), $\exists y \cdot (s \cdot t)$, so that $y \in \text{dom } (s \cdot t)\rho$. On the other hand, if $z \in \text{dom } (s \cdot t)\rho$ then (C1) gives immediately that $z \in \text{dom } (s\rho) \cdot (t\rho)$ (so that $\text{dom } (s \cdot t)\rho = \text{dom } (s\rho) \cdot (t\rho)$) and $z\rho_{s \cdot t} = (z\rho_s)\rho_t$. We conclude that $(s\rho) \cdot (t\rho) = (s \cdot t)\rho$.

Let $s \in P$; by Lemma 2.3 and the fact that $\text{dom } \alpha = \text{dom } \alpha^+$ for any $\alpha \in \mathcal{PT}_P$, we see that $\text{dom } s^+\rho = \text{dom } (s\rho)^+$. Moreover, for any x in this common domain,

$$x(s^+\rho) = x\rho_{s^+} = x \cdot s^+ = x = x(\rho_s)^+ = x(s\rho)^+,$$

so that $s^+\rho = (s\rho)^+$ and ρ is a radiant.

To see that ρ is strong, note that if $\exists (s\rho) \cdot (t\rho)$, then $\text{im } \rho_s \subseteq \text{dom } \rho_t$, so that as $s = s^+\rho_s$, we have $s \in \text{dom } \rho_t$ and so $\exists s \cdot t$ as required. □

Let $(P, \cdot, ^+)$ be a constellation with distinguished subset $E = E_P$. We wish to show that there is a radiant from P to a constellation constructed as in Example 2.4. Let E_P be ordered by the universal ordering and let $(E_P, *, ^+)$ be the associated constellation, as per the recipe of Example 2.4. By our earlier comments, $(E_P, *, ^+)$ is of course a left zero semigroup.

Proposition 2.7. *Let $(P, \cdot, ^+)$ be a constellation. Then $\nu : P \to E_P$ given by $s\nu = s^+$ is a radiant from the constellation $(P, \cdot, ^+)$ to the constellation $(E_P, *, ^+)$.*

Proof. Suppose that $\exists s \cdot t$ in P. Using Lemma 2.2, we have

$$(s \cdot t)\nu = (s \cdot t)^+ = s^+ = s^+ * t^+ = (s\nu) * (t\nu).$$

Moreover, for any $s \in P$,

$$s^+\nu = (s^+)^+ = s^+ = (s\nu)^+,$$

so that ν is indeed a radiant. □

It is clear that if $(P_i, \cdot, ^+)$ are constellations for $i \in \{0, 1, 2\}$, then $(P_1 \times P_2, \cdot, ^+)$ is a constellation, where

$$\exists (x_1, x_2) \cdot (y_1, y_2) \text{ if and only if } \exists x_1 \cdot y_1, x_2 \cdot y_2,$$

in which case

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 \cdot y_1, x_2 \cdot y_2)$$

and

$$(x, y)^+ = (x^+, y^+).$$

Moreover, if $\rho_i : P_0 \to P_i$ for $i \in \{1, 2\}$ are radiants, then $\rho : P_0 \to P_1 \times P_2$ is a radiant, where

$$p\rho = (p\rho_1, p\rho_2).$$

Corollary 2.8. *Let $(P, \cdot, ^+)$ be a constellation and let $\theta : P \to \mathcal{C}_P \times E_P$ be given by*

$$p\theta = (p\rho, p\nu),$$

where ρ and ν are as in Propositions 2.6 and 2.7, respectively. Then θ is an embedding into the product constellation $(\mathcal{C}_P \times E_P, \cdot, ^+)$.

Proof. From observations above, coupled with Propositions 2.6 and 2.7, we know that θ is a radiant. Suppose now that $\exists s\theta \cdot t\theta$; then

$$\exists (s\rho, s\nu) \cdot (t\rho, t\nu)$$

so that in particular, $\exists s\rho \cdot t\rho$, whence $\exists s \cdot t$ by Proposition 2.6. Thus θ is strong.

To see that θ is injective, suppose that $s\theta = t\theta$. Then $s\rho = t\rho$ and $s\nu = t\nu$. The latter gives us that $s^+ = t^+$ and from the former

$$s = s^+ \cdot s = s^+ \cdot t = t^+ \cdot t = t,$$

as required. □

The next observation follows easily from the fact that the composition of radiants is a radiant.

Proposition 2.9. *The class of constellations, together with radiants, forms a category* **Const**.

It is an easy exercise to show that **Const** has, for example, arbitrary products. The authors are currenly considering further properties of **Const**.

3. Inductive constellations and ordered radiants

We now introduce an ordering on a constellation, inspired by the notion of an ordered category.

Definition 3.1. Let $(P, \cdot, ^+)$ be a constellation and let $\leq$ be a partial order on P. We call $(P, \cdot, ^+, \leq)$ an *ordered* constellation if the following conditions hold:

- (O1) $a \leq c$, $b \leq d$, $\exists a \cdot b$ and $\exists c \cdot d \Rightarrow a \cdot b \leq c \cdot d$;
- (O2) $a \leq b \Rightarrow a^+ \leq b^+$;
- (O3) for $e \in E$ and $a \in P$ such that $e \leq a^+$, there exists a *restriction* $e|a$ which is the unique element with the properties $e|a \leq a$ and $(e|a)^+ = e$;
- (O4) for all $e \in E$ and all $a \in P$, there exists a *corestriction* $a|e$ which is the maximum element x with the properties $x \leq a$ and $\exists x \cdot e$;
- (O5) for $x, y \in P$ and $e \in E$, $\exists x \cdot y \Rightarrow ((x \cdot y)|e)^+ = (x|(y|e)^+)^+$;
- (O6) if $e, f \in E$, then, whenever the restriction $e|f$ is defined, it coincides with the corresponding corestriction.

We note that by (C4), $(a|e) \cdot e = a|e$.

We can deduce a useful characterisation of the partial product in an ordered constellation:

Lemma 3.2. *For $a, b \in P$,*

$$\exists a \cdot b \Longleftrightarrow a|b^+ = a.$$

Proof. First suppose that $\exists a \cdot b$. Then $\exists a \cdot b^+$, by Lemma 2.3. Recall, however, that $a|b^+$ is defined to be the maximum element x with the properties $x \leq a$ and $\exists x \cdot b^+$. We therefore deduce that $a \leq a|b^+$. Consequently, $a|b^+ = a$.

For the converse, suppose that $a|b^+ = a$. We have $\exists a \cdot b^+$, by definition of the corestriction. Then $\exists a \cdot b$, by Lemma 2.3. □

In an ordered constellation $(P, \cdot\, ^+, \leq)$, we denote by $e \wedge f$ the greatest lower bound of $e, f \in E$ with respect to $\leq$, where it exists.

