# Densely embedded ideals: a handy tool from the pages of Soviet semigroup theory

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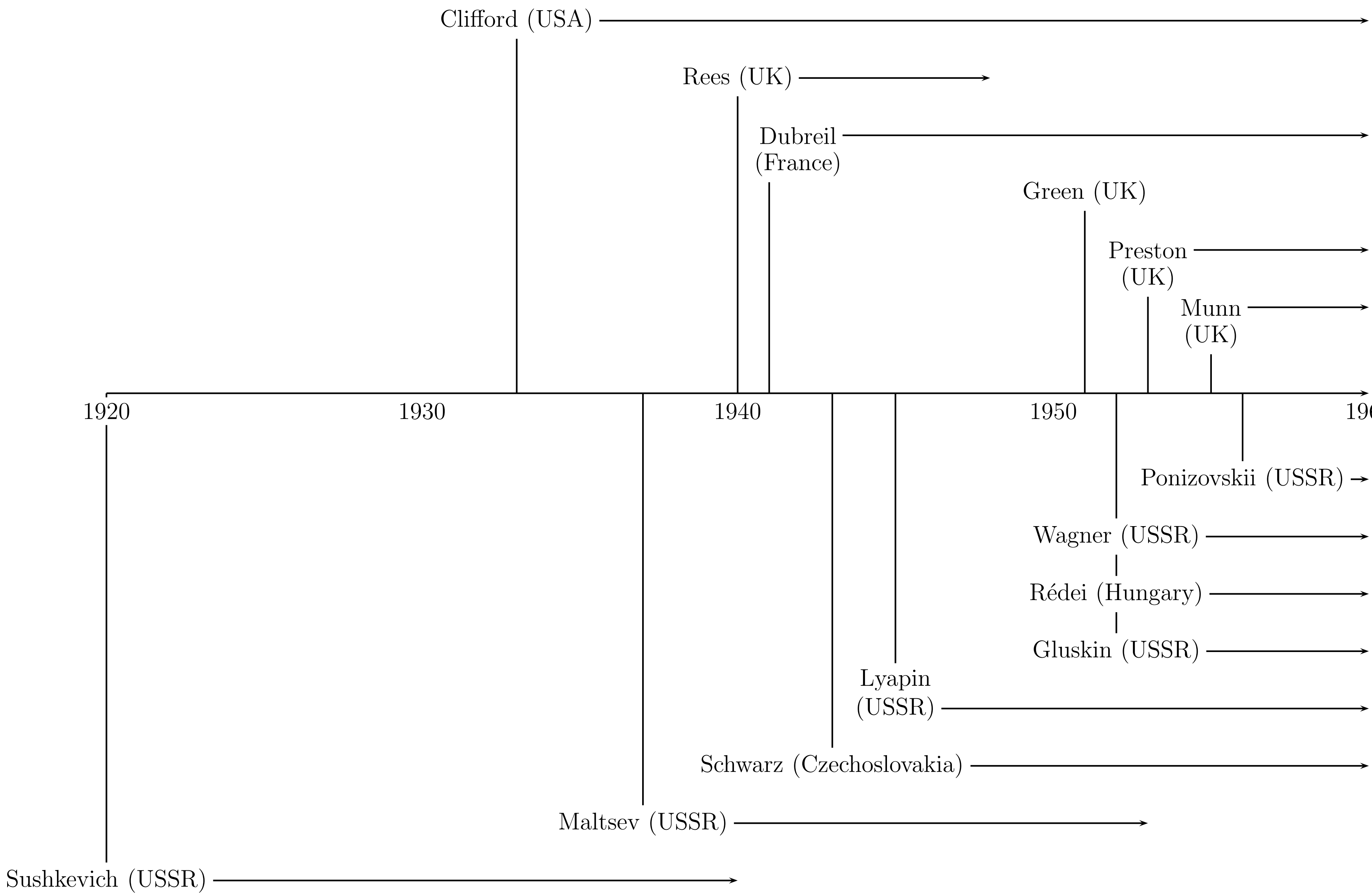
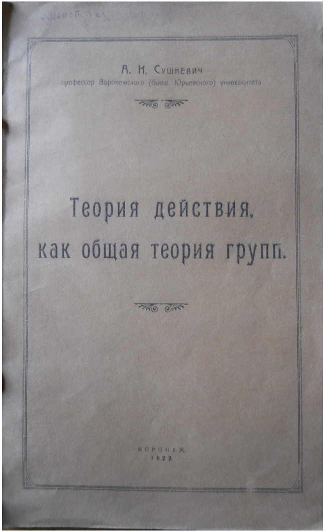
York Semigroup, 10th December 2012


Clifford (USA)
Rees (UK)
Dubreil (France)
Green (UK)
Preston (UK)
Munn (UK)
1920
1930
1940
1950
1960
Ponizovskii (USSR)
Wagner (USSR)
Rédei (Hungary)
Gluskin (USSR)
Lyapin (USSR)
Schwarz (Czechoslovakia)
Maltsev (USSR)
Sushkevich (USSR)

# Anton Kazimirovich Sushkevich (1889–1961)

# Kharkov State University

# *The theory of operations as the general theory of groups* (1922)



# Operations

An 'operation' on a set $X$ is simply a transformation $\theta : X \to X$.

