ACTIONS AND PARTIAL ACTIONS OF INDUCTIVE CONSTELLATIONS

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Abstract. *Constellations* were recently introduced by the authors as one-sided analogues of categories: a constellation is equipped with a partial multiplication for which 'domains' are defined but, in general, 'ranges' are not. *Left restriction semigroups* are the algebraic objects modelling semigroups of partial mappings, equipped with local identities in the domains of the mappings. *Inductive constellations* correspond to left restriction semigroups in a manner analogous to the correspondence between inverse semigroups and inductive groupoids.

In this paper, we define the notions of the action and partial action of an inductive constellation on a set, before introducing the *Szendrei expansion* of an inductive constellation. Our main result is a theorem which uses this expansion to link the actions and partial actions of inductive constellations, providing a global setting for results previously proved by a number of authors for groups, monoids and other algebraic objects.

Introduction

The celebrated Ehresmann-Schein-Nambooripad (ESN) Theorem gives a connection between the class of inverse semigroups and that of inductive groupoids, stating that the category of inverse semigroups and morphisms is isomorphic to the category of inductive groupoids and inductive functors [14, Theorem 4.1.8]. This result has been generalised for various classes of semigroups; see for example, [1, 13, 17, 18]. The vast majority of these generalisations, as well as the original result itself, are inherently 'two-sided', in a manner which we will describe shortly. The idea behind this approach to studying semigroups, is that in many cases much of the algebraic structure of a semigroup can be recovered from a partial order it possesses.

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This paper is concerned with a class of semigroups we refer to as *left restriction semigroups*. The term 'left restriction semigroup' is something of a misnomer, for such objects are, properly speaking, algebras of type $(2,1)$, that is, they possess a binary operation (that of the semigroup) and in addition a unary operation, denoted here by $a \mapsto a^+$. Left restriction semigroups arise in a plethora of contexts. They first appear in the work of the Russian school in the 1960s and 1970s; useful references to this being those of Schein [19, 20]. More recently they have appeared in the work of Jackson and Stokes [11] and in that of Cockett, Lack and Manes [2, 16]. The latter authors are concerned with developing a framework to handle the notion of partiality of functions, their motivation arising from questions of theoretical computer science. From the 'York' perspective, left restriction semigroups are the variety generated by the quasi-variety of left ample semigroups [5]. They were for some time referred to as 'weakly left E-ample semigroups'. We refer the reader to [10] for further details and references.

Perhaps the easiest way of getting hold of a left restriction semigroup is via its representation by partial mappings. Let $\mathcal{PT}_X$ denote the collection of all partial mappings of a set X, together with the usual left-to-right composition. We define a unary operation $^+$ in $\mathcal{PT}_X$ by putting $\alpha^+ = I_{\operatorname{dom}\alpha}$, the partial identity mapping on the domain of $\alpha \in \mathcal{PT}_X$, and regard $\mathcal{PT}_X$ as an algebra of type $(2,1)$. Then an algebra of the same type is a left restriction semigroup precisely when it is isomorphic to a subalgebra of some $\mathcal{PT}_X$. It follows that a left restriction semigroup is partially ordered by an ordering corresponding to inclusion of functions on $\mathcal{PT}_X$. *Right restriction semigroups* may be defined dually, via a unary operation denoted $a \mapsto a^*$. Any semigroup S which is simultaneously a left and a right restriction semigroup, and is such that $(a^*)^+ = (a^+)^*$ for all $a \in S$ is termed a *(two-sided) restriction semigroup*. We remark that any monoid M may be regarded as a restriction semigroup in which $a^+ = a^* = 1$, for all $a \in M$.

Lawson [13] generalised the ESN Theorem to show that the category of two-sided restriction semigroups and appropriate morphisms is isomorphic to a certain category of inductive categories (an inductive category is a special type of small, ordered category). The constructions of the ESN Theorem generalise very naturally to this case, the $^+$ operation in the semigroup corresponding directly to the domain operation $\mathbf{d}(\cdot)$ in the category, and * to range $\mathbf{r}(\cdot)$. The presence of both $^+$ and * ($\mathbf{d}(\cdot)$ and $\mathbf{r}(\cdot)$) is what was meant above when we described these generalisations of the ESN Theorem as 'two-sided'.

Left restriction semigroups are inherently one-sided objects. In a previous paper [7], the authors developed a one-sided analogue of a category which would permit an approach to *left* restriction semigroups in the spirit of the ESN Theorem. In other words, we constructed an object, which we termed an *inductive constellation*, that corresponds to a left restriction semigroup in an manner analogous to the connection between inductive groupoids and inverse semigroups. An inductive constellation possesses a unary operation (which we denote by $^+$, just as in the corresponding left restriction semigroup) that is analogous to 'domain' in a category; however, we have no notion of 'range'. Structure preserving maps between constellations are called *ordered radiants*.

Result 0.1. [7] *The category of left restriction semigroups and morphisms is isomorphic to the category of inductive constellations and ordered radiants.*

Our current paper arose from attempting to understand the notions of action and partial action of monoids and, more generally, of left restriction semigroups and inductive constellations. Partial actions of groups and monoids of various kinds are inherent to the McAlister 'P-theorem' and its extensions [12]. They appear in the work of Exel [3] where he shows that a partial action of a group on a set can be 'lifted' to the action of an inverse semigroup on the same set. The inverse semigroup he constructs is essentially that of Szendrei [21], who shows the same result in a different mathematical language. Their work was extended from groups to inverse semigroups in [15]. On the other hand Gilbert [4] studied the actions and partial actions of inductive groupoids as a way of informing that of the actions and partial actions of inverse semigroups, constructing along the way the notion of a Szendrei expansion of an inductive groupoid. Gilbert's result is analogous to those obtained for groups [12, Theorem 2.4], inverse semigroups [15, Proposition 6.20], monoids [9, Theorem 4.1] and left restriction semigroups [6, Theorem 4.2].

Our aim here is to introduce the notion of the Szendrei expansion $\mathrm{Sz}(P)$ of an inductive constellation P; $\mathrm{Sz}(P)$ is again an inductive constellation. We then prove the analogue of Gilbert's main result, formulated more precisely in Section 4.

Result 0.2. *Let P be an inductive constellation acting partially on a set X. Then the partial action of P can be lifted to an action of $\mathrm{Sz}(P)$ on X.*

Partial actions of restriction semigroups and inductive constellations are often more conveniently represented by functions termed *strong premorphisms* and *ordered pre-radiants*, just as actions may be represented by morphisms or radiants. We use Result 0.2 to prove our second theorem:

Result 0.3. *Inductive constellations and inductive pre-radiants form a category isomorphic to the category of restriction semigroups and strong premorphisms.*

The structure of the paper is as follows. In Section 1, we define, and summarise the relevant details concerning, left restriction semigroups and inductive constellations from [7]. In Section 2, we formally define two types of functions between inductive constellations: *ordered radiants*, which will be analogous to morphisms between left restriction semigroups and which will be our means of defining actions of inductive constellations, and *ordered pre-radiants*, which will be analogous to 'strong premorphisms' (see [6]) between left restriction semigroups and will allow us to define partial actions of inductive constellations. Section 3 follows, in which we introduce the notion of the Szendrei expansion of an inductive constellation and prove that this is itself an inductive constellation. Our penultimate Section 4, makes precise and proves Result 0.2. Finally in Section 5 we provide the arguments to show that Result 0.3 holds. Indeed we can say rather more, and indicate how natural are the connections between Szendrei expansions of left restriction semigroups and those of inductive constellations.

1. Restriction semigroups and inductive constellations

In this section, we give the relevant definitions and results concerning left restriction semigroups and inductive constellations. Although restriction semigroups will not feature very heavily in this paper, we nevertheless need certain useful identities afforded to us by their definition, and so we begin by giving a brief definition. As commented in the Introduction, left restriction semigroups form a variety of type (2,1) and may therefore be defined by a set of identities.

Definition 1.1. Let S be a semigroup possessing a unary operation $a \mapsto a^+$. We call S a *left restriction semigroup* if (in addition to the associative law) it satisfies the following identities, for all $s, t \in S$:

$$s^+s = s$$

$$s^+t^+ = t^+s^+; \tag{1.1}$$

$$(s^+t)^+ = s^+t^+; \tag{1.2}$$

$$st^+ = (st)^+s.$$

We may also deduce the following identities:

$$st^+ = (st^+)^+s; \tag{1.3}$$

$$(st)^+ = (st^+)^+; \tag{1.4}$$

$$(s^+t^+)^+ = s^+t^+. \tag{1.5}$$

We remark that in some articles, a 'left restriction semigroup' is referred to more simply as a 'restriction semigroup'. Let $E = \{s^+ : s \in S\}$. By putting $t = s$ in (1.2), we see that $s^+s^+ = s^+$, for any $s^+ \in E$, so $E \subseteq E(S)$. Moreover, by (1.1) and (1.5), E forms a subsemilattice of S.

We note that a left restriction semigroup S possesses a natural partial order given by

$$s \leq t \Longleftrightarrow s = s^+t. \tag{1.6}$$

This partial order corresponds exactly to inclusion of mappings in the representation of S as a subalgebra of some $\mathcal{PT}_X$ and is thus crucial to the structure of S; much of our approach is to exploit this fact. For further details on restriction semigroups, the reader is referred to [10].