Definition 3.3. Let $(P, \cdot, ^+, \leq)$ be an ordered constellation. We call $(P, \cdot, ^+, \leq)$ an *inductive* constellation if the following condition[3] holds:

(I) $e, f \in E \Rightarrow e \wedge f$ exists in E and is equal to $e|f$.

Consider now a non-trivial left zero semigroup S; as remarked in the previous section, S is a constellation where $^+$ is the identity map. If S could be ordered so that it became inductive, then for any $a, b \in S$, we have from Lemma 3.2 and Condition (I) that

$$a = a \cdot b = a|b^+ = a|b = b|a = b|a^+ = b \cdot a = b,$$

for any $a, b \in S$, a contradiction. *Thus not every constellation is inductive.*

The natural partial order on a restriction semigroup $\mathcal{PT}_X$ is easily seen to be restriction of mappings. Moreover, with this ordering, and restriction and corestriction given by

$$\epsilon|\alpha = \epsilon\alpha \text{ and } \alpha|\nu = \alpha\nu,$$

where $\epsilon, \nu \in E_X$, $\alpha \in \mathcal{PT}_X$ and $\epsilon \leq \alpha^+$, it is not hard to see that $(\mathcal{C}_X, \cdot, ^+, \leq)$ is an inductive constellation. The reader not wishing to make the necessary calculations will note that this follows from the results of the next section.

We now record some results about ordered constellations which will be of use in later sections; some of these results are one-sided analogues of those originally used by Armstrong (1984) in the (two-sided) case of cancellative categories.

Lemma 3.4. *Let P be an ordered constellation and let $a, b, c \in P$ and $e, f \in E$. Then*

(i) $a \leq b \Rightarrow a = a^+|b$;
(ii) $a \leq c$, $b \leq c$ *and* $a^+ = b^+ \Rightarrow a = b$;
(iii) $e \leq f \Rightarrow \exists e \cdot f$ *and* $e = e|f = e \cdot f$;
(iv) $f \leq e \leq a^+ \Rightarrow f|(e|a) = f|a$ *and* $f|a \leq e|a$;
(v) $f \leq e \Rightarrow (a|e)|f = a|f$ *and* $a|f \leq a|e$;
(vi) $a \leq b \Rightarrow a|e \leq b|e$.

Proof. (i) By (O2), $a^+ \leq b^+$, so $a^+|b$ is defined. This is the unique x such that $x \leq b$ and $x^+ = a^+$. Notice however that a also satisfies these conditions. Therefore, by uniqueness, $a = a^+|b$.

[3] We have been unable to show any interdependence between conditions (C1)-(C4), (O1)-(O6) and (I).

(ii) By (i), $a = a^+|c$ and $b = b^+|c$. Thus $a = a^+|c = b^+|c = b$.

(iii) We first note that $e|f$ may be regarded as either a restriction or a corestriction, by (O6), since $e \leq f$. Regarding $e|f$ as a restriction, we observe that $(e|f)^+ = e$; considering $e|f$ as a corestriction, we have $e|f \leq e$. These are properties shared by e itself, so, by (ii), $e|f = e$. We know that $\exists (e|f) \cdot f$ and that $(e|f) \cdot f = e|f$. But $e|f = e$, so $\exists e \cdot f$ and $e \cdot f = e = e|f$.

(iv) Since $f \leq e \leq a^+$, both the restrictions $f|a$ and $f|(e|a)$ are defined, as $(e|a)^+ = e$. In particular, $f|a$ is defined to be the unique $x \in P$ such that $x \leq a$ and $x^+ = f$. Notice however that $f|(e|a)$ also has these properties: $f|(e|a) \leq e|a \leq a$ and $[f|(e|a)]^+ = f$. Therefore, by uniqueness, $f|a = f|(e|a)$. The second part follows immediately from the definition of the restriction.

(v) First of all, by definition of the corestriction, we have $(a|e)|f \leq a|e \leq a$ and $\exists\, ((a|e)|f) \cdot f$, in which case,

$$(a|e)|f \leq a|f, \tag{3.1}$$

since $a|f$ is the maximum element with these properties. Next, since $\exists f \cdot e$ (by (iii)), we have $\exists\, ((a|f) \cdot f) \cdot e$ with $((a|f) \cdot f) \cdot e = (a|f) \cdot e$, hence $a|f \leq a|e$, by maximality of $a|e$. Then, since $(a|e)|f$ is the maximum element x with the properties $x \leq a|e$ and $\exists x \cdot f$, we must have $a|f \leq (a|e)|f$. Combining this with (3.1) gives the desired result.

(vi) If $a \leq b$, then we have $\exists (a|e) \cdot e$ and $a|e \leq a \leq b$. But $b|e$ is defined to be the maximum element with these properties, so $a|e \leq b|e$. □

We note that there is an alternative way of viewing the restriction of (O3):

Lemma 3.5. *For $e \in E$, if $e \leq a^+$, then $\exists e \cdot a$ and $e|a = e \cdot a$.*

Proof. Suppose that $e \leq a^+$, so that $e|a$ is defined and is the unique element with the properties $e|a \leq a$ and $(e|a)^+ = e$. By Lemma 3.4(iii), we have $\exists e \cdot a^+$. Then, by Lemma 2.3, $\exists e \cdot a$. Now, $e \cdot a \leq a^+ \cdot a = a$, and $(e \cdot a)^+ = e$, by Lemma 2.2. Therefore, by uniqueness of the restriction, we have $e|a = e \cdot a$. □

Despite this equality, we will find it useful to retain the notion of restriction which is given in Definition 3.1, since this affords us the ability to make 'uniqueness' arguments like the one used in the proof of this lemma—this is a method which will appear many times in this paper. There will be other occasions, however, when it will be more useful to consider the restriction as a product; we will switch between the two viewpoints as appropriate.

