Locally Ehresmann Semigroups

Dandan Yang
Department of Mathematics
The University of York

1. Weakly U-abundant semigroups

Suppose that S is a semigroup and U is a non-empty subset of $E(S)$. A relation $\widetilde{\mathcal{L}}_U$ on S is defined by:

$$(a, b) \in \widetilde{\mathcal{L}}_U \quad \text{if and only if} \quad (\forall u \in U) \;\; au = a \Longleftrightarrow bu = b,$$

for any $a, b \in S$.

The relation $\widetilde{\mathcal{R}}_U$ is defined dually.

The join of the relations $\widetilde{\mathcal{L}}_U$ and $\widetilde{\mathcal{R}}_U$ is denoted by $\widetilde{\mathcal{D}}_U$ and the intersection by $\widetilde{\mathcal{H}}_U$.

Then, we have the following,

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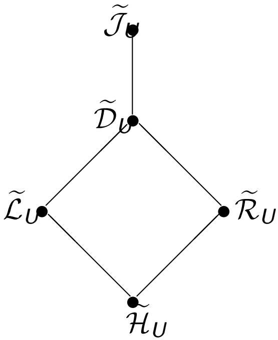
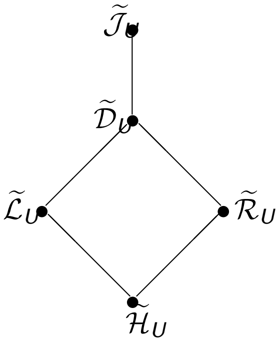


It can be verified that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_U$, $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_U$, $\mathcal{H} \subseteq \mathcal{H}^* \subseteq \widetilde{\mathcal{H}}_U$, $\mathcal{D} \subseteq \mathcal{D}^* \subseteq \widetilde{\mathcal{D}}_U$ and $\mathcal{J} \subseteq \mathcal{J}^* \subseteq \widetilde{\mathcal{J}}_U$.



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A semigroup S is said to be weakly U-abundant if every $\widetilde{\mathcal{L}}_U$-class and every $\widetilde{\mathcal{R}}_U$-class of S contains some element of U .

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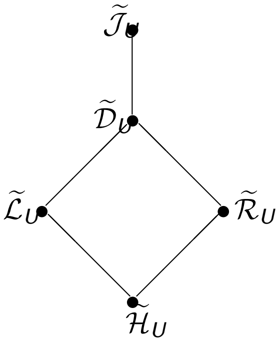

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However, in general, $\widetilde{\mathcal{R}}_U$ is not a left congruence and $\widetilde{\mathcal{L}}_U$ is not a right congruence. The following example will show this.

Let $S = \{e, 0, h, a, b\}$

	e	0	h	a	b
e	e	0	h	a	b
0	0	0	0	0	0
h	h	0	h	0	0
a	a	0	0	0	0
b	b	0	0	0	a

It is easy to verify that $\mathcal{L}^* = \mathcal{R}^* = \{\{e\},\{0\},\{h\},\{a\},\{b\}\}$. So S is not abundant. However, $\widetilde{\mathcal{L}} = \widetilde{\mathcal{R}} = \{\{e, a, b\},\{0\},\{h\}\}$. Hence S is a weakly abundant semigroup, but S does not satisfy the congruence condition.

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Let S be a weakly U-abundant semigroup and B the subsemigroup of S generated by U. For any $e \in U$, we use $\langle e \rangle$ to denote the subsemigroup of eBe generated by $eBe \cap U$. Then we introduce the following definition.

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We say that a weakly U-abundant semigroup S satisfies (WIC) if for any $a \in S$, $a^+ \in \widetilde{R}_a \cap U$ and $a^* \in \widetilde{L}_a \cap U$, there is an isomorphism $\alpha : \langle a^+ \rangle \to \langle a^* \rangle$ such that $xa = a(x\alpha)$, for any $x \in \langle a^+ \rangle$.

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Let $S = \{a, b, c, d, e, f, g, i, k\}$

	a	b	c	d	e	f	g	i	k
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b	b	b	d	d	f	f	g	i	k
c	c	c	c	c	f	f	g	i	k
d	d	d	d	d	d	d	g	i	k
e	e	e	c	c	c	c	g	i	k
f	f	f	d	d	d	d	g	i	k
g	g	g	g	g	g	g	g	i	k
i	i	i	i	i	i	i	i	i	i
k	k	k	k	k	k	k	k	i	i

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d	d	d	d	d	d	d	g	i	k
e	e	e	c	c	c	c	g	i	k
f	f	f	d	d	d	d	g	i	k
g	g	g	g	g	g	g	g	i	k
i	i	i	i	i	i	i	i	i	i
k	k	k	k	k	k	k	k	i	i

It is observed that S is not abundant, since there is no idempotent $\mathcal{L}^*$-related to the element $k \in S$.

Let $U = \{a, b, g\}$, a normal band. Then S is a weakly U-superabundant semigroup with (C). Moreover, $\widetilde{\mathcal{H}}_U$ is a congruence on S.

2. Background

In 2001, Mark Lawson gave a Rees matrix cover for a class of semigroups with local units having locally commuting idempotents. By saying local units we mean for any $a \in S$, there exists $e, f \in E(S)$ such that $a = ea = af$. The covering theorem is as following:

Theorem 2.1 Let S be a semigroup with local units having locally commuting idempotents. If S has a McAlister sandwich function, then there exists a semigroup U with local units whose idempotents commute, a Rees matrix semigroup $M = M(U; I; I; Q)$ over U, and a subsemigroup T of M that has local units, and a surjective homomorphism from T onto S that is a strict local isomorphism along which idempotents can be lifted.