We now give a brief introduction to constellations. Let P be a set and $\cdot$ be a partial binary operation on P. For $x, y \in P$, the notation '$\exists x \cdot y$' will indicate that the product $x \cdot y$ is defined in P. The notation '$\exists (x \cdot y) \cdot z$' will be understood to mean that $\exists x \cdot y$ *and* $\exists (x \cdot y) \cdot z$. An *idempotent* in $(P, \cdot)$ is an element $e \in P$ for which $\exists e \cdot e$ and $e \cdot e = e$; the collection of all idempotents of P will be denoted by $E(P)$. An idempotent $e \in P$ is a *left identity* for $x \in P$ if $\exists e \cdot x$ and $e \cdot x = x$. A unary operation $a \mapsto a^+$ on P is termed *image idempotent* if its image is contained in $E(P)$. In this case we say that

$$E = \{x^+ : x \in P\}$$

is *the distinguished subset* (of the unary operation $^+$).

Definition 1.2. Let P be a set, let $\cdot$ be a partial binary operation and let $^+$ be an image idempotent unary operation on P with distinguished subset E. We call $(P, \cdot, ^+)$ a *left constellation* if the following axioms hold:

(C1) if $\exists x \cdot (y \cdot z)$, then $\exists (x \cdot y) \cdot z$, in which case, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
(C2) $\exists x \cdot (y \cdot z)$ if, and only if, $\exists x \cdot y$ and $\exists y \cdot z$;
(C3) for each $x \in P$, x^+ is the unique left identity of x in E;
(C4) if $a \in P$, $g \in E$ and $\exists a \cdot g$, then $a \cdot g = a$.

As in [7], we omit the 'left' and refer to a left constellation simply as a *constellation*; moreover we will refer to a constellation $(P, \cdot, ^+)$ more simply as 'P' whenever both operations are clear. If $E = E(P)$, then we call P a *replete constellation.*

It is clear that if $e \in E$, then $e^+ = e$. We can therefore characterise E as the set $E = \{e \in E(P) : e^+ = e\}$.

The following lemma shows that the $^+$ operation in a constellation mimics some of the behaviour of the domain operation $\mathbf{d}(\cdot)$ in a category:

Lemma 1.3. [7, Lemma 2.2] *In a constellation P, if $\exists a \cdot b$, then $(a \cdot b)^+ = a^+$.*

Any category C with set of identities C_o is certainly a constellation with distinguished subset C_o. A constellation is thus a one-sided generalisation of a category in which the $^+$ operation serves as an analogue of domain; however, we have no notion of range.

Note also:

Lemma 1.4. [7, Lemma 2.3] *In a constellation P, $\exists a \cdot b \Longleftrightarrow \exists a \cdot b^+$.*

We now introduce an ordering on a constellation, inspired by that in an ordered category (see, for example, [1]):

Definition 1.5. Let $(P, \cdot, ^+)$ be a constellation and let $\leq$ be a partial order on P. We call $(P, \cdot, ^+, \leq)$ an *ordered* constellation if the following conditions hold:

(O1) if $a \leq c$, $b \leq d$, $\exists a \cdot b$ and $\exists c \cdot d$, then $a \cdot b \leq c \cdot d$;
(O2) if $a \leq b$, then $a^+ \leq b^+$;
(O3) for $e \in E$ and $a \in P$ such that $e \leq a^+$, there exists a *restriction* $e|a$ which is the unique element x with the properties $x \leq a$ and $x^+ = e$;
(O4) for all $e \in E$ and all $a \in P$, there exists a *corestriction* $a|e$ which is the maximum element x with the properties $x \leq a$ and $\exists x \cdot e$;

(O5) for $x, y \in P$ and $e \in E$, if $\exists x \cdot y$, then $((x \cdot y)|e)^+ = (x|(y|e)^+)^+$;
(O6) if $e, f \in E$, then, whenever the restriction $e|f$ is defined, it coincides with the corresponding corestriction.

We note that for any $a \in P$ and $e \in E$, $(a|e) \cdot e = a|e$, by (C4) and (O4). Further, we can deduce a useful characterisation of the partial product in an ordered constellation:

Lemma 1.6. [7, Lemma 3.2] *For $a, b \in P$,*

$$\exists a \cdot b \Longleftrightarrow a|b^+ = a.$$

In an ordered constellation $(P, \cdot, ^+, \leq)$, we denote by $e \wedge f$ the greatest lower bound of $e, f \in E$ with respect to $\leq$, where it exists.

Definition 1.7. Let $(P, \cdot, ^+, \leq)$ be an ordered constellation. We call $(P, \cdot, ^+, \leq)$ an *inductive* constellation if the following condition holds:

(I) for any $e, f \in E$, $e \wedge f$ exists in E and is equal to $e|f$.

In [7, §3], an example is given which demonstrates that not every constellation is inductive.

We now record some results about ordered constellations which will be of use in later sections. For ease of reference, we retain the labelling of [7].

Lemma 1.8. [7, Lemma 3.4] *Let P be an ordered constellation and let $a, b, c \in P$ and $e, f \in E$. Then*

(i) *if $a \leq b$, then $a = a^+|b$;*
(iii) *if $e \leq f$, then $\exists e \cdot f$ and $e = e|f = e \cdot f$;*
(iv) *if $f \leq e \leq a^+$, then $f|(e|a) = f|a$ and $f|a \leq e|a$;*
(vi) *if $a \leq b$, then $a|e \leq b|e$.*

We note that the restriction of (O3) can also be viewed as a product:

Lemma 1.9. [7, Lemma 3.5] *For $e \in E$, if $e \leq a^+$, then $\exists e \cdot a$ and $e|a = e \cdot a$.*

Despite the equality in Lemma 1.9, we will find it useful to retain the notion of restriction which is given in Definition 1.5, since this affords us the ability to make 'uniqueness' arguments. There will be other occasions, however, when it will be more useful to consider the restriction as a product; we will switch between the two viewpoints as appropriate.

We note some further properties of ordered constellations:

Lemma 1.10. [7, Lemma 3.6] *Let P be an ordered constellation and let $s \in P$, $e \in E$. Then $(s|e)^+|s = (s|e)^+ \cdot s = s|e$.*

(This last lemma is the analogue for (left) constellations of (1.3) for left restriction semigroups.)

Lemma 1.11. [7, Lemma 3.7] *In an ordered constellation P, if $\exists x \cdot y$, then*

$$(x \cdot y)|e = \big(x|(y|e)^+\big) \cdot (y|e).$$

As observed in the Introduction, the notion of an inductive constellation was introduced in [7] to provide an object to which left restriction semigroups would correspond in an analogue of the ESN Theorem. We summarise the main results of [7]:

Theorem 1.12. [7, Propositions 4.1 & 4.3] *Let S be a left restriction semigroup with natural partial order $\leq$ and unary operation $^+$. Then $\mathbf{P}(S) = (S, \cdot, ^+, \leq)$ is an inductive constellation, where the* restricted product $\cdot$ *in S is given by*

$$a \cdot b = \begin{cases} ab & \text{if } ab^+ = a; \\ \text{undefined} & \text{otherwise} \end{cases} \tag{1.7}$$

and where for $a \in S$ and $e, f \in E$ with $e \leq a^+$,

$$e|a = ea \text{ and } a|f = af.$$

Conversely, let $(P, \cdot, ^+, \leq)$ be an inductive constellation. Then $\mathbf{T}(P) = (P, \otimes, ^+)$ is a left restriction semigroup, where $\otimes$ is the pseudoproduct *given by*

$$a \otimes b = (a|b^+) \cdot b. \tag{1.8}$$

Moreover, $\leq$ coincides with the natural partial order of the semigroup $(P, \otimes, ^+)$, and $\otimes$ coincides with $\cdot$ whenever the latter is defined.

In particular, we note that the idempotents of an inductive constellation are precisely those of the left restriction semigroup constructed therefrom, and vice versa. For further details of these constructions, see [7].

From our comments in the Introduction, it can be seen that any partial transformation monoid $\mathcal{PT}_X$ is a left restriction semigroup. We can therefore apply Theorem 1.12 to obtain an inductive constellation which we denote by $\mathcal{C}_X$ and call the *function constellation of X*. The underlying set of $\mathcal{C}_X$ is simply that of $\mathcal{PT}_X$; the distinguished subset of $\mathcal{C}_X$ is E_X (the collection of all partial identity mappings of X), since $^+$ is the same as in $\mathcal{PT}_X$. The operation in $\mathcal{C}_X$ is simply the restricted product of (1.7) adapted to the specific case of partial transformations, so that $\exists \alpha \cdot \beta$ if and only if $\operatorname{im} \alpha \subseteq \operatorname{dom} \beta$. Function constellations are the canonical inductive constellations, in the sense that every inductive constellation may be embedded in some $\mathcal{C}_X$: see [7, Proposition 3.9].

2. Radiants and pre-radiants

In this section, we introduce two types of functions between inductive constellations: radiants and pre-radiants. The former are an analogue for constellations of (2,1)-morphisms between left restriction semigroups and were employed in [7]; the latter are analogues of strong premorphisms between left restriction semigroups (see [6]). Just as morphisms and premorphisms may be used to define actions and partial actions of semigroups, radiants and pre-radiants will be used to define actions and partial actions of constellations.