We next note a property of ordered constellations which is reminiscent of the 'left ample identity':

Lemma 3.6. *Let P be an ordered constellation and let $s \in P$, $e \in E$. Then $(s|e)^+|s = (s|e)^+ \cdot s = s|e$.*

Proof. We first observe that the restriction $(s|e)^+|s$ is defined, since $s|e \leq s$, hence $(s|e)^+ \leq s^+$. This is the unique element $x \in P$ such that $x \leq s$ and

$x^+ = (s|e)^+$. Observe, however, that $s|e$ has these properties itself. Therefore, $(s|e)^+|s = s|e$, by uniqueness of restrictions. By Lemma 3.5, we can rewrite this as $(s|e)^+ \cdot s = s|e$. □

We now record the following lemma for use in a later section:

Lemma 3.7. *In an ordered constellation P, if $\exists x \cdot y$, then*

$$(x \cdot y)|e = \big(x|(y|e)^+\big) \cdot (y|e).$$

Proof. Suppose that $\exists x \cdot y$. Then

$$\begin{aligned}
\big(x|(y|e)^+\big) \cdot (y|e) &= \Big[\big(x|(y|e)^+\big)^+ \cdot x\Big] \cdot \big((y|e)^+ \cdot y\big), \text{ by Lemma 3.6} \\
&= \Big[\Big(\big(x|(y|e)^+\big)^+ \cdot x\Big) \cdot (y|e)^+\Big] \cdot y, \text{ by (C1)} \\
&= \Big(\big(x|(y|e)^+\big)^+ \cdot x\Big) \cdot y, \text{ by (C4)} \\
&= \big(((x \cdot y)|e)^+ \cdot x\big) \cdot y, \text{ by (O5)}.
\end{aligned}$$

Now, since $\exists\, ((x \cdot y)|e)^+ \cdot x$ and $x \cdot y$, we have $\exists\, ((x \cdot y)|e)^+ \cdot (x \cdot y)$, by (C2). Then, by (C1), $\big(((x \cdot y)|e)^+ \cdot x\big) \cdot y = ((x \cdot y)|e)^+ \cdot (x \cdot y)$, hence

$$\begin{aligned}
\big(x|(y|e)^+\big) \cdot (y|e) &= ((x \cdot y)|e)^+ \cdot (x \cdot y) \\
&= (x \cdot y)|e, \text{ by Lemma 3.6},
\end{aligned}$$

as required. □

We end this section by showing that the inductive constellations $\mathcal{C}_X$ are canonical in the sense that every inductive constellation sits inside one of the form $\mathcal{C}_X$. For this to make sense, we now strengthen the notion of radiant to a type of function between inductive constellations that is the analogue of an ordered functor between inductive groupoids. We call these functions *ordered radiants*; they appear in our analogue of Theorem 0.1.

Definition 3.8. Let $\rho : P \to Q$ be a radiant of ordered constellations P and Q. We call ρ an *ordered* radiant if

(OR1) $s \le t$ in $P \Rightarrow s\rho \le t\rho$ in Q;
(OR2) $(a|e)\rho = a\rho|e\rho$, for all $a \in P$ and all $e \in E$.

(Observe that (R1) and (OR1), together with the maximality of $a\rho|e\rho$, give us $(a|e)\rho \le a\rho|e\rho$ for free.)

An ordered radiant $\rho : P \to Q$, where P and Q are inductive constellations, is *strong* if it is strong as a radiant and, in addition, if $s\rho \le t\rho$, then $s \le t$. Notice that a strong ordered radiant is necessarily injective; we refer to it as an *embedding*.

Proposition 3.9. *Let $(P, \cdot, {}^+, \le)$ be an inductive constellation. Then $\rho : P \to \mathcal{C}_P$ is an embedding.*

Proof. We know from Proposition 2.6 that ρ is a strong radiant.

Suppose that $s, t \in P$ and $s \leq t$; then by (O2) we have that $s^+ \leq t^+$ so, by Lemma 3.5, $\exists s^+ \cdot t$ and $s^+ \cdot t = s^+|t$. If $x \in \operatorname{dom} s\rho$, then $\exists x \cdot s$, so that $\exists x \cdot s^+$ also. Moreover, since $\exists s^+ \cdot t$, we deduce that $\exists (x \cdot s^+) \cdot t = x \cdot t$, i.e., $x \in \operatorname{dom} t\rho$. Next, using Lemma 3.4, we have

$$x\rho_s = x \cdot s = x \cdot (s^+|t) = x \cdot (s^+ \cdot t) = (x \cdot s^+) \cdot t = x \cdot t = x\rho_t,$$

whence $s\rho \leq t\rho$.

On the other hand, if $s\rho \leq t\rho$, then $s = s^+ \cdot t$ so that, from Lemma 3.2 and (I), we have

$$s^+ = s^+ \cdot t^+ = s^+|t^+ \leq t^+.$$

Now from (O1),

$$s = s^+ \cdot t \leq t^+ \cdot t = t.$$

Suppose now that $a \in P, e \in E$; we wish to show that (OR2) holds. As observed, $(a|e)\rho \leq a\rho|e\rho$, so we need only argue that $\operatorname{dom} a\rho|e\rho \subseteq \operatorname{dom} (a|e)\rho$. Suppose therefore that $u \in \operatorname{dom} a\rho|e\rho = \operatorname{dom} (a\rho)(e\rho)$ where the latter product is in $\mathcal{PT}_P$. This tells us that $\exists u \cdot a$ and $\exists (u \cdot a) \cdot e = u \cdot a$. It follows from (O5) that

$$u^+ = (u \cdot a)^+ = ((u \cdot a)|e)^+ = (u|(a|e)^+)^+.$$

From Lemma 3.4(ii), we deduce that $u = u|(a|e)^+$, and $\exists u \cdot (a|e)^+$ by definition of corestriction, whence finally $\exists u \cdot (a|e)$, by Lemma 2.3. Thus $u \in \operatorname{dom} (a|e)\rho$, as required. $\square$

4. Correspondence

In this section, we will establish the connection between inductive constellations and restriction semigroups. This result is a one-sided analogue of the ESN Theorem for inductive groupoids, and the theorems of Armstrong and Lawson for inductive cancellative categories and inductive (unipotent) categories, respectively. We begin by taking a restriction semigroup and constructing from it an inductive constellation.

Let S be a restriction semigroup with natural partial order relation $\leq$. We define the *restricted product* $\cdot$ in S by

$$a \cdot b = \begin{cases} ab & \text{if } ab^+ = a; \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{4.1}$$

Proposition 4.1. *If S is a restriction semigroup with natural partial order $\leq$ and unary operation $^+$, then $(S, \cdot, ^+, \leq)$ is an inductive constellation; moreover the semigroup and the constellation share the same distinguished set of idempotents.*

Proof. Let e be *any* idempotent of S (i.e., e is not necessarily in E). Then $ee^+ = (ee^+)^+e = (ee)^+e = e^+e = e$, so $\exists e \cdot e$. Moreover, $e \cdot e = e$. Thus, if e is idempotent in S, then it is idempotent in $(S, \cdot)$. Conversely, suppose that e

is an idempotent in $(S,\cdot)$ (but not necessarily in E). Then $\exists e\cdot e$ and $e\cdot e=e$, hence $ee=e$ in S. We have shown that S and $(S,\cdot)$ have precisely the same idempotents.