Sushkevich studied collections of 'operations' that are closed under composition: (generalised) groups.

He most often studied 'operations' in the form of (generalised) substitutions, usually written in 'two-row' notation, e.g.,

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}.$$

His goal was to develop an abstract theory of generalised substitutions, by analogy with the abstract theory of 'ordinary substitutions' (i.e., group theory).

# Features of Sushkevich's work

- Mostly studied associative generalised groups.
- Interplay of one- or two-sided cancellation with one- or two-sided 'invertibility'.
- Concrete representations are never very far away: substitutions in earlier papers, then matrices as well in later ones.
- Proved the finite version of the generalised Cayley Theorem.

# Sushkevich's most famous contribution

A characterisation of a finite simple semigroup $\mathfrak{K}$ as a union of isomorphic groups $\mathfrak{C}_{\kappa\lambda}$.

$$
\begin{array}{ccc|ccccccc}
\mathfrak{K} & = & & \mathfrak{A}_1 & \cup & \mathfrak{A}_2 & \cup & \cdots & \cup & \mathfrak{A}_r \\
\| & & & \| & & \| & & & & \| \\
\cline{4-10}
\mathfrak{B}_1 & = & & \mathfrak{C}_{11} & \cup & \mathfrak{C}_{21} & \cup & \cdots & \cup & \mathfrak{C}_{r1} \\
\cup & & & \cup & & \cup & & & & \cup \\
\mathfrak{B}_2 & = & & \mathfrak{C}_{12} & \cup & \mathfrak{C}_{22} & \cup & \cdots & \cup & \mathfrak{C}_{r2} \\
\cup & & & \cup & & \cup & & & & \cup \\
\vdots & & & \vdots & & \vdots & & \vdots & & \vdots \\
\cup & & & \cup & & \cup & & & & \cup \\
\mathfrak{B}_s & = & & \mathfrak{C}_{1s} & \cup & \mathfrak{C}_{2s} & \cup & \cdots & \cup & \mathfrak{C}_{rs}
\end{array}
$$

# *Theory of generalised groups* (1937)

НАУЧНО-ИССЛЕДОВАТЕЛЬСКИЙ ИНСТИТУТ МАТЕМАТИКИ И МЕХАНИКИ ХАРЬКОВСКОГО ГОСУДАРСТВЕННОГО УНИВЕРСИТЕТА

Проф. А. К. СУШКЕВИЧ

## ТЕОРИЯ ОБОБЩЕННЫХ ГРУПП

ОНТИ ГОСУДАРСТВЕННОЕ НКТП
НАУЧНО-ТЕХНИЧЕСКОЕ ИЗДАТЕЛЬСТВО УКРАИНЫ
Харьков 1937 Киев

# Sushkevich's later work

- A little on substitutions on a countably infinite set.
- A little on semigroups of matrices.
- Linear algebra.
- History of mathematics.

# Evgenii Sergeevich Lyapin (1914–2005)

# St. Petersburg/Leningrad State University

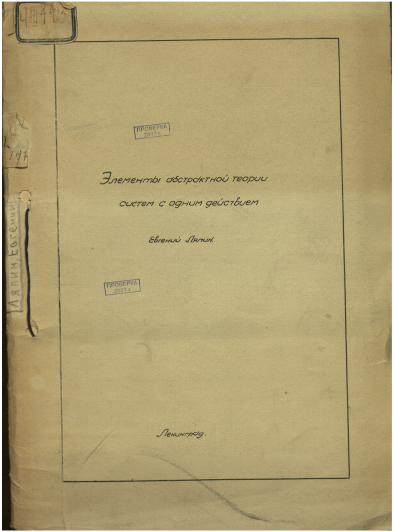
# St. Petersburg/Leningrad State University

# The Siege of Leningrad

# Lyapin and semigroups

- Apparently turned to semigroup-theoretic considerations in 1939.
- Influenced by Sushkevich and Maltsev at 1939 conference?
- Prepared doctoral dissertation on semigroups in 1945.
- Began to publish results from the thesis in 1947.
- Published the first major monograph on algebraic semigroups in 1960.