Definition 2.1. Let P and Q be constellations. A *radiant* is a function $\rho : P \to Q$ which satisfies the following conditions:

(R1) if $\exists s \cdot t$ in P, then $\exists (s\rho) \cdot (t\rho)$ in Q, in which case, $(s\rho) \cdot (t\rho) = (s \cdot t)\rho$;
(R2) $s^+\rho = s\rho^+$.

We note that a radiant $\rho : P \to Q$ is easily shown to map idempotents in P to idempotents in Q; condition (R2) is needed to ensure that E is mapped into F, where E is the distinguished subset of P and F is that of Q.

Definition 2.2. Let $\rho : P \to Q$ be a radiant of ordered constellations P and Q, and let P have distinguished subset E. We call ρ an *ordered* radiant if it satisfies the following additional conditions:

(OR1) if $s \leq t$ in P, then $s\rho \leq t\rho$ in Q;
(OR2) $(a|e)\rho = a\rho|e\rho$, for all $a \in P$ and all $e \in E$.

Notice that if P and Q are inductive, then for $e, f \in E$,

$$(e \wedge f)\rho = (e|f)\rho = e\rho|f\rho = e\rho \wedge f\rho$$

so that ρ preserves meets.

Let S and T be left restriction semigroups and let $\alpha : S \to T$ be a function. Define $\mathbf{P}(\alpha) : \mathbf{P}(S) \to \mathbf{P}(T)$ to be the same function on the underlying sets of $\mathbf{P}(S)$ and $\mathbf{P}(T)$. Similarly, let P and Q be inductive constellations and let $\beta : P \to Q$ be a function. Define $\mathbf{T}(\beta) : \mathbf{T}(P) \to \mathbf{T}(Q)$ to be the same function on the underlying sets of $\mathbf{T}(P)$ and $\mathbf{T}(Q)$.

We can now state Result 0.1 in more detail.

Theorem 2.3. [7, Theorem 4.13] *The categories of restriction semigroups and morphisms and of inductive constellations and ordered radiants are mutually isomorphic via functors* $\mathbf{P}$ *and* $\mathbf{T}$.

$$\begin{tikzcd}
S = \mathbf{T}(\mathbf{P}(S)) \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\overset{\mathbf{P}}{\dashrightarrow}"] & \mathbf{P}(S) \arrow[d, "\mathbf{P}(\alpha)"] \\
T = \mathbf{T}(\mathbf{P}(T)) \arrow[r] & \mathbf{P}(T)
\end{tikzcd}
\qquad
\begin{tikzcd}
P = \mathbf{P}(\mathbf{T}(P)) \arrow[r] \arrow[d, "\beta"'] \arrow[dr, phantom, "\overset{\mathbf{T}}{\dashrightarrow}"] & \mathbf{T}(P) \arrow[d, "\mathbf{T}(\beta)"] \\
Q = \mathbf{P}(\mathbf{T}(Q)) \arrow[r] & \mathbf{T}(Q)
\end{tikzcd}$$

It follows from the above and the constructions given in [7] that for an inductive constellation P, restrictions, corestrictions and $\wedge$ all coincide with $\otimes$ in $\mathbf{T}(P)$.

The following further property of ordered radiants will prove useful very shortly:

Lemma 2.4. [7, Lemma 4.7] *Let* $\rho : P \to Q$ *be an ordered radiant of inductive constellations* P *and* Q*. Then* ρ *preserves pseudoproducts.*

Just as an action of a semigroup may be defined as a particular morphism, we now define an action of an ordered constellation as a particular radiant:

Definition 2.5. An ordered constellation P *acts* on a set X if there is an ordered radiant $\rho : P \to \mathcal{C}_X$.

Much of our motivation comes from consideration of *partial* actions of left restriction semigroups [6]; such actions correspond to strong premorphisms (see Section 5 for the definition). We now introduce a function between constellations which is analogous to a strong premorphism, with the eventual aim of proving a result for equivalence of categories along the lines of Theorem 2.3. Such a function is also a one-sided analogue of the inductive groupoid premorphism of Gilbert [4, p. 184].

Definition 2.6. Let P and Q be inductive constellations, and let P have distinguished subset E. An *ordered pre-radiant* is a function $\psi : P \to Q$ which satisfies the following conditions:

- (P) if $\exists s\cdot t$ in P, then $(s\psi)\otimes(t\psi) = s\psi^+\otimes(s\cdot t)\psi$, where $\otimes$ is the pseudoproduct of (1.8);
- (OP1) $s\psi^+ \leq s^+\psi$;
- (OP2) if $s \leq t$ in P, then $s\psi \leq t\psi$ in Q;
- (OP3) $a\psi|e\psi = (a|e)\psi$, for all $a \in P$ and all $e \in E$.
- (IP) if $e \leq a^+$ in P, then $(e|a)\psi^+ = e\psi \wedge a\psi^+$.

We note the following:

Lemma 2.7. *Let $\psi : P \to Q$ be an ordered pre-radiant of ordered constellations P and Q with distinguished subsets E and F, respectively. If $e \in E(P)$, then $e\psi \in E(Q)$. Moreover, if $e \in E$, then $e\psi \in F$.*

Proof. Let $e \in E(P)$. Then $\exists e \cdot e$ in P, so

$$e\psi \otimes e\psi = e\psi^+ \otimes (e \cdot e)\psi = e\psi^+ \otimes e\psi = e\psi \in E(\mathbf{T}(Q)) = E(Q).$$

Now let $e \in E$, so that $e^+ = e$. By (OP1), we have $e\psi^+ \leq e^+\psi = e\psi$ in Q. Since Q has the same ordering as $\mathbf{T}(Q)$, we have $e\psi^+ \leq e\psi$ in $\mathbf{T}(Q)$ also. From (1.6), we therefore have $e\psi^+ = e\psi^+ \otimes e\psi = e\psi$, hence $e\psi \in F$. $\square$

It follows as for ordered radiants that (OP3) implies that ψ preserves meets. With this in mind, an ordered (pre-)radiant between inductive constellations is sometimes referred to as an *inductive* (pre-)radiant.

We note a property of ordered pre-radiants which we will use in Section 4:

Lemma 2.8. *Let $\psi : P \to Q$ be an ordered pre-radiant of inductive constellations P and Q. Then*

$$\exists a \cdot b \Longrightarrow a\psi^+ \wedge (a \cdot b)\psi^+ = (a\psi \otimes b\psi)^+.$$

Proof. Suppose that $\exists a \cdot b$. Then

$$\begin{aligned}(a\psi \otimes b\psi)^+ &= \left(a\psi^+ \otimes (a \cdot b)\psi\right)^+, \text{ by (P)}\\ &= \left(a\psi^+ \otimes (a \cdot b)\psi^+\right)^+, \text{ by (1.4)}\\ &= a\psi^+ \otimes (a \cdot b)\psi^+, \text{ by (1.5)}\\ &= a\psi^+ \wedge (a \cdot b)\psi^+,\end{aligned}$$

by the final sentence of Theorem 1.12. □

Just as strong premorphisms can be used to define partial actions of left restriction semigroups, we use ordered pre-radiants to define partial actions of ordered constellations:

Definition 2.9. An ordered constellation P *acts partially* on a set X if there is an ordered pre-radiant $\psi : P \to \mathcal{C}_X$.

We could also write down a definition for the (partial) action of an ordered constellation P on a set X in terms of a (partial) function $X \times P \to X$, as in [6, Definition 2.4] for the partial action of a left restriction semigroup on a set, for example. Such definitions would perhaps look somewhat like the definition of the action of a category given in [14, §10.1]. However, the above characterisations in terms of radiants and pre-radiants will suffice for our purposes.

We conclude this section with a lemma which will be used towards the end of the paper:

Lemma 2.10. *Let $\alpha : P \to Q$ be an ordered pre-radiant and $\beta : Q \to R$ be an ordered radiant, for inductive constellations P, Q and R. Then the composition $\alpha\beta : P \to R$ is an ordered pre-radiant.*

Proof. Conditions (OP1)–(OP3) are immediate. The remaining conditions, (P) and (IP), require a little more work.

(P) Suppose that $\exists s \cdot t$ in P. We have

$$\begin{aligned}s\alpha\beta \otimes t\alpha\beta &= \left(s\alpha\beta|(t\alpha\beta^+\right) \cdot t\alpha\beta\\ &= \left(s\alpha\beta|t\alpha^+\beta\right) \cdot t\alpha\beta, \text{ by (R2)}\\ &= \left(s\alpha|t\alpha^+\right)\beta \cdot (t\alpha)\beta, \text{ by (OR2)}\\ &= \left[\left(s\alpha|t\alpha^+\right) \cdot (t\alpha)\right]\beta, \text{ by (R1), since } \exists \left(s\alpha|t\alpha^+\right) \cdot (t\alpha)\\ &= (s\alpha \otimes t\alpha)\beta\\ &= \left(s\alpha^+ \otimes (s \cdot t)\alpha\right)\beta, \text{ by (P)}\\ &= s\alpha\beta^+ \otimes (s \cdot t)\alpha\beta, \text{ by Lemma 2.4, followed by (R2).}\end{aligned}$$

(IP) Suppose that $e \leq a^+$. Then, using (IP) for α,

$$(e|a)\alpha\beta^+ = (e|a)\alpha^+\beta = (e\alpha \wedge a\alpha^+)\beta = e\alpha\beta \wedge a\alpha\beta^+,$$

as required. □

3. Expansion of ordered constellations

Our goal in the following section will be to prove an 'expansion' theorem for the actions and partial actions of inductive constellations, i.e., a theorem analogous to the earlier expansion results noted in the Introduction. To that end, we first define an expansion for ordered constellations; this expansion will be analogous to that defined by Gilbert [4] for ordered groupoids. From here on, for notational simplicity, we will denote multiplication in an ordered constellation by juxtaposition. Furthermore, throughout the remainder of this paper, the distinguished subset of the constellation P will always be denoted by E.