(C1) Suppose that $\exists x\cdot(y\cdot z)$, i.e., $yz^+=y$ and $x(yz)^+=x$. Then

$$xy^+=x(yz^+)^+=x(yz)^+=x \quad\text{and}\quad (xy)z^+=xy,$$

hence $\exists(x\cdot y)\cdot z$.

(C2)($\Rightarrow$) This follows from the proof of (C1).

(C2)($\Leftarrow$) Suppose that $\exists x\cdot y$ and $\exists y\cdot z$, i.e., $xy^+=x$ and $yz^+=y$. Then

$$x(yz)^+=x(yz^+)^+=xy^+=x,$$

hence $\exists x\cdot(y\cdot z)$.

(C3) For each $x\in S$, there is a left identity $x^+\in E$, where $^+$ is defined as in the original semigroup. For, as $x^+x^+=x^+$ we have $\exists x^+\cdot x$ and $x^+\cdot x=x$. Suppose now that there is another $e\in E$ with $\exists e\cdot x$ and $e\cdot x=x$. Since $\exists e\cdot x$, we have $ex^+=e$. But $x\,\widetilde{\mathcal{R}}_E\,x^+$ so we deduce from $ex=x$ that $ex^+=x^+$. Thus $e=x^+$. Hence every $x\in S$ has a *unique* left identity x^+ in E.

(C4) If $\exists a\cdot g$, then $ag^+=a$, by definition of the restricted product. Thus $a\cdot g=ag=ag^+=a$.

The constellation also inherits properties (O1)-(O5) and (I) from the original restriction semigroup. Properties (O1) and (O2) are immediate from the comments following Lemma 1.2.

(O3) Put $e|a=ea$. Then $e|a\le a$ and $(e|a)^+=(ea)^+=(ea^+)^+=ea^+=e$, since $e\le a^+$.

Suppose now that there is another element $x\in S$ with these properties. From $x\le a$ and $x^+=e$, we have $x=x^+a=ea$. The restriction is therefore unique.

(O4) Put $a|e=ae$. Then $a|e\le a$, since $ae=(ae)^+a\le a$, and $(a|e)e^+=aee=ae=a|e$, so $\exists(a|e)\cdot e$.

Now suppose that x is another element with the properties $x\le a$ and $\exists x\cdot e$, so that $xe=x$. Then $x=xe\le ae$, by compatibility of $\le$. Hence ae is the maximum element with the specified properties.

(O5) Suppose that $\exists x\cdot y$. We have

$$\left(x|(y|e)^+\right)^+=\left(x(ye)^+\right)^+=(xye)^+=((x\cdot y)|e)^+.$$

(O6) The restriction $e|f$ (where defined) and the corestriction $e|f$ are both equal to ef.

(I) This is true in the original restriction semigroup, with $e\wedge f=ef=e|f$. $\square$

Notice that starting with the restriction semigroup $\mathcal{PT}_X$, the recipe given in Proposition 4.1 gives the inductive constellation $\mathcal{C}_X$ presented at the end of Section 2.

Let S be a restriction semigroup. We will denote the inductive constellation associated to S by $\mathbf{P}(S)$. We pause to compare $\mathbf{P}(S)$ with the category $\mathbf{C}(S)$

constructed from S in Jackson and Stokes (2003). Translating into our terminology, the set of objects of $\mathbf{C}(S)$ is the set E^1 of distinguished idempotents of S^1, where S^1 is S with an identity adjoined if necessary. For each $a \in S$ and $e \in E^1$ with $ae = a$ there is a morphism $m_{a,e}$ of $\mathbf{C}(S)$ with $\mathbf{d}(m_{a,e}) = a^+$ and $\mathbf{r}(m_{a,e}) = e$. Composition where defined is given by

$$m_{a,e}m_{b,f} = m_{ab,f};$$

notice that in this case we have that $ae = e, e = b^+$ and $bf = b$, so that $(ab)^+ = (ab^+)^+ = (ae)^+ = a^+$ and $(ab)f = a(bf) = ab$. The identity at an object e is $m_{e,e}$.

Proposition 4.2. Jackson and Stokes (2003). *If S is a restriction semigroup, then $\mathbf{C}(S)$ constructed as above is a category.*

As remarked in Section 2, $\mathbf{C}(S)$ may be regarded as a constellation with distinguished subset of idempotents E and underlying set M, the set of morphisms of $\mathbf{C}(S)$.

Lemma 4.3. *Let S be a restriction semigroup. Define $\theta : M \to S$ by $m_{a,e}\theta = a$. Then θ is a surjective radiant from the constellation $\mathbf{C}(S)$ to the constellation $\mathbf{P}(S)$.*

Proof. Suppose that $m_{a,e}, m_{b,f} \in M$ and $\exists m_{a,e}m_{b,f}$. Then $e = b^+$ and $m_{a,e}m_{b,f} = m_{ab,f}$. We also know that $a = ae = ab^+$, so that $\exists a \cdot b$ in $\mathbf{P}(S)$. Since $m_{a,e}\theta = a$ and $m_{b,f}\theta = b$, we have that $\exists m_{a,e}\theta \cdot m_{b,f}\theta$. Further,

$$(m_{a,e}m_{b,f})\theta = m_{ab,f}\theta = ab = a \cdot b = m_{a,e}\theta \cdot m_{b,f}\theta.$$

Considering the unary operation we have that for any $m_{a,e} \in M$,

$$(m_{a,e}\theta)^+ = a^+ = m_{a^+,a^+}\theta = (m_{a,e}^+)\theta.$$

Thus θ is a radiant. Clearly θ is surjective by virtue of $1 \in E^1$. $\square$

Notice that in the above result, θ will not, in general, be an injection.