# *Elements of an abstract theory of systems with one operation* (1945)



# Leningrad State Pedagogical Institute (now Herzen Russian State Pedagogical University)

# Kernels of homomorphisms and normal subsystems

Let $\mathfrak{G}$ be an associative system with identity $1_{\mathfrak{G}}$.

Kernel of the homomorphism $\varphi : \mathfrak{G} \to \mathfrak{H}$ is $\{X \in \mathfrak{G} : \varphi X = 1_{\varphi \mathfrak{G}}\}$.

Call $\mathfrak{N} \subseteq \mathfrak{G}$ a normal subsystem of $\mathfrak{G}$ if

$$ANB \in \mathfrak{N} \Leftrightarrow AB \in \mathfrak{N},$$

for any $A, B \in \mathfrak{G}$ and any $N \in \mathfrak{N}$.

**Theorem (Lyapin, 1947)**

*The kernel of a homomorphism is a normal subsystem, and every normal subsystem is the kernel of some homomorphism.*

# Normal complexes

Let $\mathfrak{A}$ be an associative system; $\mathfrak{K} \subseteq \mathfrak{A}$ is a normal complex of $\mathfrak{A}$ if

$$XKY \in \mathfrak{K} \Leftrightarrow XK'Y \in \mathfrak{K}$$

$$XK \in \mathfrak{K} \Leftrightarrow XK' \in \mathfrak{K}$$

$$KY \in \mathfrak{K} \Leftrightarrow K'Y \in \mathfrak{K}$$

for any $X, Y \in \mathfrak{A}$ and any $K, K' \in \mathfrak{K}$.

Any normal subsystem is a normal complex, as is any two-sided ideal.

# Normal complexes

## Theorem (Lyapin, 1950a)

*Let $\mathfrak{K}$ be a subset of an associative system $\mathfrak{A}$.*

1. *$\exists$ homomorphism $\varphi$ of $\mathfrak{A}$ for which $\mathfrak{K}$ is the preimage of an element from $\varphi\mathfrak{A}$ iff $\mathfrak{K}$ is a normal complex.*
2. *$\exists$ homomorphism $\varphi$ of $\mathfrak{A}$ for which $\mathfrak{K}$ is the preimage of an identity in $\varphi\mathfrak{A}$ iff $\mathfrak{K}$ is a normal subsystem.*
3. *$\exists$ homomorphism $\varphi$ of $\mathfrak{A}$ for which $\mathfrak{K}$ is the preimage of a zero element in $\varphi\mathfrak{A}$ iff $\mathfrak{K}$ is an ideal.*

# ‘Simple’ associative systems

Lyapin termed an associative system simple if it contained no proper non-trivial normal subsystems.

### Theorem (Lyapin, 1950b)

*An associative system contains no proper normal subsystems if and only if it contains a zero element, and every element has a power equal to zero.*

*An associative system with an identity, but which does not form a group, contains no proper, non-trivial normal subsystems if and only if it contains a zero element, and every non-identity element has a power equal to zero.*

Get more interesting results from the study of ‘semisimple’ associative systems: those with no proper normal subsystems **besides** ideals or singletons.

# Densely embedded ideals

Let $\mathfrak{A}$ be an associative system. A (two-sided) ideal $\mathfrak{I}$ of $\mathfrak{A}$ is said to be densely embedded in $\mathfrak{A}$ if the following two conditions are satisfied:

1. amongst all homomorphisms of $\mathfrak{A}$, only the isomorphisms induce isomorphisms on $\mathfrak{I}$;
2. any associative system $\mathfrak{A}'$, in which $\mathfrak{A}$ is properly contained, and of which $\mathfrak{I}$ is an ideal, has a homomorphism which is not an isomorphism, but which induces an isomorphism on $\mathfrak{I}$.

# Densely embedded ideals in semigroups of one-one partial transformations

Let $\Omega$ be a set. The collection of all one-one partial transformations on $\Omega$ is denoted by $\mathcal{I}_\Omega$; $\mathcal{I}_\Omega$ forms a semigroup under the composition

$$\operatorname{dom} \alpha\beta = (\operatorname{im} \alpha \cap \operatorname{dom} \beta)\, \alpha^{-1}$$

and $x(\alpha\beta) = (x\alpha)\beta$, for any $x \in \operatorname{dom} \alpha\beta$.

$\mathcal{I}_\Omega$ has a subsemigroup $\mathfrak{B}_\Omega$, consisting of the empty transformation and all transformations with singleton domains.