Let P be an ordered constellation. For each $e \in E$, let

$$\Sigma_e = \{x \in P : x^+ = e\}.$$

The set Σ_e is clearly the constellation analogue of the notion of a 'star' [4, p. 177] in a category. Let $\mathcal{P}_f^e(\Sigma_e)$ be the collection of all finite subsets of Σ_e which contain e, and put

$$\mathcal{U} = \bigcup_{e \in E} \mathcal{P}_f^e(\Sigma_e).$$

Definition 3.1. Let P be an ordered constellation. The *Szendrei expansion* of P is the set

$$\mathrm{Sz}(P) = \{(U, u) \in \mathcal{U} \times P : u \in U\},$$

together with the operation

$$(U,u)(V,v) = \begin{cases} (U, uv) & \text{if } \exists uv \text{ in } P \text{ and } uV \subseteq U; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Let $w \in V$. Then $w^+ = v^+$. By Lemma 1.4, we have

$$\exists uv \Longrightarrow \exists uv^+ \Longrightarrow \exists uw^+ \Longrightarrow \exists uw.$$

It therefore makes sense to write 'uV' in the above definition.

We note that $(F, f) \in \mathrm{Sz}(P)$ is idempotent if and only if $\exists (F, f)^2$ and $(F, f)^2 = (F, f)$. This is easily seen to be equivalent to f being idempotent in P and $fF \subseteq F$. Observe that if $f \in E$, then we automatically have $fF = F$, since $F \subseteq \Sigma_f$.

Proposition 3.2. *If P is an ordered constellation, then* $\mathrm{Sz}(P)$ *is an ordered constellation with*

$$(U, u)^+ = (U, u^+),$$

ordering

$$(U, u) \leq (V, v) \Longleftrightarrow u \leq v \text{ in } P \text{ and } u^+V \subseteq U, \tag{3.1}$$

restriction

$$(F, f)|(U, u) = (F, fu) \tag{3.2}$$

and corestriction

$$(U,u)|(F,f) = \big((u|f)F \cup (u|f)^+U, u|f\big)\,. \tag{3.3}$$

Notice that the distinguished subset of $\mathrm{Sz}(P)$ *is*

$$\mathcal{E} = \{(F,f) \in \mathcal{U} \times P : f \in E\}.$$

Moreover, if P *is inductive, then so too is* $\mathrm{Sz}(P)$*, with*

$$(H,h) \wedge (F,f) = ((h \wedge f)(H \cup F), h \wedge f)\,. \tag{3.4}$$

(We note that if Lemma 1.9 is taken into account, then the ordering given in (3.1) is essentially the same as that given by Gilbert [4] for the Szendrei expansion of an inductive groupoid.)

Proof. We first observe that in the definition of $\leq$, if $u \leq v$, then $u^+ \leq v^+$ so that $\exists u^+v^+$ and hence u^+V.

(C1) Suppose that $\exists (U,u)[(V,v)(W,w)]$, i.e.,

$$\exists vw,\ \exists u(vw),\ vW \subseteq V \text{ and } uV \subseteq U.$$

Then $\exists (uv)w$, by (C1) in P. Let $w' \in W$. Using Lemmas 1.3 and 1.6, we have

$$\exists u(vw) \Leftrightarrow u|(vw)^+ = u \Leftrightarrow u|v^+ = u \Leftrightarrow u|(vw')^+ = u \Leftrightarrow \exists u(vw').$$

It therefore makes sense to write '$u(vW)$' and we have $u(vW) \subseteq uV \subseteq U$. Then, by (C1), $(uv)W \subseteq U$, hence $\exists [(U,u)(V,v)](W,w)$. In this case,

$$\begin{aligned}(U,u)[(V,v)(W,w)] &= (U,u)(V,vw) = (U,u(vw))\\ &= (U,(uv)w) = (U,uv)(W,w) = [(U,u)(V,v)](W,w).\end{aligned}$$

(C2)($\Rightarrow$) This follows from the proof of (C1).

(C2)($\Leftarrow$) Suppose that $\exists (U,u)(V,v)$ and $\exists (V,v)(W,w)$, i.e.,

$$\exists uv,\ \exists vw,\ uV \subseteq U \text{ and } vW \subseteq V.$$

Then $\exists u(vw)$, by (C2) in P, so $\exists (U,u)[(V,v)(W,w)]$.

(C3) We claim that $(U,u)^+$ is the unique left identity in $\mathcal{E}$ of (U,u). It is easy to see that $\exists (U,u^+)(U,u)$, since $\exists u^+u$ and $u^+U = U$. Also, $(U,u^+)(U,u) = (U,u^+u) = (U,u)$.

Suppose now that there is another element $(F,f) \in \mathcal{E}$ which is a left identity for (U,u). We must have $(F,fu) = (F,f)(U,u) = (U,u)$, so $F = U$. We must also have $fu = u$, so $f = u^+$, by uniqueness of left identities in E.

(C4) Suppose that $\exists (U,u)(F,f)$, for $(U,u) \in \mathrm{Sz}(P)$ and $(F,f) \in \mathcal{E}$. Then $\exists uf$ in P. By (C4) in P, $uf = u$, so $(U,u)(F,f) = (U,uf) = (U,u)$.

Before showing that the ordering (3.1) satisfies conditions (O1)–(O6), we first prove that it is indeed a partial order. We begin by observing that, for any $(U,u) \in \mathrm{Sz}(P)$, $(U,u) \leq (U,u)$, since $u \leq u$ in P and $u^+U = U$. Now suppose that $(U,u) \leq (V,v)$ and also that $(V,v) \leq (U,u)$. We then have $u \leq v$ and $v \leq u$ in P, hence $u = v$. We also have $u^+V \subseteq U$ and $v^+U \subseteq V$. We know, however, that $u^+ = v^+$ in P, so $U \supseteq u^+V = v^+V = V$ and $V \supseteq v^+U = u^+U = U$, hence $U = V$. Therefore $(U, u) = (V, v)$.

Now suppose that $(U, u) \leq (V, v)$ and that $(V, v) \leq (W, w)$. Then $u \leq v$ and $v \leq w$, so $u \leq w$ in P. Also, $u^+V \subseteq U$ and $v^+W \subseteq V$. In order to complete the proof that $(U, u) \leq (W, w)$, we need to show that $u^+W \subseteq U$. We first note that it makes sense to write $u^+W = u^+|W$, since $u^+ \leq w^+ = x^+$, for any $x \in W$. Now, since $u^+ \leq v^+$, we have $\exists u^+v^+$ and $u^+ = u^+v^+$, by Lemma 1.8(iii). We can therefore write $u^+W = (u^+v^+)W$. Since we know that $\exists u^+v^+$ and $\exists v^+W$, we conclude that $\exists u^+(v^+W)$, by (C2) in P. Furthermore, $u^+(v^+W) = (u^+v^+)W$, by (C1). We therefore have $u^+W = u^+(v^+W) \subseteq u^+V \subseteq U$, as required. The ordering (3.1) is thus a partial order.

(O1) Suppose that $\exists (U, u)(W, w)$, $\exists (V, v)(X, x)$, $(U, u) \leq (V, v)$ and $(W, w) \leq (X, x)$, i.e.,

$$\exists uw,\ uW \subseteq U,\ \exists vx,\ vX \subseteq V,\ u \leq v,\ u^+V \subseteq U,\ w \leq x,\ w^+X \subseteq W.$$

Then $uw \leq vx$, by (O1) in P, and $(uw)^+V = u^+V \subseteq U$, using Lemma 1.3, so $(U, u)(W, w) \leq (V, v)(X, x)$.

(O2) If $(U, u) \leq (V, v)$, then $u \leq v$, hence $u^+ \leq v^+$. Then $(U, u^+) \leq (V, v^+)$.

(O3) Let $(U, u) \in \mathrm{Sz}(P)$ and $(F, f) \in \mathcal{E}$ be such that $(F, f) \leq (U, u^+)$:

$$fU \subseteq F \quad \text{and} \quad f \leq u^+.$$

Then the restriction $f|u = fu$ is defined in P. We show that the restriction in $\mathrm{Sz}(P)$ is that given in (3.2). Note that $(F, fu) \leq (U, u)$, since $fu = f|u \leq u$ and $(fu)^+U = fU \subseteq F$. Also,

$$(F, fu)^+ = (F, (fu)^+) = (F, f).$$

Suppose now that there is another element $(A, a) \in \mathrm{Sz}(P)$ such that $(A, a) \leq (U, u)$, so $a^+U \subseteq A$ and $a \leq u$, and $(A, a)^+ = (F, f)$, so $A = F$ and $a^+ = f$. But $fu = f|u$ is the unique element of P with the properties $f|u \leq u$ and $(f|u)^+ = f$, so $a = f|u$. Therefore, the restriction is unique in $\mathrm{Sz}(P)$.