Our next aim is to prove a converse to Proposition 4.1; we must start with an inductive constellation and construct a restriction semigroup. Let $(P, \cdot, ^+, \leq)$ be such a constellation. We define the *pseudoproduct* $\otimes$ on P by

$$a \otimes b = (a|b^+) \cdot b. \tag{4.2}$$

We know from (O4) that $(a|b^+) \cdot b^+$ is always defined. Then, using Lemma 2.3, we deduce that $(a|b^+) \cdot b$ is always defined, hence the pseudoproduct is everywhere defined. From Lemma 3.2, we note that whenever the product in the constellation is defined, it coincides with the pseudoproduct. Furthermore:

Lemma 4.4. *The pseudoproduct $\otimes$ is compatible with $\leq$ in P.*

Proof. Let $a, b, c, d \in P$ and suppose that $a \leq c$ and $b \leq d$. We have $a|b^+ \leq c|b^+$, by Lemma 3.4(vi). We also have $c|b^+ \leq c|d^+$, by (O2) and Lemma 3.4(v), hence $a|b^+ \leq c|d^+$. We therefore deduce from (O1) that

$$a \otimes b = (a|b^+) \cdot b \leq (c|d^+) \cdot d = c \otimes d,$$

as required. □

We have the following converse of Proposition 4.1:

Proposition 4.5. *If $(P, \cdot, {}^+, \leq)$ is an inductive constellation, then $(P, \otimes, {}^+)$ is a restriction semigroup, where $\otimes$ is the pseudoproduct of (4.2). Moreover, the natural partial order of the restriction semigroup $(P, \otimes, {}^+)$ and of the constellation $(P, \cdot, {}^+, \leq)$ coincide.*

Proof. Let E be the distinguished subset of idempotents of P. We begin by showing that E forms a semilattice with respect to $\otimes$. Let $e \in E$, so that $\exists e \cdot e$ with $e \cdot e = e$ and $e^+ = e$. Then $e \otimes e = e \cdot e = e$, since $\cdot$ and $\otimes$ coincide whenever $\cdot$ is defined. Now let $f \in E$ also. We have

$$\begin{aligned} e \otimes f &= (e|f^+) \cdot f = (e|f) \cdot f = (e \wedge f) \cdot f, \text{by (I)} \\ &= e \wedge f, \text{by (C4)} \\ &= f \wedge e = \ldots = f \otimes e. \end{aligned}$$

We show that $\otimes$ is associative. On the one hand we have

$$\begin{aligned} a \otimes (b \otimes c) &= a \otimes \big((b|c^+) \cdot c\big) \\ &= \Big[a| \big((b|c^+) \cdot c\big)^+\Big] \cdot \big[(b|c^+) \cdot c\big] \\ &= \big[a|(b|c^+)^+\big] \cdot \big[(b|c^+) \cdot c\big], \text{ by Lemma 2.2,} \\ &= \big\{\big[a|(b|c^+)^+\big] \cdot (b|c^+)\big\} \cdot c, \text{ by (C1),} \end{aligned}$$

whilst on the other, we have

$$\begin{aligned} (a \otimes b) \otimes c &= \big[(a|b^+) \cdot b\big] \otimes c \\ &= \big[\big((a|b^+) \cdot b\big) |c^+\big] \cdot c. \end{aligned}$$

Discarding the right-hand factors of c, it is sufficient to show that

$$\big[a|(b|c^+)^+\big] \cdot (b|c^+) = \big((a|b^+) \cdot b\big) |c^+.$$

By Lemma 3.7,

$$\big((a|b^+) \cdot b\big) |c^+ = \big((a|b^+)|(b|c^+)^+\big) \cdot (b|c^+).$$

However, $b|c^+ \leq b$ so $(b|c^+)^+ \leq b^+$. Therefore, by Lemma 3.4(v),

$$(a|b^+)|(b|c^+)^+ = a|(b|c^+)^+,$$

whence

$$\big((a|b^+) \cdot b\big) |c^+ = \big[a|(b|c^+)^+\big] \cdot (b|c^+).$$

Thus $\otimes$ is associative.

We now show that $a^+ \widetilde{\mathcal{R}}_E\, a$ in $(P, \otimes, ^+)$, where $^+$ is defined as in the original constellation. Firstly, a^+ is a left identity for a:

$$a^+ \otimes a = (a^+|a^+) \cdot a = a^+ \cdot a = a,$$

by Lemma 3.4(iii). Now suppose that $e \otimes a = a$, for some $e \in E$:

$$e \otimes a = a \Rightarrow (e|a^+) \cdot a = a \Rightarrow e|a^+ = a^+.$$

Then, using (C4),

$$e \otimes a^+ = (e|a^+) \cdot a^+ = e|a^+ = a^+.$$

Hence $a^+ \widetilde{\mathcal{R}}_E\, a$.

We show that the left ample identity holds:

$$(a \otimes e)^+ \otimes a = ((a|e) \cdot e)^+ \otimes a = (a|e)^+ \otimes a = \big((a|e)^+|a^+\big) \cdot a.$$

Observe, however, that $a|e \leq a$, so $(a|e)^+ \leq a^+$, whence $(a|e)^+|a^+ = (a|e)^+$, by Lemma 3.4(iii). We then have

$$\begin{aligned}(a \otimes e)^+ \otimes a &= (a|e)^+ \cdot a \\ &= a|e, \text{ by Lemma 3.6} \\ &= (a|e) \cdot e, \text{ by (O4) and (C4)} \\ &= a \otimes e,\end{aligned}$$

hence the left ample identity holds.

We next show that $\widetilde{\mathcal{R}}_E$ is a left congruence. Suppose that $a\, \widetilde{\mathcal{R}}_E\, b$ (so that $a^+ = b^+$) and $c \in S$. Then

$$\begin{aligned}(c \otimes a)^+ = [(c|a^+) \cdot a]^+ &= (c|a^+)^+, \text{ by Lemma 2.2} \\ &= (c|b^+)^+, \text{ since } a^+ = b^+ \\ &= [(c|b^+) \cdot b]^+ = (c \otimes b)^+\end{aligned}$$

Hence $c \otimes a\, \widetilde{\mathcal{R}}_E\, c \otimes b$.

We finally confirm that the ordering $\leq$ in the original inductive constellation becomes the usual ordering of the restriction semigroup $(P, \otimes, ^+)$. Suppose that $a \leq b$ in the constellation $(P, \cdot, ^+, \leq)$. By Lemma 3.4(i), $a = a^+|b$. Then, using Lemma 3.5, we have $a = a^+ \cdot b = a^+ \otimes b$ in the semigroup $(P, \otimes, ^+)$.

Now suppose that $a \leq b$ in $(P, \otimes, ^+)$, so that $a = e \otimes b$, for some idempotent $e \in E$. Then

$$a = e \otimes b = (e|b^+) \cdot b = (e \wedge b^+) \cdot b.$$

We have $e \wedge b^+ \leq b^+$ in the constellation, and so $a \leq b^+ \cdot b = b$ in $(P, \cdot, ^+, \leq)$. Thus $(P, \cdot, ^+, \leq)$ and $(P, \otimes, ^+)$ have the same ordering. $\square$

If P is an inductive constellation, then we will denote its associated restriction semigroup by $\mathbf{T}(P)$.

