On free subsemigroups in automata semigroups

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(joint work with Ivan Mitrofanov)

I. Introduction

Why? Complexity from simple rules

- Exotic properties
  - f.g. infinite periodic groups
  - intermediate growth
  - amenable but not elementary amenable
- concrete
  - action on a rooted tree
  - residually finite
  - solvable word problem

Questions
- What are they?
- What propenties do they have?
  Link between automaton properties and properties of semigroup?

Thm (Klimann '17): A group generated by a bireversible automaton has exponential growth if it contains an elmt of ∞ order.

Duality group $\leftrightsquigarrow$ free subsemigroup

II. Automata, (semi)groups and duality

Def: A Mealy automaton is $\mathcal{M} = (Q, A, \tau)$

states → Q, A ← alphabet finite sets, $\tau : Q \times A \to A \times Q$

$(q, a) \mapsto (q \cdot a,\ q@a)$

Def: Moore diagram $\mathcal{M} = (Q, A, \tau)$

graph Γ, $V(\Gamma) = Q$

edges: $q \in Q$, $a \in A$

$$(q) \xrightarrow{a \,|\, q \cdot a} (q@a)$$

$\mathcal{M} = (Q, A, \tau)$ $\quad \forall q \in Q \quad q \cdot : A \to A$

$\forall a \in A \quad @a : Q \to Q$

Def: If $q \cdot : A \to A$ $\forall q \in Q$ is bijective, $\mathcal{M}$ is <u>invertible</u>

If $@a : Q \to Q$ $\forall a \in A$ bij, $\mathcal{M}$ is <u>reversible</u>

$\mathcal{M}$ is <u>bireversible</u> if <u>invertible</u> + <u>reversible</u> + τ is bij.

$\forall q \in Q \quad q \cdot : A^* \to A^* \qquad q \cdot (aV) = (q \cdot a)(q@a) \cdot V$

Def: The semigroup gen. by $\mathcal{M} = (Q, A, \tau)$ is

$S_\mathcal{M} = Q^* / \mathrm{Ker}(Q^* \curvearrowright A^*)$. So $S_\mathcal{M} \curvearrowright A^*$

If $\mathcal{M}$ invertible, $G_\mathcal{M} = \langle S_\mathcal{M} \rangle \leq \mathrm{Aut}(A^*)$

Duality: $@a : Q^* \to Q^*$

Def: $\partial_\mathcal{M} = A^* / \mathrm{Ker}(Q^* \curvearrowleft A^*)$ dual semigroup

(Dual automaton: $\mathcal{M} = (Q, A, \tau)$, $\partial\mathcal{M} = (A, Q, \tilde{\tau})$)

$\partial_\mathcal{M}$ anti-isomorphic $S_{\partial\mathcal{M}}$

Recap

Prop:

$$\begin{array}{ccc} Q^* & \curvearrowright \curvearrowleft & A^* \\ \downarrow & & \downarrow \\ S_\mathcal{M} & \curvearrowright \curvearrowleft & \partial_\mathcal{M} \end{array}$$

If $\mathcal{M}$ invertible

$$\begin{array}{ccc} (Q \cup Q^{-1})^* & \curvearrowright \curvearrowleft & A^* \\ \downarrow & & \downarrow \\ G_\mathcal{M} & \curvearrowright \curvearrowleft & \partial_\mathcal{M} \end{array}$$

Dual action $\quad g \in S_\mathcal{M}$

Prop: $S_\mathcal{M}$ is finite $\iff$ $\partial_\mathcal{M}$ finite

Proof: Assume $\partial_\mathcal{M}$ finite. Pick $N \in \mathbb{N}$ large enough

<u>Claim</u> $g \in S_\mathcal{M}$ is uniquely determined by its action on A^N

g

$g@v = g@w$

$\square$

Prop: $\mathcal{M} = (Q, A, \tau)$ invertible, $u \in A^*$, $[u] \in \partial_\mathcal{M}$. TFAE

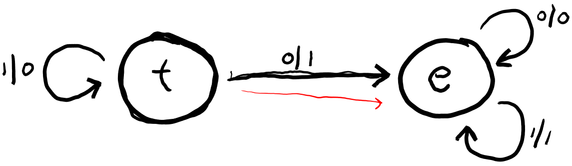
1) $|G_\mathcal{M} \cdot u^\infty| < \infty$

2) $\exists M \quad |G_\mathcal{M} \cdot [u^n]| < M \quad \forall n \in \mathbb{N}$

3) $[u] \in \partial_\mathcal{M}$ is torsion.

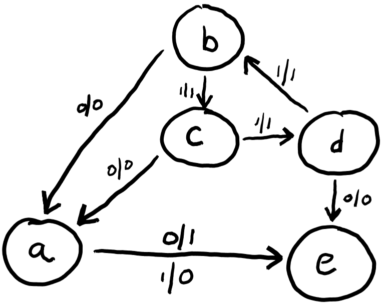
Examples

1) $Q = \{t, e\}$ $A = \{0, 1\}$

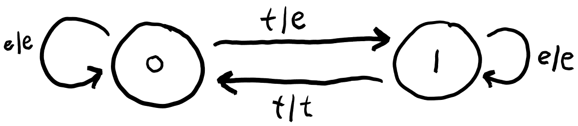


$t \cdot 1101 = 0011$

2) $Q = \{a, b, c, d, e\}$ $A = \{0, 1\}$



3) $Q = \{0, 1\}$ $A = \{t, e\}$



III. Free subsemigroups

Thm [F.-Mitrofanov 20]: If $\mathcal{M}$ is a reversible automaton, then, $S_{\mathcal{M}}$ contains a free (non commutative) subsemigroup

$\iff$ $S_{\mathcal{M}}$ contains an element of ∞ order.

Dual theorem: If $\mathcal{M} = (Q, A, \tau)$ is invertible, $\partial\mathcal{M}$ contains a free semigroup $\iff \exists\, u \in A^*$ s.t. $|G_{\mathcal{M}} \cdot u^\infty| = \infty$.

Proof: Case where $A = \{0,1\}$, $G_{\mathcal{M}} \curvearrowright A^n$ is transitive $\forall n \in \mathbb{N}$.

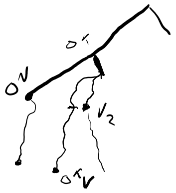
Assume no free semigroups in $\partial\mathcal{M}$

$\Rightarrow \exists\, v_1, v_2 \in A^*$ s.t. $v_1 \neq v_2$ but $[v_1] = [v_2]$

wlog, assume that $|v_1| = |v_2|$ (if not, look at $v_1 v_2$ and $v_2 v_1$)

Since $G_{\mathcal{M}}$ acts transitively, assume that $v_1 = 0^N$

$v_2 = 0^k 1 v_2'$



Let $0^k v \in A^n$, $n \geq N$

Claim: $\exists\, w \in A^n$, $w = 0^{k+1} w'$ s.t. $[0^k v] = [0^{k+1} w']$

$= [0][0^k w']$

$= [0^{k+2} w'] = \dots [0^{k+n-N+1} z]$

$\exists\, g \in G_{\mathcal{M}}$ s.t. $g \cdot 0^k v = v_2 v'$

$$g \cdot [0^k v] = [v_2][v'] = [0^N][v']$$

$$[0^k v] = g^{-1} \cdot ([0^N][v'])$$

$$= [0^{k+1} w']$$

In short, how many elements of the form $0^k v \in A^n$?

$|A|^{n-1-k}$

THANK YOU !