(O4) Let $(U, u) \in \mathrm{Sz}(P)$ and $(F, f) \in \mathcal{E}$. Note that $f = a^+$, for all $a \in F$, and that $\exists (u|f)f$, by definition of $u|f$. Hence, $\exists (u|f)a$, by Lemma 1.4. We can therefore make the product $(u|f)F$. Similarly, it makes sense to write $(u|f)^+U$, since $(u|f)^+U = (u|f)^+|U$, which is defined, as $u|f \leq u$ implies that $(u|f)^+ \leq u^+ = w^+$, for all $w \in U$.

We now show that the corestriction $(U, u)|(F, f)$ is that given in (3.3). Observe that $\exists \left((u|f)F \cup (u|f)^+U, u|f\right)(F, f)$, since $\exists (u|f)f$ and $(u|f)F \subseteq (u|f)F \cup (u|f)^+U$. We also have $u|f \leq u$ and $(u|f)^+U \subseteq (u|f)F \cup (u|f)^+U$, so $((u|f)F \cup (u|f)^+U, u|f) \leq (U, u)$.

Now suppose that (B, b) is another element of $\mathrm{Sz}(P)$ with $(B, b) \leq (U, u)$, so that $b \leq u$ and $b^+U \subseteq B$, and $\exists (B, b)(F, f)$, so that $\exists bf$ and $bF \subseteq B$. Since $b \leq u$, we have $b|f \leq u|f$, by Lemma 1.8(vi). Also, since $\exists bf$, we have $b|f = b$, by Lemma 1.6. Therefore $b \leq u|f$.

Let $x \in b^+ ((u|f)F) = b^+| ((u|f)F)$, so that $x = b^+| ((u|f)g)$, for some $g \in F$. It is clear that $x^+ = b^+$ and that $x \leq (u|f)g$. Note, however, that these are properties shared by the element bg (which is defined, by Lemma 1.4, since $\exists bf$ and $f = g^+$). Therefore, by uniqueness of restrictions, $x = bg$, hence $b^+ ((u|f)F) \subseteq bF \subseteq B$. Note also that

$$b^+ \left((u|f)^+U\right) = \left(b^+(u|f)^+\right) U = b^+U \subseteq B,$$

using Lemma 1.8(iii). We have shown that

$$b^+ \left((u|f)F \cup (u|f)^+U\right) \subseteq B,$$

hence $(B, b) \leq ((u|f)F \cup (u|f)^+U, u|f)$.

(O5) Suppose that $\exists(U, u)(V, v)$, i.e., $\exists uv$ and $uV \subseteq U$. Then

$$[(U, u)(V, v)]\,|(F, f) = (U, uv)|(F, f) = \left((uv|f)F \cup (uv|f)^+U, uv|f\right),$$

so that

$$\{[(U, u)(V, v)]\,|(F, f)\}^+ = \left((uv|f)F \cup (uv|f)^+U, (uv|f)^+\right).$$

Also,

$$(V, v)|(F, f) = \left((v|f)F \cup (v|f)^+V, v|f\right),$$

so that if $\triangle = (U, u)|\,[(V, v)|(F, f)]^+$, then

$$\begin{aligned}
\triangle &= (U, u)|\left((v|f)F \cup (v|f)^+V, v|f\right)^+ \\
&= (U, u)|\left((v|f)F \cup (v|f)^+V, (v|f)^+\right) \\
&= \left(\left[u|(v|f)^+\right]\left[(v|f)F \cup (v|f)^+V\right] \cup \left[u|(v|f)^+\right]^+ U, u|(v|f)^+\right) \\
&= \left(\left[\left(u|(v|f)^+\right)(v|f)\right] F \cup \left[u|(v|f)^+\right]\left[(v|f)^+V\right] \cup \left[u|(v|f)^+\right]^+ U, u|(v|f)^+\right),
\end{aligned}$$

using (C1) in P. Hence

$$\triangle^+ = \left((uv|f)F \cup \left[u|(v|f)^+\right]\left[(v|f)^+V\right] \cup (uv|f)^+U, (uv|f)^+\right),$$

using Lemma 1.11 and (O5) in P. For brevity, we will put

$$W = (uv|f)F \cup \left[u|(v|f)^+\right]\left[(v|f)^+V\right] \cup (uv|f)^+U.$$

To obtain the desired equality, we need to show that

$$W = (uv|f)F \cup (uv|f)^+U.$$

We do this by showing that

$$\left[u|(v|f)^+\right]\left[(v|f)^+V\right] \subseteq (uv|f)^+U. \tag{3.5}$$

Consider $x \in (u|(v|f)^+)\,[(v|f)^+V]$, so that $x = (u|(v|f)^+)\,[(v|f)^+w]$, for some $w \in V$. Observe that since $\exists uw$ in P, we have

$$x \leq uw \qquad \text{and} \qquad x^+ = (u|(v|f)^+)^+ = (uv|f)^+ \qquad \text{(by (O5))}.$$

We have

$$(uv|f)^+ \leq (uv)^+ = u^+ = (uw)^+$$

so that $(uv|f)^+|uw$ is defined, hence by uniqueness of restrictions,

$$x = (uv|f)^+|(uw) = (uv|f)^+(uw) \in (uv|f)^+U,$$

since $uV \subseteq U$, and the inclusion (3.5) holds.

(O6) Let $(G, g), (F, f) \in \mathcal{E}$ and consider $(G, g)|(F, f)$. Regarding this first as a corestriction, we have

$$(G, g)|(F, f) = ((g|f)F \cup (g|f)^+G, g|f), \tag{3.6}$$

from (3.3). We now suppose that the corresponding restriction is defined; we have $(G, g) \leq (F, f)$, so that $g \leq f$ and $gF \subseteq G$. The restriction $g|f$ is therefore defined in P, and, by Lemma 1.8(iii), $g|f = g$. The corestriction of (3.6) can therefore be rewritten

$$\begin{aligned}(G, g)|(F, f) = (gF \cup gG, g) &= (gF \cup G, g), \text{ since } gG = G \\ &= (G, g), \text{ since } gF \subseteq G,\end{aligned}$$

which is precisely the appropriate restriction, since $gf = g|f = g$. We have therefore shown that $\mathrm{Sz}(P)$ is an ordered constellation.

We now suppose that P is inductive and show that condition (I) holds in $\mathrm{Sz}(P)$. Given $(H, h), (F, f) \in \mathcal{E}$, we put

$$(G, g) := ((h \wedge f)(H \cup F), h \wedge f)\,.$$

Since $h \in H$, we have $(h \wedge f)|h = (h \wedge f)h \in G$. But $(h \wedge f)h = h \wedge f$, by (C4). Therefore $h \wedge f \in G$. It is also clear that $x^+ = h \wedge f$, for any $x \in G$, hence $(G, g) \in \mathrm{Sz}(P)$.

We claim that (G, g) is the greatest lower bound of (H, h) and (F, f). It is certainly true that $(G, g) \leq (H, h)$, since $g = h \wedge f \leq h$ and $(h \wedge f)H \subseteq G$. Similarly, $(G, g) \leq (F, f)$. Thus (G, g) is a lower bound for (H, h) and (F, f).

Suppose now that $(L, l) \in \mathcal{E}$ is another lower bound for (H, h) and (F, f). Then $l \leq h$ and $l \leq f$, so $l \leq h \wedge f$. Also, $l|H = lH \subseteq L$ and $l|F = lF \subseteq L$, hence $l|(H \cup F) \subseteq L$. By Lemma 1.8(iv), we have

$$lG = l|G = l|\,[(h \wedge f)(H \cup F)] = l|\,[(h \wedge f)|(H \cup F)] = l|(H \cup F).$$

Thus $lG \subseteq L$, and $(L, l) \leq (G, g)$. We have shown that any lower bound of (H, h) and (F, f) is less than or equal to (G, g), hence $(G, g) = (H, h) \wedge (F, f)$.

Recall now that for $(H, h), (F, f) \in \mathcal{E}$, we have the corestriction

$$\begin{aligned}(H, h)|(F, f) &= \big((h|f)F \cup (h|f)^+H, h|f\big) \\ &= \big((h \wedge f)F \cup (h \wedge f)^+H, h \wedge f\big)\,, \text{ by (I) in } P \\ &= ((h \wedge f)(H \cup F), h \wedge f) = (H, h) \wedge (F, f),\end{aligned}$$

as required. □

Corollary 3.3. *If P is a replete ordered constellation, then $\mathrm{Sz}(P)$ is a replete ordered constellation.*

Proof. It is easy to see that if $E = E(P)$, then $\mathcal{E} = E(\mathrm{Sz}(P))$. □

We now define a function $\iota : P \to \mathrm{Sz}(P)$ by $s\iota = (\{s^+, s\}, s)$. The function ι is an ordered pre-radiant, but in fact we can show rather more.