Proposition 4.6. *Let S be a restriction semigroup and P be an inductive constellation. Then $\mathbf{T}(\mathbf{P}(S)) = S$ and $\mathbf{P}(\mathbf{T}(P)) = P$.*

Proof. Let the operation in S be denoted by juxtaposition. By Proposition 4.1, $\mathbf{P}(S)$ is an inductive constellation under the restricted product:

$$a \cdot b = \begin{cases} ab & \text{if } ab^+ = a; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Further, in $\mathbf{P}(S)$, the unary operation of $^+$ coincides with that of S, $e|a = ea$, $a|e = ae$ and $e \wedge f = ef$.

We now construct $\mathbf{T}(\mathbf{P}(S))$ by defining the pseudoproduct:

$$a \otimes b = (a|b^+) \cdot b.$$

By Proposition 4.5, $\mathbf{T}(\mathbf{P}(S))$ is a restriction semigroup under $\otimes$. It is clear that S and $\mathbf{T}(\mathbf{P}(S))$ share the same underlying set. Observe further that

$$a \otimes b = (a|b^+) \cdot b = (ab^+) \cdot b = (ab^+)b = ab,$$

so the operations in S and $\mathbf{T}(\mathbf{P}(S))$ are the same. Hence $S = \mathbf{T}(\mathbf{P}(S))$.

We turn now to the second part of the proposition. Let $\cdot$ denote the operation in P. We construct the restriction semigroup $\mathbf{T}(P)$ by defining the pseudoproduct $\otimes$ of (4.2).

We next define the restricted product:

$$\begin{aligned} a \odot b &= \begin{cases} a \otimes b & \text{if } a \otimes b^+ = a \\ \text{undefined} & \text{otherwise} \end{cases} \\ &= \begin{cases} (a|b^+) \cdot b & \text{if } (a|b^+) \cdot b^+ = a \\ \text{undefined} & \text{otherwise} \end{cases} \\ &= \begin{cases} (a|b^+) \cdot b & \text{if } a|b^+ = a \\ \text{undefined} & \text{otherwise} \end{cases} \\ &= a \cdot b, \end{aligned}$$

from Lemma 3.2. We know that $\mathbf{P}(\mathbf{T}(P))$ is an inductive constellation under $\odot$, and we see that $\odot$ and $\cdot$ coincide.

It is again clear that P and $\mathbf{P}(\mathbf{T}(P))$ share the same underlying set. From Propositions 4.1 and 4.5, P and $\mathbf{P}(\mathbf{T}(P))$ share the same partial order and operation of $^+$. We must now show they have the same restriction and corestriction.

Let '$|$' denote restriction and corestriction in P, and '$\|$' denote the same in $\mathbf{P}(\mathbf{T}(P))$. We then have

$$a \| e = a \otimes e = (a|e) \cdot e = a|e,$$

by (C4) in P.

We now consider the restriction. Let $e \in E$ be such that $e \leq a^+$, for some $a \in \mathbf{P}(\mathbf{T}(P))$. Then from Lemma 3.5,

$$e \| a = e \odot a = e \cdot a = e | a.$$

Thus $P = \mathbf{P}(\mathbf{T}(P))$. □

We now proceed to establish a category-theoretic correspondence between inductive constellations and restriction semigroups. Clearly:

Proposition 4.7. *The class of inductive constellations, together with ordered radiants, forms a category* **Iconst***.*

We can show that an ordered radiant ρ automatically satisfies a condition dual to (OR2):

Lemma 4.8. *Let $\rho : P \to Q$ be an ordered radiant of ordered constellations P and Q. Then $(e|a)\rho = e\rho|a\rho$, for $a \in P$ and $e \in E$ with $e \leq a^+$.*

Proof. If $e \leq a^+$, then $e\rho \leq a^+\rho = (a\rho)^+$, so the restriction $e\rho|a\rho$ is defined. We note that $e\rho|a\rho$ is the unique element x of Q with the properties $x \leq a\rho$ and $x^+ = e\rho$. Observe however that $(e|a)\rho \leq a\rho$, by (OR1), and that $(e|a)\rho^+ = (e|a)^+\rho = e\rho$. Therefore, by uniqueness, $(e|a)\rho = e\rho|a\rho$. □

The following further property of ordered radiants will prove useful very shortly:

Lemma 4.9. *Let $\rho : P \to Q$ be an ordered radiant of inductive constellations P and Q. Then ρ preserves pseudoproducts.*

Proof. Let $s, t \in P$. Then

$$\begin{aligned}
(s \otimes t)\rho = \big((s|t^+) \cdot t\big)\,\rho &= (s|t^+)\rho \cdot (t\rho), \text{ by (R1)} \\
&= (s\rho|t^+\rho) \cdot (t\rho), \text{ by (OR2)} \\
&= \big(s\rho|(t\rho)^+\big) \cdot (t\rho), \text{ by (R2)} \\
&= (s\rho) \otimes (t\rho),
\end{aligned}$$

as required. □

An ordered radiant between inductive constellations is analogous to a (2,1)-morphism between restriction semigroups. We will make this statement more precise via the following pair of propositions:

Proposition 4.10. *Let $\rho : P \to Q$ be an ordered radiant of inductive constellations P and Q. We define $\mathbf{T}(\rho) := \rho_{\mathbf{T}} : \mathbf{T}(P) \to \mathbf{T}(Q)$ to be the same function on the underlying sets. Then $\rho_{\mathbf{T}}$ is a (2,1)-morphism between the restriction semigroups $\mathbf{T}(P)$ and $\mathbf{T}(Q)$.*

Proof. We have $(a\rho_{\mathbf{T}})^+ = (a\rho)^+ = a^+\rho = a^+\rho_{\mathbf{T}}$. It follows from Lemma 4.9 that $\rho_{\mathbf{T}}$ preserves pseudoproducts. □

Proposition 4.11. *Let $\varphi : S \to T$ be a (2,1)-morphism between restriction semigroups S and T. We define $\mathbf{P}(\varphi) := \varphi_{\mathbf{P}} : \mathbf{P}(S) \to \mathbf{P}(T)$ to be the same function on the underlying sets. Then $\varphi_{\mathbf{P}}$ is an ordered radiant between the constellations $\mathbf{P}(S)$ and $\mathbf{P}(T)$.*

Proof. Suppose that $\exists s \cdot t$ in $\mathbf{P}(S)$, so that $st^+ = s$ in S. Then

$$(s\varphi)(t\varphi)^+ = (s\varphi)(t^+\varphi) = (st^+)\varphi = s\varphi$$

in T, so $\exists (s\varphi) \cdot (t\varphi)$ in $\mathbf{P}(T)$, and

$$(s\varphi_{\mathbf{P}}) \cdot (t\varphi_{\mathbf{P}}) = (s\varphi) \cdot (t\varphi) = (s\varphi)(t\varphi) = (st)\varphi = (s \cdot t)\varphi_{\mathbf{P}}.$$

It is clear that $\varphi_{\mathbf{P}}$ preserves $^+$ and order, since these are unchanged in the passage from semigroups to constellations.