In fact, $\mathfrak{B}_\Omega$ is a densely embedded ideal of $\mathcal{I}_\Omega$.

# Densely embedded ideals in semigroups of one-one partial transformations

Abstracting certain properties of $\mathfrak{B}_\Omega$, we say that an associative system $\mathfrak{A}$ belongs to class $\Sigma_1$ if

1. $\mathfrak{A}$ has a zero element 0;
2. for every $A \in \mathfrak{A}$, there exists a pair of idempotents $E, J \in \mathfrak{A}$ such that $EA = AJ = A$;
3. for every pair of non-zero idempotents $E, J \in \mathfrak{A}$, there exists a non-zero element $A \in \mathfrak{A}$ such that $EA = AJ = A$;
4. the product of any two distinct idempotents in $\mathfrak{A}$ is equal to 0.

$\mathfrak{B}_\Omega$ clearly belongs to class $\Sigma_1$; any associative system belonging to $\Sigma_1$ is isomorphic to some $\mathfrak{B}_\Omega$.

# Densely embedded ideals in semigroups of one-one partial transformations

**Theorem (Lyapin, 1953)**

*An associative system $\mathfrak{A}$ is isomorphic to a system of all one-one partial transformations of some set if and only if it contains a densely embedded ideal belonging to the class $\Sigma_1$.*

(If an associative system $\mathfrak{A}$ is isomorphic to some $\mathcal{I}_\Omega$, then its densely embedded ideal is of course isomorphic to $\mathfrak{B}_\Omega$.)

# Full transformations

An associative system $\mathfrak{A}$ is said to belong to class $\Sigma_2$ if, for all $X, Y \in \mathfrak{A}$, $XY = X$ ($\mathfrak{A}$ is a left zero semigroup).

**Theorem (Lyapin, 1953, 1955)**

*An associative system $\mathfrak{A}$ is isomorphic to a system of all self-mappings of some set if and only if it contains a densely embedded ideal belonging to the class $\Sigma_2$.*

# Lazar Matveevich Gluskin (1922–1985)

# Gluskin and semigroups of transformations

$\Omega$ — set; $C$ — circle; $V$ — vector space; $F$ — field.

| **Semigroup** | **Transformations** |
|---|---|
| $W(\Omega)$ | all partial transformations of $\Omega$ |
| $V(\Omega)$ | all 1-1 partial transformations of $\Omega$ |
| $S(\Omega)$ | all full transformations of $\Omega$ |
| $B(\Omega)$ | all 1-1 full transformations of $\Omega$ |
| $C(\Omega)$ | all conformal transformations of $C$ |
| $mP(F, \Omega)$ | all endomorphisms of $V$ over $F$ |
| $S'(F, \Omega)$ | all semi-linear transformations of $V$ over $F$ |

# Gluskin and densely embedded ideals

### Theorem (Gluskin, 1961)

*If $A$ is a semigroup without equi-acting elements, then there exists a semigroup $S$ which contains $A$ as a densely embedded ideal.*

($a, a'$ are equi-acting (or like) if $ax = a'x$ and $xa = xa'$, for all other elements $x$.)

(One choice for $S$ is a subsemigroup of the translational hull $\Psi(A) \times \Phi(A)$.)

### Theorem (Gluskin, 1961)

*If $A$ is a densely embedded ideal of a semigroup $S$, then any automorphism of $A$ may be extended uniquely to an automorphism of $S$.*

# Gluskin and densely embedded ideals

- Used densely embedded ideals to establish conditions for isomorphism of two abstract semigroups containing such ideals.
- Used one-sided versions of densely embedded ideals ($l$-dense ideals) to obtain necessary and sufficient conditions for isomorphism to semigroup of left translations.
- Used generalised densely embedded ideals ($d$-ideals) to tackle the case of arbitrary partial transformations.

# Densely embedded ideals in semigroups of matrices

Let $F$ be a non-commutative field. Denote by $G_n^r(F)$ the collection of all $n \times n$ matrices over $F$ with rank $\leq r$. $G_n(F)$ is the collection of *all* $n \times n$ matrices over $F$.

$G_n^1(F)$ is a densely embedded ideal of $G_n(F)$.

### Theorem (Gluskin, 1958)

*A semigroup $S$ is isomorphic to some $G_n(F)$ if and only if $S$ contains a densely embedded ideal which is isomorphic to $G_n^1(F)$.*

$G_n^1(F)$ is completely simple, so study of general matrix semigroups is reduced to that of matrix semigroups of a better-known class.

Leads also to conditions for $G_n^r(F)$ and $G_m^s(H)$ to be isomorphic (namely that $n = m$, $r = s$ and $F \cong H$).