Lemma 3.4. *For any $s, t \in P$,*

$$s\iota \otimes t\iota = s\iota^+ \otimes (s \otimes t)\iota.$$

Proof. Let $s, t \in P$. Bearing in mind that $s|t^+ \leq s$, so that $(s|t^+)^+ \leq s^+$, Lemma 1.10 and calculations of a now familiar kind yield:

$$(s|t^+)t^+ = s|t^+, (s|t^+)t = s \otimes t, (s|t^+)^+ s^+ = (s|t^+)^+ \text{ and } (s|t^+)^+ s = s|t^+.$$

Consequently,

$$\begin{aligned} s\iota \otimes t\iota &= (s\iota|t\iota^+)t\iota \\ &= \big((\{s^+, s\}, s)|(\{t^+, t\}, t^+)\big)(\{t^+, t\}, t) \\ &= \big((s|t^+)\{t^+, t\} \cup (s|t^+)^+\{s^+, s\}, s|t^+\big)(\{t^+, t\}, t) \\ &= (\{(s|t^+)^+, s|t^+, s \otimes t, \}, s \otimes t). \end{aligned}$$

On the other hand,

$$\begin{aligned} s\iota^+ \otimes (s \otimes t)\iota &= (\{s^+, s\}, s^+) \otimes (\{(s|t^+)^+, s \otimes t\}, s \otimes t) \\ &= \big((\{s^+, s\}, s^+)|(\{(s|t^+)^+, s \otimes t\}, (s|t^+)^+)\big)(\{(s|t^+)^+, s \otimes t\}, s \otimes t) \\ &= ((s|t^+)^+\{s^+, s, (s|t^+)^+, s \otimes t\}, s \otimes t) \\ &= (\{(s|t^+)^+, s|t^+, s \otimes t\}, s \otimes t), \end{aligned}$$

as required. □

Lemma 3.5. *The function ι is an ordered pre-radiant.*

Proof. Condition (P) is immediate from Lemma 3.4 .

(OP1) Let $s \in P$. We have $s\iota^+ = (\{s^+, s\}, s^+)$. On the other hand, $s^+\iota = (\{s^+\}, s^+)$. Since $s^+\{s^+\} \subseteq \{s^+, s\}$, we have $(\{s^+, s\}, s^+) \leq (\{s^+\}, s^+)$ in $\mathrm{Sz}(P)$.

(OP2) Let $s, t \in P$ with $s \leq t$. Then $s\iota = (\{s^+, s\}, s)$ and $t\iota = (\{t^+, t\}, t)$. We consider the set $s^+\{t^+, t\}$. Since $s \leq t$, we have $s^+t^+ = s^+$, by Lemma 1.8(iii), and $s^+t = s^+|t = s$, by Lemma 1.8(i). Hence $s^+\{t^+, t\} = \{s^+, s\}$, so $(\{s^+, s\}, s) \leq (\{t^+, t\}, t)$.

(OP3) Let $s \in P$ and $e \in E$. Then $(s|e)\iota = (\{(s|e)^+, s|e\}, s|e)$. On the other hand,

$$s\iota|e\iota = (\{s^+, s\}, s)\,|\,(\{e\}, e) = (\,(s|e)\{e\} \,\cup\, (s|e)^+\{s^+, s\}, s|e\,).$$

We know that $(s|e)e = s|e$, by (C4) in P. Also, $(s|e)^+s^+ = (s|e)^+$, by Lemma 1.8(iii). Finally, $(s|e)^+s = s|e$, from Lemma 1.10. We therefore have $s\iota|e\iota = (s|e)\iota$, as required.

(IP) Let $a \in P$, $e \in E$ with $e \leq a^+$. On the one hand, we have

$$(e|a)\iota^+ = (\{e, e|a\}, e|a)^+ = (\{e, e|a\}, e),$$

whilst on the other, we have

$$\begin{aligned} e\iota \wedge a\iota^+ &= (\{e\}, e) \wedge (\{a^+, a\}, a^+) = (e\{e, a^+, a\}, e) \\ &= (\{e, ea\}, e) = (\{e, e|a\}, e), \end{aligned}$$

as required. □

Proposition 3.6. *The Szendrei expansion* $\mathrm{Sz}(P)$ *of an inductive constellation* P *is generated by* $P\iota$ *via multiplication,* $^+$ *and meet.*

Proof. Let $(U, u) \in \mathrm{Sz}(P)$. Notice that

$$(U, u) = (U, u^+)(\{u^+, u\}, u) = (U, u^+)(u\iota).$$

Now let $e \in E$ and let $X \in \mathcal{P}_f^e(\Sigma_e)$ with $|X| = n$. We claim that

$$\bigwedge_{x \in X} (\{e, x\}, e) = (X, e), \tag{3.7}$$

and prove this by induction. The claim is certainly true for $n = 1$ and $n = 2$, by (3.4).

Suppose now that (3.7) holds for $n = k$. Let $Y \in \mathcal{P}_f^e(\Sigma_e)$ with $|Y| = k$. We choose $z \in \Sigma_e \setminus Y$ and put $Z = Y \cup \{z\}$ so that $|Z| = k + 1$. Consider

$$\begin{aligned} \bigwedge_{x \in Z} (\{e, x\}, e) &= \left(\bigwedge_{x \in Y} (\{e, x\}, e) \right) \wedge (\{e, z\}, e) \\ &= (Y, e) \wedge (\{e, z\}, e), \text{ by assumption} \\ &= (e(Y \cup \{z\}), e), \text{ from (3.4)} \\ &= (Z, e), \text{ since } e \text{ is a left identity for } \Sigma_e. \end{aligned}$$

Therefore, (3.7) holds for all $n \in \mathbb{N}$, by induction. Observe further that

$$(X, e) = \bigwedge_{x \in X} (\{e, x\}, e) = \bigwedge_{x \in X} (\{e, x\}, x)^+ = \bigwedge_{x \in X} x\iota^+. \tag{3.8}$$

Returning to the task at hand, we use (3.8) to write (U, u^+) as

$$(U, u^+) = \bigwedge_{y \in U} y\iota^+,$$

hence

$$(U, u) = \left(\bigwedge_{y \in U} y\iota^+ \right) (u\iota),$$

as required. □

4. Expansion of partial actions of inductive constellations

We finally present our 'expansion' theorem for the partial actions of inductive constellations:

Theorem 4.1. *Let P and Q be inductive constellations with distinguished subsets E and F, respectively. If $\psi : P \to Q$ is an ordered pre-radiant, then there exists a unique ordered radiant $\overline{\psi} : \mathrm{Sz}(P) \to Q$ such that $\psi = \iota\overline{\psi}$, i.e., such that the following diagram commutes:*

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "\iota"'] & Q \\ \mathrm{Sz}(P) \arrow[ur, "\overline{\psi}"'] & \end{tikzcd}$$

Conversely, if $\overline{\psi} : P \to Q$ is an ordered radiant, then $\psi = \iota\overline{\psi}$ is an ordered pre-radiant.

Proof. Let $\psi : P \to Q$ be an ordered pre-radiant. We define $\overline{\psi} : \mathrm{Sz}(P) \to Q$ by

$$(U,u)\overline{\psi} = \bigwedge_{y\in U} y\psi^+ \Big| u\psi = \left(\bigwedge_{y\in U} y\psi^+\right) u\psi.$$

Note that we do not necessarily need to include the $y = u^+$ 'factor': $u\psi^+ \leq (u^+\psi)^+$, by (OP1) and (O2), so $u\psi^+ \wedge (u^+\psi)^+ = u\psi^+$. Observe also that if $(F,f) \in \mathcal{E}$, then

$$(F,f)\overline{\psi} = \left(\bigwedge_{y\in F} y\psi^+\right) f\psi = \bigwedge_{y\in F} y\psi^+, \tag{4.1}$$

by Lemma 1.8(iii), since $\bigwedge_{y\in F} y\psi^+ \leq f\psi^+ = f\psi$, from Lemma 2.7.

First of all, it is easy to see that $\psi = \iota\overline{\psi}$. Let $s \in P$. Then

$$s\iota\overline{\psi} = (\{s^+, s\}, s)\overline{\psi} = s\psi^+ s\psi = s\psi.$$

We now show that $\overline{\psi}$ preserves ordering. Suppose that $(U,u) \leq (V,v)$ in $\mathrm{Sz}(P)$. Then $u \leq v$ and $u^+V \subseteq U$, so $u^+ \leq v^+ = x^+$, for any $x \in V$, and the restriction $u^+|x = u^+x$ is defined, with $u^+x \leq x$. We then have $(u^+x)\psi \leq x\psi$, hence

$(u^+x)\psi^+ \leq x\psi^+$. Thus

$$\begin{aligned}
(V,v)\overline{\psi} &= \bigwedge_{x\in V} x\psi^+ \Big| v\psi \\
&\geq \bigwedge_{x\in V} \big((u^+x)\psi\big)^+ \Big| v\psi, \text{ by Lemma 1.8(iv)} \\
&\geq \bigwedge_{y\in U} y\psi^+ \Big| v\psi, \text{ since } u^+V \subseteq U \\
&\geq \bigwedge_{y\in U} y\psi^+ \Big| u\psi, \text{ by Lemma 1.9, since } u\psi \leq v\psi \\
&= (U,u)\overline{\psi},
\end{aligned}$$

as required.