Finally, we have

$$a\varphi_{\mathbf{P}}|e\varphi_{\mathbf{P}} = a\varphi|e\varphi = (a\varphi)(e\varphi) = (ae)\varphi = (a|e)\varphi_{\mathbf{P}},$$

as required. □

The following result is easy to see, given Propositions 4.10 and 4.11.

Proposition 4.12. *Let $\varphi : S \to T$ be a (2,1)-morphism of restriction semigroups and $\rho : P \to Q$ be an ordered radiant of inductive constellations. Then $\mathbf{T}(\mathbf{P}(\varphi)) = \varphi$ and $\mathbf{P}(\mathbf{T}(\rho)) = \rho$.*

We can now gather together the results of Propositions 4.1, 4.5, 4.6, 4.10, 4.11 and 4.12 into the following theorem, which is our promised analogue of Theorem 0.1.

Theorem 4.13. *The category of restriction semigroups and (2,1)-morphisms is isomorphic to the category of inductive constellations and ordered radiants.*

It is illustrative to revist Proposition 3.9 in the light of the above. Let $(P, \cdot, ^+, \leq)$ be an inductive constellation. From Theorem 6.2 of Gould (2007), a result originally due to Trokhimenko (1973),

$$\nu : \mathbf{T}(P) \to \mathcal{PT}_{\mathbf{T}(P)} = \mathcal{PT}_P,\ s \mapsto \nu_s$$

is an embedding, where

$$\text{dom } \nu_s = P \otimes s^+$$

and for each $x \in \text{dom } \nu_s$,

$$x\nu_s = x \otimes s.$$

From Proposition 4.11,

$$\nu_{\mathbf{P}} : \mathbf{P}(\mathbf{T}(P)) \to \mathbf{P}(\mathcal{PT}_P) = \mathcal{C}_P$$

is an ordered radiant, where $\nu_{\mathbf{P}}$ is the same function as ν on the underlying sets; clearly $\nu_{\mathbf{P}}$ remains injective. We claim that $\nu = \nu_{\mathbf{P}}$ coincides with ρ as introduced in Proposition 2.6. Let $s \in P$; then

$$\begin{array}{rcll} x \in \text{dom } \nu_s & \Leftrightarrow & x \in P \otimes s^+ & \\ & \Leftrightarrow & x = p \otimes s^+ & \text{for some } p \in P \\ & \Leftrightarrow & x = (p|s^+) \cdot s^+ & \text{for some } p \in P \\ & \Leftrightarrow & x = p|s^+ & \text{for some } p \in P \\ & \Leftrightarrow & \exists x \cdot s^+ & \\ & \Leftrightarrow & \exists x \cdot s & \\ & \Leftrightarrow & x \in \text{dom } \rho_s. & \end{array}$$

Moreover, for $x \in \text{dom } \nu_s = \text{dom } \rho_s$,

$$x\nu_s = x \otimes s = (x|s^+) \cdot s = x \cdot s = x\rho_s.$$

Hence $\nu_s = \rho_s$ so that $\nu = \nu_{\mathbf{P}} = \rho$ as required. However, Proposition 2.6 says something rather stronger — that $\nu_{\mathbf{P}}$ is a *strong* ordered radiant.

5. Some special cases

We end this article by briefly describing the inductive constellations that correspond to some special classes of restriction semigroups.

Our first result is immediate.

Corollary 5.1. *The category of weakly left ample semigroups and $(2,1)$-morphisms is isomorphic to the category of replete inductive constellations and ordered radiants.*

At the other extreme, we consider the case where $|E| = 1$. It is clear that a category with one object is a monoid; we see that the analogous situation holds for inductive constellations. First, let M be a restriction semigroup with distinguished semilattice $E = \{1\}$. Since $a^+ = 1$ for any $a \in M$, it is clear that 1 is a left identity for M; further, for any $a \in M$ we have that

$$a\,1 = (a\,1)^+a = 1\,a = a,$$

so that M is a monoid with identity 1. It follows that in $\mathbf{P}(M)$, multiplication is everywhere defined and coincides with that of M. It is easy to see that any monoid is a restriction semigroup with trivial distinguished semilattice $\{1\}$, the natural partial order is equality, and a $(2,1)$-morphism between monoids regarded as restriction semigroups with trivial distinguished semilattices coincides with a monoid morphism.

On the other hand, let $P = (P, \cdot, ^+, \leq)$ be an inductive constellation with trivial distinguished subset $\{1\}$. Then $\mathbf{T}(P)$ is a restriction semigroup with trivial distinguished semilattice $\{1\}$, so that from the above, multiplication in $P = \mathbf{P}(\mathbf{T}(P))$ is everywhere defined and hence coincides with that of $\mathbf{T}(P)$; further, P is a monoid, and $\leq$ is equality. Moreover, it is now clear from (O4) that for any $a \in P$, $a|1 = a$. Consequently, an ordered radiant between two inductive constellations having trivial distinguished subsets is the same thing as a monoid morphism.

Corollary 5.2. *The category of restriction semigroups having trivial distinguished semilattices and* $(2,1)$*-morphisms coincides with the category of monoids and monoid morphisms, and also with the category of inductive constellations having trivial distinguished subsets and ordered radiants.*

We now turn our attention to left ample semigroups. We say that an inductive constellation P is *right cancellative* if for any $a, b, c \in P$, if $\exists b \cdot a$, $\exists c \cdot a$ and $b \cdot a = c \cdot a$, then $b = c$. Note that a right cancellative inductive constellation is necessarily replete.

Corollary 5.3. *The category of left ample semigroups and* $(2,1)$*-morphisms is isomorphic to the category of inductive right cancellative constellations and ordered radiants.*

Proof. Let S be a left ample semigroup. Then S is weakly left ample, so that $\mathbf{P}(S)$ is an inductive constellation. Suppose now that in $\mathbf{P}(S)$, $\exists b \cdot a, \exists c \cdot a$ and $b \cdot a = c \cdot a$. It follows that in S, $ba = ca, b = ba^+$ and $c = ca^+$. Since $a \,\mathcal{R}^*\, a^+$ we have that

$$b = ba^+ = ca^+ = c,$$

so that $\mathbf{P}(S)$ is right cancellative.