2. Main Theorems

Theorem 2.2 Let S be a locally Ehresmann semigroup with a McAlister sandwich function. Then there exists an admissible strict local isomorphism from some locally Ehresmann semigroup $WM(T; I, \Lambda; Q)$ onto S along which idempotents can be lifted , where T is an Ehresmann semigroup and $Q = (q_{u,v})$.

Corollary 2.3 Let S be a locally Ehresmann semigroup with regularity condition. Then there exists an admissible strict local isomorphism from some locally Ehresmann semigroup $WM(T; I, \Lambda; Q)$ with regularity condition onto S along which idempotents can be lifted, where T is an Ehresmann semigroup and each entry of Q is regular in T.

Corollary 2.4 Let S be a locally Ehresmann semigroup with a McAlister sandwich function and (WIC). Then there exists an admissible strict local isomorphism from some locally Ehresmann semigroup $WM(T; I, \Lambda; Q)$ with (WIC) onto S along which idempotents can be lifted, where T is an Ehresmann semigroup and $Q = (q_{u,v})$.

3. Constructions

In this paper, we will talk about weakly U-abundant semigroup S with (C) in which $U = E(S)$. So, for convenience, we directly call S a weakly abundant semigroup with (C).

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Definition 3.1 A weakly abundant semigroup S with (C) is called a locally Ehresmann semigroup if the idempotents in each submonoid commute.

Definition 3.2 A function $p : E(S) \times E(S) \rightarrow S$, where we write $p(u, v) = p_{u,v}$, is called a McAlister sandwich function if the following conditions hold:
(i) $p_{u,v} \in uSv$ and $p_{u,u} = u$;
(ii) $p_{u,v} \in V(p_{v,u})$;
(iii) $p_{u,v}p_{v,f} \leqslant p_{u,f}$.

Proposition 3.3 Let S be a locally Ehresmann semigroup with a McAlister sandwich function. Then the set

$$W = \{(e, x, f) \in E \times S \times E : x \in eSf\}$$

with the multiplication given by $(e, x, f)(i, y, j) = (e, xp_{f,i}y, j)$ is a weakly abundant semigroup with (C), and in fact $E(W)$ is a normal band.

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Now we define a relation ρ on $W = E \times S \times E$ by

$$(e, x, f)\rho(i, y, j) \text{ iff}$$
$$(e, x, f) = (e, x, f)^{+}(i, y, j)(e, x, f)^{*},\ (i, y, j) = (i, y, j)^{+}(e, x, f)(i, y, j)^{*},$$

where $(e, x, f)^{+}\widetilde{\mathcal{R}}(e, x, f)\widetilde{\mathcal{L}}(e, x, f)^{*}$, $(i, y, j)^{+}\widetilde{\mathcal{R}}(i, y, j)\widetilde{\mathcal{L}}(i, y, j)^{*}$.

Then we have the following result.

Proposition 3.4 Let S be a locally Ehresmann semigroup. Then ρ is an admissible Ehresmann congruence on W, and the natural homomorphism $\rho^\natural$ from W onto W/ρ is an admissible strict local isomorphism, and $E(W/\rho) = E(W)\rho$. Hence, $T = W/\rho$ is an Ehresmann semigroup with (C) and the set of distinguished elements is $E(T)$.

Let $I = \Lambda = E(S)$, $q_{u,v} = (u, uv, v)\rho \in T$, $Q = (q_{u,v})$ and $M = (T; I, \Lambda; Q)$ be the Rees matrix semigroup over T. If we denote the set of all weakly abundant element of $M(T; I, \Lambda; Q)$ by $WM(T; I, \Lambda; Q)$, then $WM(T; I, \Lambda; Q)$ is a locally Ehresmann semigroup.

3. Constructions

The main result we get is as following.

Theorem 3.5 Let S be a locally Ehresmann semigroup with a McAlister sandwich function. Then there exists an admissible strict local isomorphism from some locally Ehresmann semigroup $WM(T; I, \Lambda; Q)$ onto S along which idempotents can be lifted, where T is an Ehresmann semigroup and $Q = (q_{u,v})$.

Now we consider some special cases of locally Ehresmann semigroup with a McAlister sandwich function.

Let S be a locally Ehresmann semigroup satisfying the regularity condition, that is, $RegS$ is a regular subsemigroup of S. Fix an idempotent $e \in E(S)$, for each $f \in E(S)$, define $f^* \in S(e, f)$. Then the function $p : E(S) \times E(S) \to S$ defined by $p_{u,v} = u^* v$ is in fact a McAlister sandwich function.

Corollary 3.6 Let S be a locally Ehresmann semigroup with regularity condition. Then there exists an admissible strict local isomorphism from some locally Ehresmann semigroup $WM(T; I, \Lambda; Q)$ with regularity condition onto S along which idempotents can be lifted, where T is an Ehresmann semigroup, $Q = (q_{u,v})$ and each entry of Q is regular in T.

Lemma 3.7 Let S and T locally Ehresmann semigroups. If there exists an admissible strict local isomorphism from S onto T, then S is (WIC) if and only if T is (WIC).

Corollary 3.8 Let S be a locally Ehresmann semigroup with a McAlister sandwich function and (WIC). Then there exists an admissible strict local isomorphism from some locally Ehresmann semigroup $WM(T; I, \Lambda; Q)$ with (WIC) onto S along which idempotents can be lifted, where T is an Ehresmann semigroup and $Q = (q_{u,v})$.