We next show that $\overline{\psi}$ preserves corestrictions. If, with reference to (3.3), we put $W = (u|f)F \cup (u|f)^+U$ and $\diamondsuit = [(U,u)|(F,f)]\overline{\psi}$, then we have

$$\begin{aligned}
\diamondsuit &= \left(\bigwedge_{x\in W} x\psi^+\right)(u|f)\psi \\
&= \left((u|f)\psi^+ \wedge \bigwedge_{y\in F}[(u|f)y]\psi^+ \wedge \bigwedge_{z\in U}[(u|f)^+z]\psi^+\right)(u|f)\psi. \qquad (4.2)
\end{aligned}$$

Notice that we have pulled an extra 'factor' of $(u|f)\psi^+$ out of the left-hand side. Further copies of this factor appear in $\bigwedge_{y\in F}[(u|f)y]\psi^+$ for $y = f$, by (C4), and in $\bigwedge_{z\in U}[(u|f)^+z]\psi^+$ for $z = u$, by Lemma 1.10.

Consider the expression $(u|f)\psi^+ \wedge [(u|f)y]\psi^+$, for $y \in F$. By Lemma 2.8 with $a = u|f$ and $b = y$, we can rewrite this as $((u|f)\psi \otimes y\psi)^+$. We now take the first two terms on the right-hand side of (4.2) and introduce $|F| - 2$ additional 'factors' of $(u|f)\psi^+$. By repeatedly applying Lemma 2.8, we obtain:

$$(u|f)\psi^+ \wedge \bigwedge_{y\in F}[(u|f)y]\psi^+ = \bigotimes_{y\in F}((u|f)\psi \otimes y\psi)^+, \qquad (4.3)$$

since $\wedge$ and $\otimes$ coincide.

Recall that restriction (where it exists) and corestriction also coincide with the pseudoproduct in an inductive constellation. We can use this fact, together with (4.3), to write:

$$\diamondsuit = \left(\bigotimes_{y\in F}((u|f)\psi \otimes y\psi)^+\right) \otimes \left(\bigotimes_{z\in U}[(u|f)^+z]\psi^+\right) \otimes u\psi \otimes f\psi,$$

where we have also used (OP3) to write $(u|f)\psi = u\psi|f\psi = u\psi \otimes f\psi$.

Notice now that the hypothesis of (IP) holds for the factors $[(u|f)^+z]\psi^+ = [(u|f)^+|z]\psi^+$, since $(u|f)^+ \leq u^+ = z^+$. We can therefore replace each factor $[(u|f)^+z]\psi^+$ by $(u|f)^+\psi \wedge z\psi^+ = (u|f)^+\psi \otimes z\psi^+$ to obtain:

$$\begin{aligned}\diamondsuit &= \left(\bigotimes_{y\in F}((u|f)\psi \otimes y\psi)^+\right) \otimes \left(\bigotimes_{z\in U}((u|f)^+\psi \otimes z\psi^+)\right) \otimes u\psi \otimes f\psi \\ &= \left(\bigotimes_{y\in F}((u|f)\psi \otimes y\psi)^+\right) \otimes (u|f)^+\psi \otimes \left(\bigotimes_{z\in U} z\psi^+\right) \otimes u\psi \otimes f\psi,\end{aligned}$$

where we have retained only a single factor of $(u|f)^+\psi$ (this factor lies in F, by Lemma 2.7).

Notice that every factor in the above expression is an idempotent in F, with the exception of the factor of $u\psi$. We can therefore pull out an extra factor of $u\psi^+$ from the $\bigotimes_{z\in U} z\psi^+$ term and repeatedly commute idempotents to bring this to the front of the expression. We now consider the factor $u\psi^+ \otimes ((u|f)\psi \otimes y\psi)^+$, for $y \in F$. We know that $y\psi^+ \leq y^+\psi = f\psi$. We therefore have:

$$\begin{aligned}u\psi^+ \otimes ((u|f)\psi \otimes y\psi)^+ &= \left(u\psi^+ \otimes ((u|f)\psi \otimes y\psi)^+\right)^+, \text{ by (1.5)} \\ &= \left(u\psi^+ \otimes (u|f)\psi \otimes y\psi\right)^+, \text{ by (1.4)} \\ &= (u\psi^+ \otimes u\psi \otimes f\psi \otimes y\psi)^+, \text{ by (OP3)} \\ &= (u\psi \otimes f\psi \otimes y\psi)^+ \\ &= (u\psi \otimes f\psi \otimes y\psi^+)^+, \text{ by (1.4)} \\ &= (u\psi \otimes y\psi^+)^+, \text{ since } y\psi^+ \leq f\psi \\ &= (u\psi \otimes y\psi)^+, \text{ by (1.4).}\end{aligned}$$

So, by bringing out further factors of $u\psi^+$, we can repeat this process and replace the factor

$$\bigotimes_{y\in F}((u|f)\psi \otimes y\psi)^+$$

in the expression for $\diamondsuit$ by

$$\bigotimes_{y\in F}(u\psi \otimes y\psi)^+. \tag{4.4}$$

Notice also that $u\psi \otimes f\psi = u\psi|f\psi = (u|f)\psi$ and that $(u|f)\psi^+ \leq (u|f)^+\psi$, so that $(u\psi \otimes f\psi)^+ \leq (u|f)^+\psi$. Since $(u\psi \otimes f\psi)^+$ appears as a factor in (4.4), we can dispense with the factor of $(u|f)^+\psi$ and write:

$$\diamondsuit = \left(\bigotimes_{y\in F}(u\psi \otimes y\psi)^+\right) \otimes \left(\bigotimes_{z\in U} z\psi^+\right) \otimes u\psi \otimes f\psi.$$

We now apply the (1.3) to the final two terms in this expression to obtain

$$\begin{aligned}\diamondsuit &= \left(\bigotimes_{y\in F}(u\psi\otimes y\psi)^+\right)\otimes\left(\bigotimes_{z\in U} z\psi^+\right)\otimes(u\psi\otimes f\psi)^+\otimes u\psi\\ &= \left(\bigotimes_{y\in F}(u\psi\otimes y\psi)^+\right)\otimes\left(\bigotimes_{z\in U} z\psi^+\right)\otimes u\psi, \qquad (4.5)\end{aligned}$$

since the factor of $(u\psi\otimes f\psi)^+$ already occurs as part of the first term.

Let us now consider $(U,u)\overline{\psi}|(F,f)\overline{\psi}$:

$$\begin{aligned}(U,u)\overline{\psi}|(F,f)\overline{\psi} &= \left[\left(\bigwedge_{z\in U} z\psi^+\right)u\psi\right]\Bigg|\bigwedge_{y\in F} y\psi^+, \text{ using (4.1)}\\ &= \left(\bigotimes_{z\in U} z\psi^+\right)\otimes u\psi\otimes\left(\bigotimes_{y\in F} y\psi^+\right). \qquad (4.6)\end{aligned}$$

For $g\in F$, the product $u\psi\otimes g\psi^+$ can be rewritten $(u\psi\otimes g\psi)^+\otimes u\psi$, by (1.3). We therefore repeatedly apply (1.3) to $u\psi$ and the left-most remaining factor of the final term of (4.6) to obtain (after also applying (1.1)):

$$(U,u)\overline{\psi}|(F,f)\overline{\psi} = \left(\bigotimes_{z\in U} z\psi^+\right)\otimes\left(\bigotimes_{y\in F}(u\psi\otimes y\psi)^+\right)\otimes u\psi,$$

which is precisely (4.5). Thus $\overline{\psi}$ preserves corestrictions.

We show that $\overline{\psi}$ preserves $^+$:

$$(U,u)\overline{\psi}^+ = \left[\left(\bigwedge_{y\in U} y\psi^+\right)u\psi\right]^+ = \bigwedge_{y\in U} y\psi^+ = (U,u^+)\overline{\psi} = (U,u)^+\overline{\psi},$$

by (4.1).

It remains to show that $\overline{\psi}$ preserves multiplication. Let $(U,u),(V,v)\in \mathrm{Sz}(P)$ with $\exists(U,u)(V,v)$, i.e., $\exists uv$ in P and $uV\subseteq U$. We have the following rule, from Lemma 1.6:

$$\begin{aligned}\exists(U,u)\overline{\psi}(V,v)\overline{\psi} &\iff (U,u)\overline{\psi}\big|\,(V,v)\overline{\psi}^+ = (U,u)\overline{\psi}\\ &\iff (U,u)\overline{\psi}\big|\,(V,v^+)\overline{\psi} = (U,u)\overline{\psi}.\end{aligned}$$

The right-hand side is easily shown:

$$\begin{aligned}(U,u)\overline{\psi}|(V,v^+)\overline{\psi} &= [(U,u)|(V,v^+)]\overline{\psi}, \text{ since } \overline{\psi} \text{ preserves corestrictions}\\ &= (U,u)\overline{\psi}, \text{ by Lemma 1.6 in } \mathrm{Sz}(P), \text{ since } \exists(U,u)(V,v).\end{aligned}$$