Conversely, suppose that P is an inductive right cancellative constellation; we know that $\mathbf{T}(P)$ is weakly left ample; it remains to show that $\mathcal{R}^* = \widetilde{\mathcal{R}}$ in $\mathbf{T}(P)$. Suppose then that $a, b, c \in P$ and $b \otimes a = c \otimes a$. Hence

$$(b|a^+) \cdot a = (c|a^+) \cdot a,$$

and so $b|a^+ = c|a^+$ since P is right cancellative. Consequently,

$$b \otimes a^+ = (b|a^+) \cdot a^+ = b|a^+ = c|a^+ = (c|a^+) \cdot a^+ = c \otimes a^+.$$

It follows that $a \,\mathcal{R}^*\, a^+$ and it is then clear that $\widetilde{\mathcal{R}} = \mathcal{R}^*$ as required. □

Finally we define an inductive constellation P to have *right inverses* if, for any $a \in P$, there exists $\bar{a} \in P$ with $\exists a \cdot \bar{a}$ and $a \cdot \bar{a} = a^+$. It is easy to see that if P has right inverses then it is right cancellative, and hence replete. We remark that any semigroup morphism between inverse semigroups preserves the unary operation of inverse, hence also that of $a \mapsto a^+ = aa^{-1}$.

Corollary 5.4. *The category of inverse semigroups and semigroup morphisms is isomorphic to the category of inductive constellations with right inverses and ordered radiants.*

Proof. Let S be an inverse semigroup. Certainly $\mathbf{P}(S)$ is an inductive constellation. Let $a \in P$; then $(a^{-1})^+ = a^{-1}a$ and in S, $a(a^{-1})^+ = a$. Consequently, in P, we have that $\exists a \cdot a^{-1}$ and $a \cdot a^{-1} = aa^{-1} = a^+$.

Conversely, suppose that P is an inductive constellation with right inverses. Then $\mathbf{T}(P)$ is a weakly left ample semigroup, hence in particular, $E(\mathbf{T}(P))$ is a semilattice. Let $a \in \mathbf{T}(P)$. In P we know there is an element $\bar{a}$ such that $\exists a \cdot \bar{a}$ and $a \cdot \bar{a} = a^+$. We therefore have $\exists (a \cdot \bar{a}) \cdot a = a^+ \cdot a = a$. Recall that if $u \cdot v$ is defined in P, then $u \cdot v = u \otimes v$. Consequently, $a = a \otimes \bar{a} \otimes a$ in $\mathbf{T}(P)$ so that $\mathbf{T}(P)$ is regular, hence inverse, thus completing the proof. □

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References

Armstrong, S. (1984). The structure of type A semigroups. *Semigroup Forum* 29:319-336.

Batbedat, A. (1981). γ-demi-groupes, demi-modules, produit demi-direct. In *Semigroups Proceedings: Oberwolfach 1979.* Lecture Notes in Mathematics 855, Springer-Verlag, 1981, pp. 1-18.

Batbedat, A., Fountain, J. (1981). Connections between left adequate semigroups and γ-semigroups. *Semigroup Forum* 22:59-65.

Cockett, J. R., Lack, S. (2002). Restriction categories I: categories of partial maps. *Theoretical Computer Science* 270:223-259.

Cockett, J. R. (2008). *Private communication.*

Desharnais, J. and Struth, G. (2008). Domain axioms for a family of near-semirings. In *Lecture Notes in Computer Science* 5140:330–345.

Exel, R. (2006). Semigroupoid C^*-algebras. *Preprint: arXiv:math/0611929.*

Fountain, J. (1977). A class of right *PP* monoids. *Quart. J. Math. Oxford (2)* 28:285-300.

Fountain, J. (1979). Adequate semigroups. *Proc. Edinb. Math. Soc. (2)* 22:113-125.

Fountain, F., Gomes, G. M. S., Gould, V. (1999). A Munn type representation for a class of E-semiadequate semigroups. *J. Algebra* 218:693-714.

Gilbert, N. D. (2005). Actions and expansions of ordered groupoids. *J. Pure Appl. Algebra* 198:175-195.

Gould, V. (2007). (Weakly) left E-ample semigroups. Notes available at *http://www-users.york.ac.uk/∼varg1/finitela.ps*

Gould, V., Hollings, C. (to appear). Partial actions of inverse and weakly left E-ample semigroups. *J. Australian Math. Soc.*

Hollings, C. (2007a). *Partial actions of semigroups and monoids.* PhD thesis, University of York.

Hollings, C. (2007b). Partial actions of monoids. *Semigroup Forum* 75:293-316.

Jackson, M., Stokes, T. (2001). An invitation to C-semigroups. *Semigroup Forum* 62:279-310.

Jackson, M., Stokes, T. (2003). Agreeable semigroups. *J. Algebra* 226:393–417.

Jackson, M., Stokes, T. (to appear). Algebras of partial maps. In *Proceedings of the Special Interest Meeting on Semigroups and Related Mathematics*, University of Sydney 2005.

Kellendonk, J., Lawson, M. V. (2004). Partial actions of groups. *I.J.A.C.* 14:87-114.

Lawson, M. V. (1985). *The structure theory of abundant semigroups.* DPhil thesis, University of York.

Lawson, M. V. (1991). Semigroups and ordered categories I: the reduced case. *J. Algebra* 141:422-462.

Lawson, M. V. (1998). *Inverse Semigroups: The Theory of Partial Symmetries.* World Scientific.

Lawson, M. V., Margolis, S. W., Steinberg, B. (2006). Expansions of inverse semigroups. *J. Australian Math. Soc.* 80:205-228.

Manes, E. (2006). Guarded and banded semigroups. *Semigroup Forum* 72:94-120.

Nambooripad, K. S. S. (1979). Structure of regular semigroups I. *Mem. Amer. Math. Soc.* 22, No. 224.

Schein, B. M. (1970a). Restrictively multiplicative algebras of transformations. *Izv. Vysš. Učebn. Zaved. Matematika* 4(95):91-102 (Russian).

Schein, B. M. (1970b). Relation algebras and function semigroups. *Semigroup Forum* 1:1-62.

Schweizer, B., Sklar, A. (1960). The algebra of functions. *Math. Ann.* 139:366-382.

Schweizer, B., Sklar, A. (1961). The algebra of functions II. *Math. Ann.* 143:440-447.

Schweizer, B., Sklar, A. (1965). The algebra of functions III. *Math. Ann.* 161:171-196.

Schweizer, B., Sklar, A. (1967). Function systems. *Math. Ann.* 172:1-16.

Trokhimenko, V. S. (1973). Menger's function systems. *Izv. Vysš. Učebn. Zaved. Matematika* 11(138):71-78 (Russian).

Department of Mathematics, University of York, Heslington, York, YO10 5DD, UK

E-mail address: varg1@york.ac.uk

Centro de Álgebra da Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal

E-mail address: cdh500@cii.fc.ul.pt