We now show that $(U,u)\overline{\psi}(V,v)\overline{\psi} = [(U,u)(V,v)]\overline{\psi}$. On the one hand, we have

$$[(U,u)(V,v)]\overline{\psi} = (U,uv)\overline{\psi} = \left(\bigwedge_{y\in U} y\psi^+\right)(uv)\psi = \left(\bigwedge_{y\in U} y\psi^+\right)\otimes (uv)\psi,$$

whilst, on the other, we have

$$\begin{aligned}(U,u)\overline{\psi}(V,v)\overline{\psi} &= \left(\bigwedge_{y\in U} y\psi^+\right)(u\psi)\left(\bigwedge_{x\in V} x\psi^+\right)(v\psi)\\ &= \left(\bigwedge_{y\in U} y\psi^+\right)\otimes u\psi\otimes\left(\bigotimes_{x\in V} x\psi^+\right)\otimes v\psi,\end{aligned}$$

where we have deliberately retained the initial '$\wedge$', since this factor requires no manipulation. Consider the factor $u\psi\otimes x\psi^+$, for $x\in V$. We have

$$\begin{aligned}u\psi\otimes x\psi^+ &= (u\psi\otimes x\psi^+)^+\otimes u\psi, \text{ by (1.3)}\\ &= (u\psi\otimes x\psi)^+\otimes u\psi, \text{ by (1.4)}\\ &= (u\psi^+\otimes (ux)\psi)^+\otimes u\psi, \text{ by (P)}\\ &= (u\psi^+\otimes (ux)\psi^+)^+\otimes u\psi, \text{ by (1.4)}\\ &= u\psi^+\otimes (ux)\psi^+\otimes u\psi, \text{ by (1.5)}\\ &= (ux)\psi^+\otimes u\psi^+\otimes u\psi, \text{ by (1.1)}\\ &= (ux)\psi^+\otimes u\psi.\end{aligned}$$

By repeating this process for each $x\in V$, we have

$$\begin{aligned}(U,u)\overline{\psi}(V,v)\overline{\psi} &= \left(\bigwedge_{y\in U} y\psi^+\right)\otimes\left(\bigotimes_{x\in V}(ux)\psi^+\right)\otimes u\psi\otimes v\psi\\ &= \left(\bigwedge_{y\in U} y\psi^+\right)\otimes\left(\bigwedge_{x\in V}(ux)\psi^+\right)\otimes u\psi^+\otimes (uv)\psi\\ &= \left(\bigwedge_{y\in U} y\psi^+\right)\otimes (uv)\psi,\end{aligned}$$

since $uV\subseteq U$, and the fact that $u\psi^+$ appears as a 'factor' in the first term. Hence $(U,u)\overline{\psi}(V,v)\overline{\psi} = [(U,u)(V,v)]\overline{\psi}$, as required. We have shown that $\overline{\psi}$ is an ordered radiant.

We must now show that $\overline{\psi}$ is unique. Suppose that $\varphi : \mathrm{Sz}(P)\to Q$ is another ordered radiant such that $\psi = \iota\varphi$. Then $s\iota\overline{\psi} = s\iota\varphi$, for all $s\in P$, so $\overline{\psi}=\varphi$ on $P\iota$. We must therefore have $\overline{\psi}=\varphi$ on the whole of $\mathrm{Sz}(P)$, by Proposition 3.6.

Conversely, suppose that $\overline{\psi}$ is an ordered radiant, and put $\psi = \iota\overline{\psi}$. It follows from Lemmas 3.5 and 2.10 that ψ is an ordered pre-radiant. $\square$

5. The category of inductive constellations and inductive pre-radiants

In this final section, we use Theorem 4.1 to show that inductive constellations, together with ordered pre-radiants, form a category. Further, this category is isomorphic to the category of left restriction semigroups and strong premorphisms. It is now convenient to define the latter.

Let S and T be left restriction semigroups and let $\theta : S \to T$. Then θ is a *strong premorphism* if

(i) $s\theta t\theta = (s\theta)^+(st)\theta$ and

(ii) $s\theta^+ \leq s^+\theta$

for all $s, t \in S$.

Notice that in a left restriction semigroup S, if $s \leq t$, then as $s = s^+t$ we have that $s^+ = (s^+t)^+ = s^+t^+ \leq t^+$. It follows that in (ii) above, $s\theta^+ \leq (s^+\theta)^+$ and $s\theta^+ = s\theta^+\, s^+\theta$.

Lemma 5.1. *Let S, T, U be left restriction semigroups, and let $\theta : S \to T, \psi : T \to S$ be strong premorphisms. Then $\theta\psi$ is a strong premorphism.*

Proof. Let $s, t \in S$. First, $(s\theta\psi)^+ \leq (s\theta^+)\psi$. By [6, Lemma 2.10], ψ preserves order. As $s\theta^+ \leq s^+\theta$ we therefore have that $(s\theta\psi)^+ \leq s^+\theta\psi$.

For products, we calculate

$$\begin{aligned}
s\theta\psi\, t\theta\psi &= (s\theta\psi)^+(s\theta\, t\theta)\psi \\
&= (s\theta\psi)^+(s\theta^+(st)\theta)\psi \\
&= (s\theta\psi)^+(s\theta^+\psi)^+\big(s\theta^+(st)\theta\big)\psi \\
&= (s\theta\psi)^+\big(s\theta^+\psi\big)\big((st)\theta\psi\big) \\
&= (s\theta\psi)^+(st)\theta\psi
\end{aligned}$$

so that $\theta\psi$ is a premorphism as claimed.

□

Corollary 5.2. *The class of left restriction semigroups, together with strong premorphisms, forms a category.*

The next result is immediate from the definition of ordered pre-radiant and Lemmas 3.4 and 3.5. We use the same notation as in Theorem 2.3, without danger of ambiguity.

Corollary 5.3. *Let P be an inductive constellation and let $\iota : P \to \mathrm{Sz}(P)$. Then $\mathbf{T}(\iota) : \mathbf{T}(P) \to \mathbf{T}(\mathrm{Sz}(P))$ is a strong premorphism.*

The proof of the next result is routine, and may be found in the thesis of the second author [8, Proposition 9.7.5].

Proposition 5.4. *Let S and T be left restriction semigroups, and let $\theta : S \to T$ be a strong premorphism. Then $\mathbf{P}(\theta) : \mathbf{P}(S) \to \mathbf{P}(T)$ is an ordered pre-radiant.*

Theorem 4.1 enables us to prove the converse to the above.

Lemma 5.5. *Let P and Q be inductive constellations and let $\theta : P \to Q$ be an ordered pre-radiant. Then $\mathbf{T}(\theta) : \mathbf{T}(P) \to \mathbf{T}(Q)$ is a strong premorphism.*

Proof. We need only consider products. Let $s, t \in \mathbf{T}(P)$. Then with $\overline{\theta} : \mathrm{Sz}(P) \to Q$ defined as in Theorem 4.1, we have that

$$\begin{aligned} s\theta \otimes t\theta &= s\iota\overline{\theta} \otimes t\iota\overline{\theta} & \\ &= (s\iota \otimes t\iota)\overline{\theta} & \text{by Lemma 2.4} \\ &= \big(s\iota^+ \otimes (s \otimes t)\iota\big)\overline{\theta} & \text{by Lemma 3.4} \\ &= (s\iota\overline{\theta})^+ \otimes (s \otimes t)\iota\overline{\theta} & \\ &= (s\theta)^+ \otimes (s \otimes t)\theta. & \end{aligned}$$

□

It follows from Lemma 5.1, Proposition 5.4 and Lemma 5.5 that the class of inductive constellations, together with ordered pre-radiants, forms a category, and, moreover, the following is clear.

Theorem 5.6. *The categories of left restriction semigroups and strong premorphisms and of inductive constellations and ordered pre-radiants are mutually isomorphic via functors $\mathbf{P}$ and $\mathbf{T}$.*

Our notion of the Szendrei expansion of an inductive constellation was motivated by those of the Szendrei expansions of other algebraic structures such as groups [21], inductive groupoids [4] and restriction semigroups [8]. We note that laborious calculations, detailed in [8, Proposition 10.2.6], show that if P is an inductive constellation, then $\mathbf{T}(\mathrm{Sz}(P)) = \mathrm{Sz}(\mathbf{T}(P))$ and if S is a left restriction semigroup, then $\mathbf{P}(\mathrm{Sz}(S)) = \mathrm{Sz}(\mathbf{P}(S))$. Here $\mathrm{Sz}(\mathbf{T}(P))$ and $\mathrm{Sz}(S)$ are the Szendrei expansions of the restriction semigroups $\mathbf{T}(P)$ and S, respectively.

We summarise our findings connecting the actions and partial actions of left restriction semigroups and the actions and partial actions of inductive constellations in the diagram below. Here $\theta : S \to \mathcal{PT}_X$ is a strong premorphism from a left restriction semigroup S to some $\mathcal{PT}_X$, or equivalently, S acts partially on X, $P = \mathbf{P}(S)$ and $\psi = \mathbf{P}(\theta)$. The lower square commutes by Theorem 5.6, the triangles by Theorem 4.1 and [6, Theorem 4.2] and the upper 'square' by [8, Proposition 10.2.6], applying $\mathbf{P}$ or $\mathbf{T}$.

$$\begin{tikzcd}
\mathrm{Sz}(P) \arrow[rrr, leftrightarrow, bend left] \arrow[dr, "\overline{\psi}"'] & P \arrow[l, "\iota'"'] \arrow[r, leftrightarrow] \arrow[d, "\psi"'] & S \arrow[r, "\iota"] \arrow[d, "\theta"] & \mathrm{Sz}(S) \arrow[dl, "\overline{\theta}"] \\
& \mathcal{C}_X \arrow[r, leftrightarrow] & \mathcal{PT}_X &
\end{tikzcd}$$

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